

Theorem: Let $X_1, X_2, \dots, X_n \sim N(\mu, 1/\tau)$ and suppose you have **priors** on $\mu|\tau$ and τ ,

$$\begin{aligned}\mu|\tau &\sim N(\mu_0, 1/(\lambda_0\tau)) \\ \tau &\sim \text{Gamma}(\alpha_0, \beta_0)\end{aligned}$$

Then, the **posteriors** on $\mu|\tau$ and τ are,

$$\begin{aligned}\mu|\tau &\sim N(\mu_1, 1/(\lambda_1\tau)) \\ \tau &\sim \text{Gamma}(\alpha_1, \beta_1)\end{aligned}$$

where $\mu_1 = \frac{\lambda_0\mu_0 + n\bar{x}}{\lambda_0 + n}$, $\lambda_1 = \lambda_0 + n$, $\alpha_1 = \alpha_0 + \frac{n}{2}$, $\beta_1 = \beta_0 + \frac{1}{2} \sum_{i=1}^n (x_i - \bar{x})^2 + \frac{n\lambda_0(\bar{x} - \mu_0)^2}{2(\lambda_0 + n)}$.

Proof:

$$f(\underline{x}|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left[-\frac{1}{2}\tau \sum_{i=1}^n (x_i - \mu)^2\right]$$

$$h_1(\mu|\tau) = \left(\frac{\lambda_0\tau}{2\pi}\right)^{1/2} \exp\left[-\frac{1}{2}\lambda_0\tau(\mu - \mu_0)^2\right]$$

$$h_2(\tau) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \tau^{\alpha_0-1} e^{-\beta_0\tau}$$

Note, μ and τ are **not** independent, and $h(\mu, \tau) = h_1(\mu|\tau) h_2(\tau)$

$$\begin{aligned}\varphi(\mu, \tau|\underline{x}) &\propto f(\underline{x}|\mu, \tau) h_1(\mu|\tau) h_2(\tau) \\ &\propto \tau^{\alpha_0 + (n+1)/2 - 1} \exp\left[-\frac{\tau}{2}\left(\lambda_0[\mu - \mu_0]^2 + \sum_{i=1}^n (x_i - \mu)^2\right) - \beta_0\tau\right]\end{aligned}\quad (1)$$

As seen in class, we can add and subtract \bar{x} inside $(x_i - \mu)^2$ to get:

$$\sum_{i=1}^n (x_i - \mu)^2 = \sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2\quad (2)$$

By adding and subtracting μ_1 :

$$n(\bar{x} - \mu)^2 + \lambda_0(\mu - \mu_0)^2 = (\lambda_0 + n)(\mu - \mu_1)^2 + \frac{n\lambda_0(\bar{x} - \mu_0)^2}{\lambda_0 + n}\quad (3)$$

Combining (2) and (3) we get:

$$\sum_{i=1}^n (x_i - \mu)^2 + \lambda_0(\mu - \mu_0)^2 = (\lambda_0 + n)(\mu - \mu_1)^2 + \sum_{i=1}^n (x_i - \bar{x})^2 + \frac{n\lambda_0(\bar{x} - \mu_0)^2}{\lambda_0 + n}\quad (4)$$

By plugging (4) into (1) we get:

$$\varphi(\mu, \tau|\underline{x}) \propto \left\{ \tau^{1/2} \exp\left[-\frac{1}{2}\lambda_1\tau(\mu - \mu_1)^2\right] \right\} (\tau^{\alpha_1-1} e^{-\beta_1\tau})\quad (5)$$

$$\varphi(\mu, \tau|\underline{x}) = \varphi_1(\mu|\tau, \underline{x})\varphi_2(\tau|\underline{x})\quad (6)$$

□

Theorem: Let $X_1, X_2, \dots, X_n \sim N(\mu, 1/\tau)$ and suppose you have priors on $\mu|\tau$ and τ ,

$$\begin{aligned}\mu|\tau &\sim N(\mu_0, 1/(\lambda_0\tau)) \\ \tau &\sim \text{Gamma}(\alpha_0, \beta_0)\end{aligned}$$

Then, the marginal posterior distribution of μ can be written as:

$$\left(\frac{\lambda_1\alpha_1}{\beta_1}\right)^{1/2} (\mu - \mu_1) \sim t_{2\alpha_1}$$

where $\mu_1, \lambda_1, \alpha_1$, and β_1 are given in the previous theorem.

Proof:

First, let

$$\begin{aligned}z &= (\lambda_1\tau)^{1/2}(\mu - \mu_1) = u(\mu) \\ \mu &= z(\lambda_1\tau)^{-1/2} + \mu_1 = w(z)\end{aligned}$$

We know (from the previous theorem):

$$\varphi(\mu, \tau|\underline{x}) = \varphi_1(\mu|\tau, \underline{x}) \varphi_2(\tau|\underline{x}) \quad (7)$$

$$\text{So, } \varphi(z, \tau|\underline{x}) = \varphi_1(w(z)|\tau, \underline{x}) \left| \frac{\partial w(z)}{\partial z} \right| \varphi_2(\tau|\underline{x}) \quad (8)$$

$$= \varphi_1(z(\lambda_1\tau)^{-1/2} + \mu_1|\tau, \underline{x}) |(\lambda_1\tau)^{-1/2}| \varphi_2(\tau|\underline{x}) \quad (9)$$

$$= \sqrt{\frac{\lambda_1\tau}{2\pi}} \exp\left\{-\frac{(z(\lambda_1\tau)^{-1/2} + \mu_1 - \mu_1)^2}{2(\lambda_1\tau)^{-1}}\right\} (\lambda_1\tau)^{-1/2} \varphi_2(\tau|\underline{x}) \quad (10)$$

$$= \sqrt{\frac{1}{2\pi}} \exp(-z^2/2) \varphi_2(\tau|\underline{x}) \quad (11)$$

$$= \Phi(z|\underline{x}) \varphi_2(\tau|\underline{x}) \quad (12)$$

$$(13)$$

Which gives us:

$$\begin{aligned}Z|\underline{x} &\sim N(0, 1) \quad \tau|\underline{x} \sim \text{Gamma}(\alpha_1, \beta_1) \quad (\text{Independent!}) \\ \text{Let } Y &= 2\beta_1\tau \rightarrow Y|\underline{x} \sim \text{Gamma}(\alpha_1, 1/2) \equiv \chi_{2\alpha_1}^2\end{aligned}$$

So, creating a t random variable:

$$U = \frac{Z}{\sqrt{Y/2\alpha_1}} = \frac{(\lambda_1\tau)^{1/2}(\mu - \mu_1)}{\sqrt{2\beta_1\tau/2\alpha_1}} = \left(\frac{\lambda_1\alpha_1}{\beta_1}\right)^{1/2} (\mu - \mu_1) \quad (14)$$

Which gives:

$$\left(\frac{\lambda_1\alpha_1}{\beta_1}\right)^{1/2} (\mu - \mu_1) \sim t_{2\alpha_1} \quad (15)$$

□