Let $X_1, X_2, \ldots, X_n \sim N(\theta, \theta)$, independently. Find the Fisher Information for $\theta$ in the random sample.

Solution:

$$f(x|\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\{- (x - \theta)^2 / 2\theta\}$$

$$\lambda(x|\theta) = \frac{-1}{2} \ln 2\pi - \frac{1}{2} \ln \theta - \frac{(x - \theta)^2}{2\theta}$$

$$= \frac{-1}{2} \ln 2\pi - \frac{1}{2} \ln \theta - \frac{x^2}{2\theta} + \frac{2x\theta}{2\theta} - \frac{\theta^2}{2\theta}$$

$$\lambda'(x|\theta) = \frac{-1}{2\theta} + \frac{x^2}{2\theta^2} + 0 - \frac{1}{2}$$

$$\lambda''(x|\theta) = \frac{1}{2\theta^2} - \frac{2x^2}{2\theta^3}$$

$$I(\theta) = -E[\lambda''(X|\theta)]$$

$$= \frac{-1}{2\theta^2} + \frac{E[X^2]}{\theta^3}$$

$$= \frac{-1}{2\theta^2} + \frac{\theta + \theta^2}{\theta^3}$$

$$= \frac{1 + 2\theta}{\theta^2}$$

$$I_n(\theta) = \frac{n(1 + 2\theta)}{\theta^2}$$

Note, by differentiating the log-likelihood (and using the quadratic formula), you can find that the MLE is:

$$\hat{\theta} = \frac{-1 + \sqrt{1 + 4 \sum_{i=1}^{n} x_i^2}}{2}$$

Using the CRLB and the theorems in the book, we can show that the MLE has an approximate normal distribution centered around $\theta$ with variance given above by the Fisher Information.