Math 152 – Fall 2008
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warm-up # 11

Name: ________________________________

Let $X_1, X_2, \ldots, X_n \sim N(\theta, \theta)$, independently. Find the Fisher Information for θ in the random sample.

Solution:

$$
\begin{aligned}
f(x|\theta) &= \frac{1}{\sqrt{2\pi\theta}} exp\{-(x-\theta)^2/2\theta\} \\
\lambda(x|\theta) &= \frac{-1}{2}\ln 2\pi - \frac{1}{2}\ln\theta - \frac{(x-\theta)^2}{2\theta} \\
&= \frac{-1}{2}\ln 2\pi - \frac{1}{2}\ln\theta - \frac{x^2}{2\theta} + \frac{2x\theta}{2\theta} - \frac{\theta^2}{2\theta} \\
\lambda'(x|\theta) &= \frac{-1}{2\theta} + \frac{x^2}{2\theta^2} + 0 - \frac{1}{2} \\
\lambda''(x|\theta) &= \frac{1}{2\theta^2} - \frac{2x^2}{2\theta^3} \\
I(\theta) &= -E[\lambda''(X|\theta)] \\
&= \frac{-1}{2\theta^2} + \frac{E[X^2]}{\theta^3} \\
&= \frac{-1}{2\theta^2} + \frac{(\theta+\theta^2)}{\theta^3} \\
&= \frac{1+2\theta}{\theta^2} \\
I_n(\theta) &= \frac{n(1+2\theta)}{\theta^2}
\end{aligned}
$$

Note, by differentiating the log-likelihood (and using the quadratic formula), you can find that the MLE is:

$$\hat{\theta} = \frac{-1 + \sqrt{1 + 4\frac{\sum x_i^2}{n}}}{2}$$

Using the CRLB and the theorems in the book, we can show that the MLE has an approximate normal distribution centered around θ with variance given above by the Fisher Information.