Math 152 – Fall 2018
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warm-up # 11 - UMP simple tests

Name: ________________________________

Let's say we have 2 batches of paint one of which is quick-dry. The paint is unlabeled, and we forgot which was which! We paint 5 boards from batch 1 and record the drying time. We think batch 1 is quick dry. We also believe that the drying times are normally distributed with a standard deviation of 5 min.

$$H_0 : \theta = 25 \text{ min}$$
$$H_1 : \theta = 10 \text{ min}$$

Find δ^* that minimizes $\beta(\delta^*)$ subject to $\alpha(\delta^*) < .05$.

Solution

For funzies, let's find δ^* that minimizes $a\alpha(\delta^*) + b\beta(\delta^*)$.

$$\begin{aligned}
f_0(\underline{x}) &= \frac{1}{\sqrt{2\pi 25}} \exp\left(\frac{-1}{2 \cdot 25} \sum (x_i - 25)^2\right) \\
f_1(\underline{x}) &= \frac{1}{\sqrt{2\pi 25}} \exp\left(\frac{-1}{2 \cdot 25} \sum (x_i - 10)^2\right) \\
\frac{f_1(\underline{x})}{f_0(\underline{x})} &= \exp\left(\frac{-1}{2 \cdot 25} \sum ((x_i - 10)^2 - (x_i - 25)^2)\right) \\
&= \exp\left(\frac{-1}{2 \cdot 25} \sum (x_i^2 - 20x_i + 100 - x_i^2 + 50x_i - 625)\right) \\
&= \exp\left(\frac{-1}{50} \sum (30x_i - 525)\right) \\
&= \exp\left(\frac{-30}{50} \sum (x_i - 17.5)\right) \\
&= \exp\left(\frac{-3n}{5} (\overline{x} - 17.5)\right) \\
&> \frac{a}{b} \\
\overline{x} - 17.5 &< \frac{-5}{3n} \ln\left(\frac{a}{b}\right) \\
\overline{x} &< 17.5 - \frac{5}{3n} \ln\left(\frac{a}{b}\right)
\end{aligned}$$

- If a type I error is worse ($\alpha(\delta) <<$) then $\frac{a}{b} > 1$, and your rejection rule would be when $\overline{x} < c_1$ where $c_1 < 17.5$ (we reject H_0 less often).

- If a type II error is worse ($\beta(\delta) <<$) then $\frac{a}{b} < 1$, and your rejection rule would be when $\overline{x} < c_2$ where $c_2 > 17.5$ (we reject H_0 more often).

$$\delta^* = \{\text{reject } H_0 \text{ if } \overline{x} < 17.5 - \frac{5}{3n}\ln(k)\}$$

$$\begin{aligned}
P(\overline{X} < 17.5 - \frac{5}{3n}\ln(k)|\theta = 25) &= 0.05 \\
P(Z < \frac{17.5 - 5/3n\ln(k) - 25}{5/\sqrt{n}}) &= 0.05 \\
\frac{17.5 - 5/3n\ln(k) - 25}{5/\sqrt{n}} &= -1.645 \\
\ln(k) &= -11.47 \\
\delta^* = \{\text{reject } H_0 \text{ if } \overline{x} < 21.32\} & \\
\text{note: } P(\overline{X} > 21.32|\theta = 10) &= 0
\end{aligned}$$

The problem can be done without so much algebra. The test must be:

$$\delta^* = \{\text{reject } H_0 \text{ if } \overline{x} < \text{ some constant }\}$$

So, we set the probability of rejecting H_0 when H_0 is true to 0.05.

$$\begin{aligned}
P(\overline{X} < c|\theta = 25) &= 0.05 \\
P(Z < \frac{c - 25}{5/\sqrt{n}}) &= 0.05 \\
\frac{c - 25}{5/\sqrt{n}} &= -1.645 \\
c &= -1.645 \cdot 5/\sqrt{5} + 25 \\
\delta^* = \{\text{reject } H_0 \text{ if } \overline{x} < 21.32\} & \\
\text{note: } P(\overline{X} > 21.32|\theta = 10) &\approx 0
\end{aligned}$$