Math 152 – Fall 2018
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warm-up # 12 - MLR & UMP

Name: ______________________________

Suppose $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} N(0, \sigma^2)$, and it is desired to test:

$$H_0 : \sigma^2 \leq 2$$
$$H_1 : \sigma^2 > 2$$

1. Find the MLR statistic. For which types of values of T would you reject H_0 (i.e., big ones or small ones)?

2. Assume $n = 20, \alpha_0 = 0.01$, find the UMP (Uniformly Most Powerful) test.

Solution:
Let $\sigma_1^2 < \sigma_2^2, \sigma_1^2, \sigma_2^2 \in \Omega$.

$$\begin{aligned}
\frac{f(\underline{x}|\sigma_2^2)}{f(\underline{x}|\sigma_1^2)} &= \left(\frac{\sigma_1}{\sigma_2}\right)^n \frac{e^{-\sum x_i^2/2\sigma_2^2}}{e^{-\sum x_i^2/2\sigma_1^2}} \\
&= \left(\frac{\sigma_1}{\sigma_2}\right)^n e^{\sum x_i^2(1/2\sigma_1^2 - 1/2\sigma_2^2)} \\
(1/2\sigma_1^2 - 1/2\sigma_2^2) &> 0 \\
T &= \sum X_i^2 \\
\delta : & \quad \{\text{reject } H_0 \text{ if } \sum x_i^2 \geq c\}
\end{aligned}$$

Note that the exact same work leads to answers for the following slightly more interesting set of questions:

1. Find the UMP (Uniformly Most Powerful) test at level α_0.

2. Assume $n = 20, \alpha_0 = 0.01$, find the actual cutoff for the test (draw a picture).

3. Find the power function for the test.

Solution:
Let $\sigma_1^2 < \sigma_2^2, \sigma_1^2, \sigma_2^2 \in \Omega$.

$$\begin{aligned}
\frac{f(\underline{x}|\sigma_2^2)}{f(\underline{x}|\sigma_1^2)} &= \left(\frac{\sigma_1}{\sigma_2}\right)^n \frac{e^{-\sum x_i^2/2\sigma_2^2}}{e^{-\sum x_i^2/2\sigma_1^2}} \\
&= \left(\frac{\sigma_1}{\sigma_2}\right)^n e^{\sum x_i^2(1/2\sigma_1^2 - 1/2\sigma_2^2)} \\
(1/2\sigma_1^2 - 1/2\sigma_2^2) &> 0 \\
T &= \sum X_i^2 \\
\delta : & \quad \{\text{reject } H_0 \text{ if } \sum x_i^2 \geq c\}
\end{aligned}$$

is UMP at α_0. For $n = 20, \alpha_0 = 0.01$, find c.

$$\begin{aligned}
P(\sum X_i^2 \geq c|\sigma^2 = 2) &= 0.01 \\
P(\sum X_i^2/2 \geq c/2|\sigma^2 = 2) &= 0.01 \\
c/2 &= 37.57 \\
c &= 75.14 \\
\delta : &\quad \{\text{reject } H_0 \text{ if } \sum x_i^2 \geq 75.14\}
\end{aligned}$$

The power function is given by the cdf of a χ^2_{20}. We can calculate the probability of rejecting H_0 for various values of σ^2.

$$\begin{aligned}
\pi(\sigma^2|\delta) &= P(\text{reject } H_0|\sigma^2) \\
&= P(\sum X_i^2 \geq 75.14|\sigma^2) \\
&= P(\sum X_i^2/\sigma^2 \geq 75.14/\sigma^2) \\
&= 1 - \chi^2_{20}(75.14/\sigma^2) \\
\pi(\sigma^2 = 6|\delta) &= 1 - \chi^2_{20}(12.52) \approx 0.80 \\
\pi(\sigma^2 = 3|\delta) &= 1 - \chi^2_{20}(25.05) \approx 0.20 \\
\pi(\sigma^2 = 4|\delta) &= 1 - \chi^2_{20}(18.79) \approx 0.45
\end{aligned}$$