Math 152 – Fall 2008
Jo Hardin
warm-up # 15

Name: ________________________________

Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(0, \sigma^2)$, and it is desired to test:

$$H_0 : \sigma^2 \leq 2$$
$$H_1 : \sigma^2 > 2$$

1. Find the UMP test at level α_0.
2. Assume $n = 20, \alpha_0 = 0.01$, find the power function for the test in part 1.

Let $\sigma_1^2 < \sigma_2^2, \sigma_1^2, \sigma_2^2 \in \Omega$.

$$\begin{aligned}
\frac{f(\underline{x}|\sigma_2^2)}{f(\underline{x}|\sigma_1^2)} &= \left(\frac{\sigma_1}{\sigma_2}\right)^n \frac{e^{-\sum x_i^2/2\sigma_2^2}}{e^{-\sum x_i^2/2\sigma_1^2}} \\
&= \left(\frac{\sigma_1}{\sigma_2}\right)^n e^{\sum x_i^2(1/2\sigma_1^2 - 1/2\sigma_2^2)} \\
(1/2\sigma_1^2 - 1/2\sigma_2^2) &> 0 \\
T &= \sum X_i^2 \\
\delta : & \quad \{\text{reject } H_0 \text{ if } \sum x_i^2 \geq c\}
\end{aligned}$$

is UMP at α_0. For $n = 20, \alpha_0 = 0.01$, find c.

$$\begin{aligned}
P(\sum X_i^2 \geq c|\sigma^2 = 2) &= 0.01 \\
P(\sum X_i^2/2 \geq c/2|\sigma^2 = 2) &= 0.01 \\
c/2 &= 37.57 \\
c &= 75.14 \\
\delta : & \quad \{\text{reject } H_0 \text{ if } \sum x_i^2 \geq 75.14\}
\end{aligned}$$

The power function is found by calculating the probability of rejecting H_0 for various values of σ^2.

$$\begin{aligned}
\pi(\sigma^2|\delta) &= P(\text{reject } H_0|\sigma^2) \\
&= P(\sum X_i^2 \geq 75.14|\sigma^2) \\
&= P(\sum X_i^2/\sigma^2 \geq 75.14/\sigma^2) \\
&= 1 - \chi^2_{20}(75.14/\sigma^2) \\
\pi(\sigma^2 = 6|\delta) &= 1 - \chi^2_{20}(12.52) \approx 0.80 \\
\pi(\sigma^2 = 3|\delta) &= 1 - \chi^2_{20}(25.05) \approx 0.20 \\
\pi(\sigma^2 = 4|\delta) &= 1 - \chi^2_{20}(18.79) \approx 0.45
\end{aligned}$$