Name: ________________

Let $X_1, X_2, \ldots, X_{10}$ be a random sample from an $\exp(\theta)$ distribution, $E[X] = \theta$. Consider the following hypotheses:

$$H_0 : \theta \leq 2$$
$$H_1 : \theta > 2$$

Notice:

- If $X_1, X_2, \ldots, X_n \sim \exp(2)$ then $\sum X_i \sim \chi^2_{2n}$.
- The MLE for $\theta$ in an exponential distribution is $\bar{X}$.

1. Find the likelihood ratio test at a level of $\alpha = 0.05$. (Hint: find $\ln(\Lambda)$ and look at it as a function of $\sum X_i$ or $\bar{X}$.)

2. Let’s say you collect some data and find $\sum X_i = 29$, what is your conclusion?

3. What type of error might you have made?

Solution

1. $\delta : \{\text{reject } H_0 \text{ if } \sup_{L_0} \frac{L_1}{L_0} \geq k\}$

$$\sup_{L_0} \frac{L_1}{L_0} = \left(\frac{1}{\bar{X}}\right)^n e^{-\sum X_i / \bar{X}} \frac{(\frac{1}{\theta_0})^n e^{-\sum X_i / \theta_0}}{\left(\frac{2}{\bar{X}}\right)^n e^{\sum X_i / 2 - n}} \geq k$$

$$\Leftrightarrow -n \ln(\bar{X}) + n \ln(2) + \frac{\sum X_i}{2} - n \geq k$$

$$\Leftrightarrow -n \ln(\bar{X}) + \frac{\sum X_i}{2} \geq k$$

$$\Leftrightarrow -n \ln(\sum X_i) + \frac{\sum X_i}{2} \geq k$$
Let \( g(y) = -n \ln(y) + \frac{y}{2} \).

Then \( \frac{\partial g(y)}{\partial y} = -\frac{n}{y} + \frac{1}{2} \).

And \( \frac{\partial^2 g(y)}{\partial y^2} = \frac{n}{y^2} > 0 \).

So, we know that \( g(y) \) is minimized at \( y = 2n \). However, we also know that \( \sum X_i > 2n \) (because it makes sense that the MLE is in the alternative parameter space). So, if we want \( g(\sum X_i) \) to be greater than some value, we need \( \sum X_i \) to be bigger than some other value.

\[
\delta : \{ \text{reject } H_0 \text{ if } \sum X_i \geq c \}
\]

\[
P(\sum X_i \geq c|\theta = 2) = P(\chi^2_{20} \geq c) = 0.05, \ c = 31.41.
\]

\[
\delta : \{ \text{reject } H_0 \text{ if } \sum X_i \geq 31.41 \}
\]

2. \( \sum X_i = 29 \) → do not reject \( H_0 \).

3. It’s possible that we’ve made a type II error (that is, we didn’t reject \( H_0 \), and it’s possible that we should have.) It is not possible that we made a type I error.