You are provided the marginal distribution of $X$ and the conditional distribution of $Y$ given $X$. Your task is to find their joint distribution and the marginal distribution of $Y$.

$$f_X(x) = e^{-x} \quad 0 < x < \infty$$
$$f_{Y|X=x}(Y|X = x) = e^{-(y-x)} \quad 0 < x < y < \infty$$

1. Find the joint distribution of $X$ and $Y$, $f_{X,Y}(x, y)$.

2. Find the marginal distribution of $Y$, $f_Y(y)$.

3. Why doesn’t $f_{X,Y}(x, y) = f_X(x)f_Y(y)$?

Solution:

1. 

$$f_{X,Y}(x, y) = f_{Y|X=x}(y|X = x)f_X(x)$$
$$= e^{-(y-x)}e^{-x} \quad 0 < x < y < \infty$$

2. 

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx$$
$$= \int_{0}^{y} e^{-y}dx$$
$$= e^{-y} \cdot \bigg|_0^y$$
$$= e^{-y}(y - 0)$$
$$= ye^{-y} \quad 0 < y < \infty$$

3. Because $X$ and $Y$ are not independent random variables.