

Name: _____

Let θ denote the average number of defects per 100 feet of a particular type of tape. θ is unknown, but the prior on θ is a gamma distribution with $E[\theta] = \alpha/\beta = 2/10$, $\alpha = 2$, $\beta = 10$. When a 1200 foot roll of tape is inspected, exactly 4 defects are found.

What is your best guess of [the Bayes' estimate of] the average number of defects per 100 feet? Note: first find the posterior.

Solution:

Prior:

$$\xi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} = \frac{10^2}{\Gamma(2)} \theta e^{-10\theta}$$

Likelihood:

$$f(\underline{x}|\theta) = \prod_{i=1}^n \frac{e^{-\theta} \theta^{x_i}}{x_i!} = \frac{e^{-n\theta} \theta^{\sum x_i}}{\prod(x_i!)}$$

Posterior:

$$\begin{aligned} \xi(\theta|\underline{x}) &\propto \frac{\theta e^{-10\theta} e^{-n\theta} \theta^{\sum x_i}}{\Gamma(2) 10^2 \prod(x_i!)} \\ &\propto e^{-\theta(n+10)} \theta^{\sum x_i + 1} \end{aligned}$$

$$\theta|\underline{x} \sim \text{Gamma}(\sum x_i + 2 = 6, n + 10 = 22)$$

$$\begin{aligned} \delta^*(\underline{X}) &= \frac{\sum X_i + 2}{n + 10} \\ \delta^*(\underline{x}) &= \frac{6}{22} = \frac{3}{11} \end{aligned}$$

Note: The Gamma distribution is parameterized slightly differently in DeGroot and on your sheet (as is the exponential). Make sure the expected value matches what you've been given in the problem.