Math 152 – Fall 2018
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warm-up # 3

Name: ________________________________

Let $\theta$ denote the average number of defects per 100 feet of a particular type of tape. $\theta$ is unknown, but the prior on $\theta$ is a gamma distribution with $E[\theta] = \alpha/\beta = 2/10, \alpha = 2, \beta = 10$. When a 1200 foot roll of tape is inspected, exactly 4 defects are found.

What is your best guess of [the Bayes' estimate of] the average number of defects per 100 feet? Hint: first find the posterior.

**Solution:**

$$\begin{aligned}
&\text{Prior:} && \\
\xi(\theta) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} = \frac{10^2}{\Gamma(2)}\theta e^{-10\theta} \\
&\text{Likelihood:} && \\
f(\underline{x}|\theta) &= \prod_{i=1}^{n}\frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod(x_i!)} \\
&\text{Posterior:} && \\
\xi(\theta|\underline{x}) &\propto \frac{\theta e^{-10\theta}e^{-n\theta}\theta^{\sum x_i}}{\Gamma(2)10^2\prod(x_i!)} \\
&\propto e^{-\theta(n+10)}\theta^{\sum x_i+1} \\
\theta|\underline{x} &\sim \text{Gamma}\left(\sum x_i + 2 = 6, n + 10 = 22\right) \\
\delta^*(\underline{X}) &= \frac{\sum X_i + 2}{n+10} \\
\delta^*(\underline{x}) &= \frac{6}{22} = \frac{3}{11}
\end{aligned}$$

Note: The Gamma distribution is parameterized slightly differently in DeGroot and on your sheet (as is the exponential). Make sure the expected value matches what you've been given in the problem.