BCa CI

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In the percentile method, we've assumed that there exists a transformation of θ, $\phi(\theta)$, such that

$$\phi(\hat{\theta}) - \phi(\theta) \sim N(0,1)$$

The transformation assumes that neither θ nor ϕ are biased, and it assumes that the variance is constant for all values of the parameter. That is, in the percentage intervals, we assume the normalizing transformation creates a sampling distribution that is unbiased and variance stabilizing. Consider a monotone transformation that *normalizes* the sampling distribution (we no longer assume unbiased or constant variance).

$$\phi(\hat{\theta}) - \phi(\theta) \sim N(-z_0\sigma_\phi, \sigma_\phi), \qquad \sigma_\phi = 1 + a\phi$$

That is, there must exist a monotone transformation ϕ such that $\phi(\hat{\theta}) \sim N$ where

$$E(\phi(\hat{\theta})) = \phi(\theta) - z_0[1 + a\phi(\theta)] \qquad SE(\phi(\hat{\theta})) = 1 + a\phi(\theta)$$

(Note: in the expected value and SE we've assumed that $c = 1$. If $c \neq 1$, then we can always choose a different transformation, ϕ' so that $c = 1$.) Then

$$P(z_{\alpha/2} \leq \frac{\phi(\hat{\theta}) - \phi(\theta)}{1 + a\phi(\theta)} + z_0 \leq z_{1-\alpha/2}) = 1 - \alpha$$

A $(1-\alpha)100\%$ CI for $\phi(\theta)$ is

$$\left[\frac{\phi(\hat{\theta}) - (z_{1-\alpha/2} - z_0)}{1 + a(z_{1-\alpha/2} - z_0)}, \frac{\phi(\hat{\theta}) - (z_{\alpha/2} - z_0)}{1 + a(z_{\alpha/2} - z_0)}\right]$$

Let's consider an interesting probability question:

$$\begin{aligned}
P\bigg(\phi(\hat{\theta}^*) \quad &\leq \quad \frac{\phi(\hat{\theta}) - (z_{1-\alpha/2} - z_0)}{(1 + a(z_{1-\alpha/2} - z_0))}\bigg) = ? \\
= P\bigg(\frac{\phi(\hat{\theta}^*) - \phi(\hat{\theta})}{1 + a\phi(\hat{\theta})} \quad &\leq \quad \frac{\phi(\hat{\theta}) - (z_{1-\alpha/2} - z_0) - \phi(\hat{\theta}) - \phi(\hat{\theta})a(z_{1-\alpha/2} - z_0)}{(1 + a(z_{1-\alpha/2} - z_0))(1 + a\phi(\hat{\theta}))}\bigg) \\
= P\bigg(\frac{\phi(\hat{\theta}^*) - \phi(\hat{\theta})}{1 + a\phi(\hat{\theta})} \quad &\leq \quad \frac{-(z_{1-\alpha/2} - z_0) - \phi(\hat{\theta})a(z_{1-\alpha/2} - z_0)}{(1 + a(z_{1-\alpha/2} - z_0))(1 + a\phi(\hat{\theta}))}\bigg) \\
= P\bigg(\frac{\phi(\hat{\theta}^*) - \phi(\hat{\theta})}{1 + a\phi(\hat{\theta})} \quad &\leq \quad \frac{-(1 + a\phi(\hat{\theta}))(z_{1-\alpha/2} - z_0)}{(1 + a(z_{1-\alpha/2} - z_0))(1 + a\phi(\hat{\theta}))}\bigg) \\
= P\bigg(\frac{\phi(\hat{\theta}^*) - \phi(\hat{\theta})}{1 + a\phi(\hat{\theta})} \quad &\leq \quad \frac{-(z_{1-\alpha/2} - z_0)}{(1 + a(z_{1-\alpha/2} - z_0))}\bigg) \\
= P\bigg(\frac{\phi(\hat{\theta}^*) - \phi(\hat{\theta})}{1 + a\phi(\hat{\theta})} \quad &\leq \quad \frac{(z_{\alpha/2} + z_0)}{(1 - a(z_{\alpha/2} + z_0))}\bigg) \\
= P\bigg(\frac{\phi(\hat{\theta}^*) - \phi(\hat{\theta})}{1 + a\phi(\hat{\theta})} + z_0 \quad &\leq \quad \frac{(z_{\alpha/2} + z_0)}{(1 - a(z_{\alpha/2} + z_0))} + z_0\bigg) \\
= P\bigg(Z \quad &\leq \quad \frac{(z_{\alpha/2} + z_0)}{(1 - a(z_{\alpha/2} + z_0))} + z_0\bigg) = \gamma_1 \\
\text{where } \gamma_1 \quad &= \quad \Phi\bigg(\frac{(z_{\alpha/2} + z_0)}{(1 - a(z_{\alpha/2} + z_0))} + z_0\bigg) \\
&= \quad \texttt{pnorm}\bigg(\frac{(z_{\alpha/2} + z_0)}{(1 - a(z_{\alpha/2} + z_0))} + z_0\bigg)
\end{aligned}$$

What we've shown is that the γ_1 quantile of the $\phi(\hat{\theta}^*)$ sampling distribution will be a good estimate for the lower bound of the confidence interval for $\phi(\theta)$. Using the same argument on the upper bound, we find a $(1-\alpha)100\%$ confidence interval for $\phi(\theta)$ to be:

$$[\phi(\hat{\theta}^*)_{\gamma_1}, \phi(\hat{\theta}^*)_{\gamma_2}]$$

$$\begin{aligned}
\text{where } \gamma_1 &= \Phi\left(\frac{(z_{\alpha/2}+z_0)}{(1-a(z_{\alpha/2}+z_0))}+z_0\right) \\
\gamma_2 &= \Phi\left(\frac{(z_{1-\alpha/2}+z_0)}{(1-a(z_{1-\alpha/2}+z_0))}+z_0\right)
\end{aligned}$$

Using the transformation respecting property of percentile intervals, we know that a $(1-\alpha)100\%$ confidence interval for θ is:

$$[\hat{\theta}^*_{\gamma_1}, \hat{\theta}^*_{\gamma_2}]$$

How do we estimate a and z_0?

bias: z_0 is a measure of the bias. Recall:

$$\begin{aligned}
bias &= E(\hat{\theta}) - \theta \\
\hat{bias} &= \hat{\theta}^*(\cdot) - \hat{\theta}
\end{aligned}$$

But remember that z_0 represents the bias for $\phi(\hat{\theta})$, not for $\hat{\theta}$ (and we don't know ϕ!). So, we use θ to see what proportion of θ values are too low, and we can map it back to the ϕ space using the normal distribution:

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\#\hat{\theta}^*(b) < \hat{\theta}}{B}\right)$$

That is, if $\hat{\theta}^*$ underestimates $\hat{\theta}$, then $\hat{\theta}$ likely underestimates θ; $z_0 > 0$. We think of z_0 and the normal quantile associated with the proportion of BS replicates less than $\hat{\theta}$.

skew: a is a measure of skew.

$$\begin{aligned}
bias &= E(\hat{\theta} - \theta) \\
var &= E(\hat{\theta} - \theta)^2 = \sigma^2 \\
skew &= E(\hat{\theta} - \theta)^3/\sigma^3
\end{aligned}$$

We can think of the skew as the rate of chance of the standard error on a normalized scale. If there is no skew, we will estimate $a = 0$. Our estimate of a comes from a procedure known as the jackknife.

$$\hat{a} = \frac{\sum_{i=1}^n (\hat{\theta}(\cdot) - \hat{\theta}_{(i)})^3}{6[\sum_{i=1}^n (\hat{\theta}(\cdot) - \hat{\theta}_{(i)})^2]^{3/2}}$$