Taylor Polynomials and Taylor Series

Math 31 - Jo Hardin - Spring 2011

Some friendly series to know (*and know how to derive them . . .*) :

- *Geometric series:* $\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$, converges for $|x| < 1$.
- *Binomial Series:* $(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$, converges for $-1 < x < 1$. Note that if p is a positive integer, this sum is finite: all coefficients are zero for powers of x greater than p.
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, converges for all x.
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, converges for all x.
- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, converges for all x.
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$. Note what happens if you take the derivative of this one: you get the geometric series, with $-x$ in place of x. This series converges for $-1 < x \le 1$.

Some facts:

- The general formula for the *Taylor series about the point* $x = a$ for a function f with infinitely many derivatives is:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{n=1}^{\infty} \frac{f^{(n)}}{n!}(x-a)^n$$

  A function may converge to its Taylor series at some values of x, one value of x or all values of x.

- If the Taylor series is derived around the point a, then it is a power series, and its radius of convergence can be determined by finding:

$$R = \lim_{n\to\infty} \frac{C_n}{C_{n+1}}, \quad \text{where the } C_n \quad \text{is the coefficient of } x^n$$

- If a Taylor series converges you can differentiate it, multiply it, and integrate it to get a new series.
- The nth degree Taylor polynomial for $f(x)$ around the point $x = a$ is the Taylor series for $f(x)$ around $x = a$ truncated after the first term containing x^n:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- If the function f and all of its derivatives are continuous, then the difference between f and its nth degree Taylor polynomial around the point a is given by:

$$|E_n(x)| = |f(x) - P_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

  where M is the maximum of $|f^{n+1}|$ on the interval between a and x.

Example: Find the error in the nth degree Taylor polynomial for $\cos(x)$ around $a = 0$ when $x = \pi/2$:
From the error estimate in the last fun fact above,

$$|E_n(\pi)| \leq \frac{M}{(n+1)!}\pi^{n+1} = \frac{\pi^{n+1}}{(n+1)!}$$

so, for $n = 10$, the absolute value of the error is $< \frac{\pi^{11}}{11!} \approx 0.007370431$.

Show that a Taylor series converges: Show that the Taylor series for $\cos(x)$ converges for all x.
Answer: Using the ratio test, and using only the even terms, we get:

$$\lim_{n\to\infty} \frac{C_n}{C_{n+2}} = \lim_{n\to\infty} \frac{1/n!}{1/(n+2)!} = \lim_{n\to\infty} (n+2)(n+1) \to \infty$$

so $R \to \infty$ and the interval of convergence is all of $\mathbb{R}$. *Sneaky trick!!* This doesn't actually show that the Taylor series converges to the function $\cos(x)$ ... it might converge to something else! To make sure, we can show that the error terms go to zero:

$$|E_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

To see that this goes to zero for every value of x, we will pick an x (and "fix" it, i.e. we won't change it). For any $\epsilon > 0$, let $k > 2x$ be an integer, and let m be another integer such that $\left(\frac{1}{2}\right)^m < \frac{\epsilon}{x^k}$. Then, if $n = m + k$ we have:

$$\frac{x^n}{n!} = \frac{x^{m+k}}{n(n-1)\dots(k+1)k(k-1)\dots(2)(1)} < \left(\frac{x}{k+1}\right)^m \frac{x^k}{k!} < \left(\frac{1}{2}\right)^m x^k < \epsilon$$

so $\lim_{n\to\infty} |E_n| = 0$, and the series converges to $\cos(x)$ for all x.

Practice:

1. Find the second degree Taylor polynomial around the given point:
   (a) $\ln(x)$ around $x = 2$.
   (b) $\sin(x)$ around $x = \pi/4$.
2. Find the fourth degree Taylor polynomial around zero of $\frac{1}{1-4z^2}$.
3. Find the exact value of the following infinite sum:
   $47 + 47(0.1)^2 + \frac{47(.10)^4}{2} + \frac{47(.01)^6}{3!} + \dots$
4. Find the exact value of the following finite or infinite series:
   (a) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} \dots$
   (b) $47 - \frac{4}{2} + \frac{8}{3} - \frac{16}{4} + \frac{32}{5} - \dots$
   (c) $8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{10}}$
5. A function f has $f(3) = 1$, $f'(3) = 5$, $f''(3) = -10$. Find the best estimate you can for $f(3.1)$.
6. Suppose x is positive but very small. Arrange the following expressions in increasing order:

$$x, \quad \sin(x), \quad \ln(1+x), \quad 1-\cos(x), \quad e^x - 1, \quad \arctan(x), \quad x\sqrt{1-x}$$

7. (a) Find the Taylor series for $f(t) = te^t$ about $t = 0$.
   (b) Using your answer to part (a), find a Taylor series expansion about $x = 0$ for

$$\int_0^x te^t \, dt.$$

   (c) Using your answer to part (b), show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4(2!)} + \frac{1}{5(3!)} + \frac{1}{6(4!)} + \dots = 1$$