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Approaching Cauchy's Theorem

Stephan Ramon Garcia and William T. Ross

Abstract: We hope to initiate a discussion about various methods for introducing Cauchy's Theorem. Although Cauchy's Theorem is the fundamental theorem upon which complex analysis is based, there is no "standard approach." The appropriate choice depends upon the prerequisites for the course and the level of rigor intended. Common methods include Green's Theorem, Goursat's Lemma, Leibniz' Rule, and homotopy theory, each of which has its positives and negatives.

Keywords: Complex analysis, pedagogy, Cauchy's Theorem, Green's Theorem, Goursat's Lemma, homotopy, Leibniz' Rule

1. INTRODUCTION

For those of us who regularly teach an undergraduate course in complex variables, there are always the looming concerns about the appropriate level of technical detail and rigor to include. What do the students hope to get out of the course? Will the physics and engineering students get what they need out of it? Will the pure math students heading toward graduate school be satisfied?

One side of the debate argues that technical details are what makes mathematics so beautiful, concise, and complete. It all fits together so well. We are occasionally shocked by the loose and imprecise thinking of our colleagues from other disciplines. Mathematics, on the other hand, is a pure form of rational thought. In addition, some argue that it is this attention to detail and rigor that trains students to be precise and careful thinkers.

Another side of the debate argues that the majority of our students do not go on to graduate school in pure mathematics. Focusing on the technical details is not needed, nor is it appreciated, by the majority of our students. Instead, teachers should emphasize the general ideas and the interconnections

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between these ideas. Let the graduate school-bound students sweat over the precise definition of a contour integral, fret over whether the curve is C^1 or not, and worry about passing the limit under the integral sign. One should emphasize the beauty and applicability of the subject. In other words, do not let mathematics get in the way of a good idea.

Then, of course, there is the issue that presentations and proofs of classic theorems are sometimes too slick. They can leave the student with no real appreciation or understanding of the material. How many of us deliberately go through a more difficult proof, which uses the definitions directly, so that our students obtain some facility with these definitions, rather than use a slick "one liner" that gives them little insight about what is going on?

Since the two of us frequently teach undergraduate complex analysis, we often struggle with "detail" versus "not enough detail," with "plodding proofs" versus "*deus ex machina* proofs." Adding even more confusion to the debate is the fact that our students come from varied backgrounds. Some of our complex analysis students do not have the analysis background to appreciate the subtleties needed to make some of the arguments in complex variables legitimate. Others have proficiency in real analysis and want to appreciate the fine details. Unfortunately, both types of students are often in the same class.

In this paper, we focus on Cauchy's Theorem, and look at both sides of this struggle. Since the reader is presumably a professional mathematician with experience in complex variables, we will assume a working knowledge of the field. We do not attempt to grind through the details of the well-known proofs and techniques discussed below. We focus instead on several standard approaches to Cauchy's Theorem, and weigh the pros and cons of each from a pedagogical viewpoint.

Since Cauchy's name is attached to half of the results in elementary complex variables, let us be precise about which version of Cauchy's Theorem we are referring to!

Cauchy's Theorem: *Suppose* Ω *is a nonempty, simply connected, open set in* $\mathbb{C}$ *and* γ *is a simple, closed curve in* Ω. *If* f *is an analytic function on* Ω, *then*

$$\int_\gamma f(z)dz = 0.$$

As we all know, Cauchy's Theorem is *the* cornerstone of complex analysis and there are many proofs of it that emphasize varying starting points and generality. We focus on how to *introduce* students to Cauchy's Theorem, and we do not dwell on the advanced homological approaches to Cauchy theory that are typically encountered in a graduate-level course, or towards the end of an advanced undergraduate course. Furthermore, this is *not* meant to be an exhaustive list of all possible approaches to Cauchy's Theorem. We intend only to focus on some of the more common approaches that we have seen in standard textbooks.

2. GREEN'S THEOREM

This is one of the most common approaches that we have seen [3, 5, 6, 7]. Under the hypotheses of Cauchy's Theorem, write $f = u + iv$, in which u and v are real-valued harmonic functions. Then

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma (u + iv)\,(dx + i\,dy) \\ &= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx) \\ &= \iint_\Omega \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\; dy + i\iint_\Omega \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\; dy \\ &= 0.\end{aligned}$$

The final equality follows from the Cauchy–Riemann equations.

There are several obvious advantages to this method. First, it does not require any real analysis. The proof is easy and short. It also makes use of the Cauchy–Riemann equations, which are sometimes presented early on, but used for little else other than a brief digression on harmonic functions.

Another possible advantage is that this approach clearly makes complex analysis subordinate to multivariable calculus, which every student in the class will have already taken. Moreover, this brings things more in line with the modern approach to complex variables as a research area, in which differential geometry, potential theory, $\bar{\partial}$-problems, Cauchy transforms, and the like take center stage.

The Green's Theorem approach has several notable disadvantages, however. First of all, the proof requires the continuity of f', as the application of Green's Theorem requires the continuity of the partial derivatives u_x, u_y, v_x, and v_y. Of course, we all know that this restriction is illusory, as analytic functions are automatically infinitely differentiable. From a pedagogical viewpoint, this is an obstacle, as proofs of the continuity of f' typically rest upon Cauchy's Integral Formula! However, one can use Green's Theorem to sell students on the idea, and then patch up the logical gap later by appealing to some version of Goursat's Lemma (see Section 3).

There are other possible issues. Students may not remember Green's Theorem, since many of them will be juniors or seniors who are a year or two removed from any form of calculus. Furthermore, most of them have never seen a proof of Green's Theorem (although special cases may have been discussed). Indeed, Green's Theorem usually appears towards the end of multivariable calculus when the teacher is breathlessly rushing to finish the course, and so it is not often proved with much care. Another major objection is that students may only have a formal (that is, "symbol pushing") understanding of the differential forms dx, dy, $dx\,dy$, and $dy\,dx$. This threatens to make their understanding of Cauchy's Theorem incomplete and shallow.

One can also argue that the subordination of complex analysis to multivariable calculus is undesirable. There is something special about complex variables, something unique that makes it "work." It is completely unlike calculus in any other context, and to make complex variables appear to be nothing more than glorified calculus is selling this beautiful subject short.

3. GOURSAT'S LEMMA

As noted in Section 2, the standard Green's Theorem approach to Cauchy's Theorem requires the continuity of f'. However, the definition of an analytic function requires only that

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h},$$

exist for every $z\in\Omega$. There is no apriori assumption that f' is continuous. Goursat's Lemma is often employed to address this, in order to make the theory more complete, to put things on a more solid foundation, and to avoid making any extraneous assumptions [1, 2, 4, 9]. The text [6] uses a hybrid of Green's Theorem and Goursat's Lemma.

To be more specific, Goursat's Lemma establishes that

$$\int_\gamma f(z)dz=0,$$

for any triangle (sometimes rectangles are used) γ in Ω and for any analytic function f on Ω (the continuity of f' is *not* assumed). The argument involves a clever recursive construction, linear approximation, compactness, and continuity.

The most obvious advantage of Goursat's Lemma is that it does not require the continuity of f'. It permits us to eliminate a hypothesis that we know to be unnecessary. Once Goursat's Lemma has been established, one can easily prove Cauchy's Theorem for convex regions. This is sufficient to obtain the existence of power series expansions and some of the other standard gems of complex analysis. A more comprehensive version of Cauchy's Theorem often comes later in the course. In most presentations along these lines, Goursat's Lemma is often used "after the fact" to eliminate the hypothesis that f' is continuous in other approaches to Cauchy's Theorem ("what we did using Green's Theorem was legal after all").

One can also argue that the proof of Goursat's Lemma is beautiful. Indeed, it involves an elegant, "fractal"-like construction and many of the tools of undergraduate analysis: compactness, continuity, and linear approximation. Its combination of technique and elegance make it stand out among the other theorems of introductory complex variables. After all, most of the other theorems from basic complex analysis are straightforward consequences of Cauchy's Integral Formula and/or power series manipulations.

Of course, this strength can also be viewed as a weakness. The proof of Goursat's Lemma requires significant amounts of analysis, and this might not be in the skill set of every complex variables student. Does a physics or engineering student care about the logical subtlety here? Is assuming the continuity of f' really a big deal (the classic book of Conway [4] does not seem to think so)? After putting in a lot of boardwork on Goursat's Lemma, one only has Cauchy's Theorem for triangles and/or rectangles. Is it worth slowing down the pace of the course just to satisfy our inner need for rigor? Will the students appreciate the proof, or will they feel that it is too much of a "trick"? These are all important questions that the instructor must face.

4. LEIBNIZ' RULE

Recall that Leibniz' Rule tells us that if $F(x,t)$ and $F_x(x,t)$ are continuous on the closed rectangle

$$\{(x,t) : x_0 \le x \le x_1, a \le t \le b\},$$

then

$$\frac{d}{dx}\int_a^b F(x,t)dt = \int_a^b F_x(x,t)dt.$$

From here one can show that

$$\int_0^{2\pi} \frac{e^{it}\,dt}{e^{it}-z} = 2\pi, \quad |z| < 1.$$

Another appeal to Leibniz' Rule reveals that

$$\int_0^{2\pi} \left[\frac{f(e^{it})e^{it}dt}{e^{it}-z} - f(z)\right] dt = 0,$$

which is Cauchy's Integral Formula for the unit circle. This approach heads straight for the punchline and bypasses Cauchy's Theorem altogether. After all, is it not it the integral formula that we are after? Once we have Cauchy's integral formula for a disk, the road quickly opens up and we may progress to all manner of more advanced topics. Another advantage: this is a short calculus-based proof that students can understand.

This approach is not without obvious disadvantages. For instance, where is the geometric intuition? There is a lot of analysis being swept under the rug here. Like the Green's Theorem approach, this too requires the continuity of a certain derivative. How many of us use Leibniz' Rule when we teach? How many of us prove it? Is this proof too much like a trick?

5. HOMOTOPY AND DEFORMATION

One can also approach Cauchy's Theorem from a topological perspective from the outset. Suppose that γ_1 and γ_2 are simple closed curves that are homotopic to each other with respect to the region Ω. There are (at least) two significantly different ways to go from here:

1. (Non-rigorous.) Start from the Cauchy–Riemann equations and hand-wave about conservative vector fields. Based on physical principles, argue that

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Now "shrink γ_2 to a point" to get Cauchy's Theorem.

2. (Rigorous.) Use compactness to partition the domain of the homotopy $H : [0, 1]^2 \to \Omega$ connecting γ_1 and γ_2 into small rectangles whose images are contained in open disks in Ω. Then apply Cauchy's Theorem for a disk and patch the results together.

There are some obvious advantages to these sorts of approaches. They are physically motivated, visual, and intuitive. Moreover, they highlight topological aspects of complex analysis and provide the students with a taste of topology.

The disadvantages are obvious. One needs serious analysis and topology to do this properly. Indeed, homotopy is easy to motivate, but hard to do rigorously. For the purists, pretty pictures just do not cut it. Furthermore, the second (more rigorous) approach requires Cauchy's Theorem for disks. Does this not make things seem circular?

6. FOR FURTHER THOUGHT

In preparing your own course, you may wish to consider the following questions:

- What approach do you use and why?
- How much rigor is too much?
- How much hand-waving is too much?

- Do we place rigor above pedagogy?
- Do your students appreciate a rigorous approach?
- Do your students want more applications?

If anything, our study of the approaches found in a number of textbooks suggests a few things. If you have a class full of budding young analysts who are ready to jump on any logical gap, then certainly Goursat's Lemma is a necessity to get things flying. For a quick and dirty approach, it is hard to beat Green's Theorem, so long as one does not mind playing fast-and-loose with the rules. If your students could not care less about whether you assume f' is continuous or not, Green's Theorem is probably the way to go. The Leibniz' Rule method is not far behind, but (to us) the Green's Theorem approach seems more natural; we believe that students will feel that it is less of a trick. A suitable compromise between rigor and accessibility might be to go with Green's Theorem while providing, for the students who have a strong background in analysis, supplementary notes or bonus problems that cover Goursat's Lemma and its consequences.

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REFERENCES

1. Ahlfors, L. 2013. *Complex Analysis*, *Third Edition*. McGraw Hill.
2. Boas, R. 1988. *Invitation to Complex Analysis*. Random House.
3. Brown, J. and R. Churchill. 2013. *Complex Variables and Applications, Ninth Edition*. McGraw-Hill Education.
4. Conway, J. B. 1973. *Functions of One Complex Variable*. New York, NY: Springer-Verlag.
5. Hoffman, M. and J. Marsden. 1998. *Basic Complex Analysis*, *Third Edition*. New York, NY: Freeman.
6. Howell, J. and R. Matthews. 2000. *Complex Analysis for Mathematics and Engineering*. Jones and Bartlett
7. Krantz, S. 2007. *Complex Variables: A Physical Approach with Applications and MATLAB*. Boca Raton, FL: Chapman and Hall/CRC.
8. Saff, E. and A. Snider. 2003. *Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics*, *Third Edition*, Upper Saddle River, NJ: Pearson.
9. Sarason, D. 2007. *Complex Function Theory*, *Second Edition*. Providence, RI: American Mathematical Society.

BIOGRAPHICAL SKETCHES

Stephan Ramon Garcia grew up in San Jose, California before attending U.C. Berkeley for his B.A. and Ph.D. After graduating, he worked at U.C. Santa Barbara for several years before moving to Pomona College in 2006. He has earned three NSF research grants and five teaching awards from three different institutions. He was also twice nominated by Pomona College for the prestigious US Professors of the Year Award. He is the author of two books and over 70 research articles in operator theory, complex analysis, matrix analysis, number theory, discrete geometry, and other fields.

William T. Ross grew up in the Bronx and Yonkers, New York. After attending Fordham University, he obtained his Ph.D. at the University of Virginia. He is currently the Richardson Professor of Mathematics at the University of Richmond. He has written papers and several books on complex analysis and operator theory and loves to teach the subject as often as he can.