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Linearly Independent Spanning Sets

Suppose that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s$ are two linearly independent lists in $\mathbb{C}^n$ that have the same span (i.e., form bases for the same subspace of $\mathbb{C}^n$). We show that $r = s$.

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_r] \in \mathsf{M}_{n\times r}(\mathbb{C})$ and $B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_s] \in \mathsf{M}_{n\times s}(\mathbb{C})$. Each column of A is a linear combination of the columns of B, so there are $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r \in \mathbb{C}^s$ such that $\mathbf{a}_i = B\mathbf{x}_i$ for $i = 1, 2, \ldots, r$. Then $A = BX$, in which $X = [\mathbf{x}_1\, \mathbf{x}_2\, \cdots\, \mathbf{x}_r] \in \mathsf{M}_{s\times r}(\mathbb{C})$. Similarly, $B = AY$, in which $Y \in \mathsf{M}_{r\times s}(\mathbb{C})$. Thus, $A(I_r - YX) = A - AYX = A - BX = A - A = 0$. Since A has linearly independent columns, each column of $I_r - YX$ is zero; that is, $YX = I_r$. Similarly, $XY = I_s$, and hence, $r = \operatorname{tr} I_r = \operatorname{tr} YX = \operatorname{tr} XY = \operatorname{tr} I_s = s$. Thus, $r = s$. ■

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