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Lattice Theory and Toeplitz Determinants

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Abstract. This is a survey of our recent joint investigations of lattices that are generated by finite Abelian groups. In the case of cyclic groups, the volume of a fundamental domain of such a lattice is a perturbed Toeplitz determinant with a simple Fisher–Hartwig symbol. For general groups, the situation is more complicated, but it can still be tackled by pure matrix theory. Our main result on the lattices under consideration states that they always have a basis of minimal vectors, while our results in the other direction concern exact and asymptotic formulas for perturbed Toeplitz determinants. The survey is a slightly modified version of the talk given by the first author at the Humboldt Kolleg and the IWOTA in Tbilisi in 2015. It is mainly for operator theorists and therefore also contains an introduction to the basics of lattice theory.

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1. Introduction

The determinant of the $n \times n$ analogue A_n of the matrix

$$A_6 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 1 \\ -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 1 & 0 & 0 & 1 & -4 & 6 \end{pmatrix}$$

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is $\det A_n = (n+1)^3 \sim n^3$, whereas the determinant of the $n \times n$ analogue T_n of the matrix

$$T_6 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 1 & -4 & 6 \end{pmatrix}$$

equals

$$\det T_n = \frac{(n+1)(n+2)^2(n+3)}{12} \sim \frac{n^4}{12}.$$

(The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$.) The determinants $\det A_n$ emerge in a problem of lattice theory [6] and the formula $\det A_n = (n+1)^3$ was established only in [6], while the determinants $\det T_n$ are special cases of the well-known Fisher–Hartwig determinants one encounters in statistical physics [11, 12]. The matrices T_n are principal truncations of an infinite Toeplitz matrix. This is not true of the matrices A_n, but these are simple corner perturbations of T_n.

The observations made above motivated us to undertake studies into two directions. First, the ability to compute the determinants of A_n, which arise when considering lattices associated to cyclic groups, encouraged us to turn to lattices that are generated by arbitrary finite Abelian groups. And secondly, intrigued by the question why the corner perturbations lower the growth of the determinants from n^4 to n^3, we explored the determinants of perturbed Toeplitz matrices with more general Fisher–Hartwig symbols.

Our investigations resulted in the two papers [5, 6], and here we want to give a survey of these papers. This survey is intended for operator theorists. We are therefore concise when dealing with Toeplitz operators and matrices, but we consider it as useful to devote due space to some basics of lattice theory. Sections 1 to 6 are dedicated to lattice theory, and in the remaining Sections 7 to 9 we embark on Toeplitz determinants.

2. Examples of lattices

By an n-dimensional lattice we mean a discrete subgroup $\mathcal{L}$ of the Euclidean space $\mathbf{R}^n$. The lattice is said to have full rank if

$$\mathrm{span}_{\mathbf{R}} \mathcal{L} = \mathbf{R}^n,$$

where $\mathrm{span}_{\mathbf{R}} \mathcal{L}$ is the intersection of all linear subspaces of $\mathbf{R}^n$ which contain $\mathcal{L}$. Unless otherwise stated, all lattices considered in this paper are of full rank and hence we omit the attribute "full-rank". Of course, $\mathbf{Z}^n$ is the simplest example of an n-dimensional lattice.

The 1-dimensional lattices are just the sets $b\mathbf{Z}$ where b is a nonzero real number. Figure 1 shows three examples of 2-dimensional lattices. In these examples, the lattice consists of the dots, one of which is the origin of $\mathbf{R}^2$.

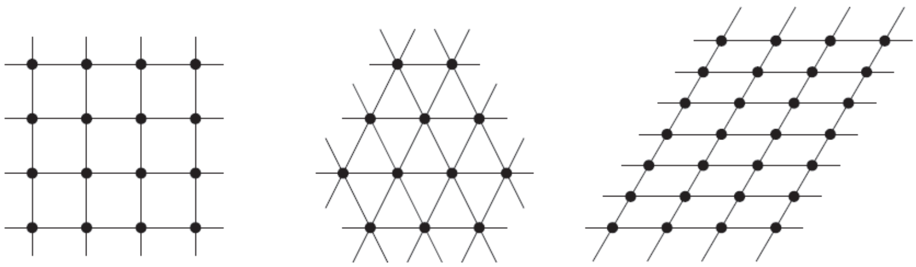

FIGURE 1. Three 2-dimensional lattices.

Two prominent 3-dimensional lattices are the face-centered cubic (fcc) lattice and the body-centered cubic (bcc) lattice. These emerge from periodically repeating the boxes shown in Figure 2. The fcc lattice is usually denoted by A_3 or by D_3, while the bcc lattice goes under the notation A_3^*. In formulas,

$$A_3 = D_3 = \{(x,y,z) \in \mathbf{Z}^3 : x+y+z \equiv 0 \mod 2\},$$
$$A_3^* = \{(x,y,z) \in \mathbf{Z}^3 : x \equiv y \equiv z \mod 2\}.$$

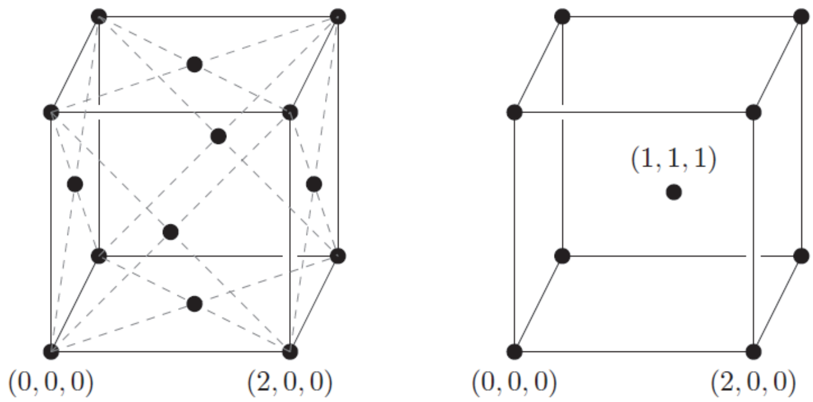

FIGURE 2. The fcc lattice (left) and the bcc lattice (right).

The so-called root lattices A_n are defined by

$$A_n = \{(x_0, x_1, \ldots, x_n) \in \mathbf{Z}^{n+1} : x_0 + x_1 + \cdots + x_n = 0\}.$$

Clearly, $\operatorname{span}_{\mathbf{R}} A_n$ is a proper subset of $\mathbf{R}^{n+1}$ and hence A_n is not of full rank in $\mathbf{R}^{n+1}$. However, we view A_n as a subset of the n-dimensional Euclidean space

$$E_n := \{(x_0, x_1, \ldots, x_n) \in \mathbf{R}^{n+1} : x_0 + x_1 + \cdots + x_n = 0\},$$

and after identifying E_n with $\mathbf{R}^n$ in the natural way, that is, as a subspace of the surrounding Euclidean $\mathbf{R}^{n+1}$, the lattice A_n becomes an n-dimensional full-rank lattice. Figure 3 shows A_1.

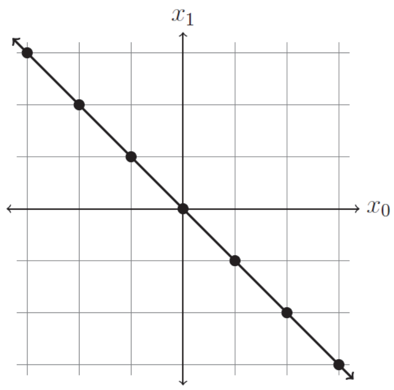

FIGURE 3. The lattice $A_1 = \{(x_0, x_1) \in \mathbb{Z}^2 : x_0 + x_1 = 0\}$.

The lattice A_2 is plotted in Figure 4. We see that A_2 is actually the 2-dimensional honeycomb lattice formed by the vertices of the regular triangles tiling the plane.

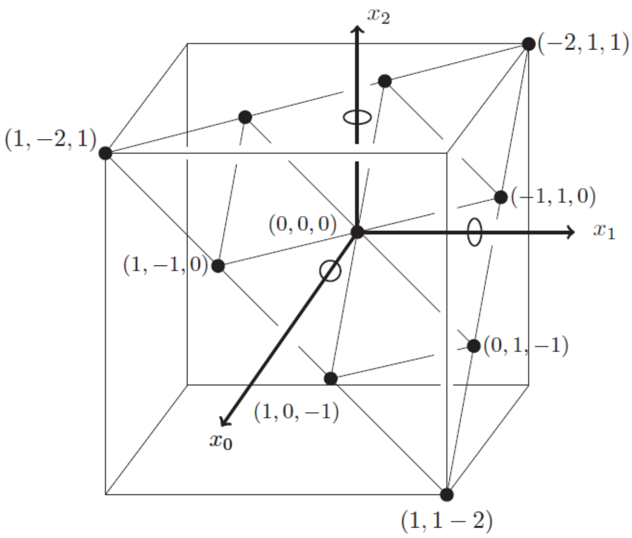

FIGURE 4. The lattice $A_2 = \{(x_0, x_1, x_2) \in \mathbb{Z}^3 : x_0 + x_1 + x_2 = 0\}$.

Figure 5 shows the 3-dimensional lattice

$$A_3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 : x_0 + x_1 + x_2 + x_3 = 0\} = \text{fcc}.$$

(Of course, we could not draw the surrounding $\mathbf{Z}^4$.) The lattice A_3 consists of the full dots and the circles in Figure 5. It is clearly seen that A_3 is nothing but the fcc lattice.

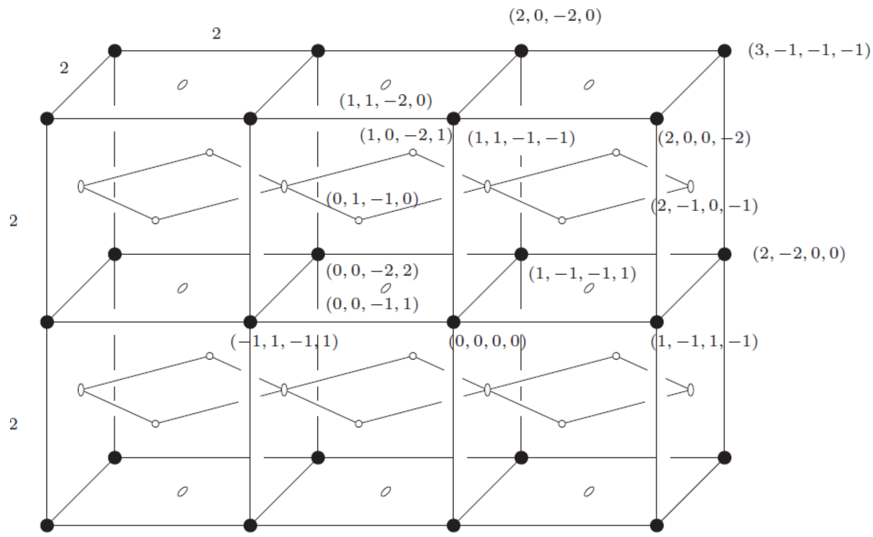

FIGURE 5. The lattice A_3.

3. Basis matrices, fundamental domains, and sphere packings

Every n-dimensional lattice $\mathcal{L}$ has a basis $\{b_1, \dots, b_n\}$. This is a set of lattice vectors $b_1, \dots, b_n$ which are linearly independent over $\mathbf{R}$ and satisfy

$$\mathcal{L} = \{t_1 b_1 + \cdots + t_n b_n : t_j \in \mathbf{Z}\}.$$

After choosing coordinates we may write $b_1, \dots, b_n$ as columns. The matrix $B = (b_1, \dots, b_n)$ formed by these columns is called the corresponding basis matrix of the lattice. Thus, $\mathcal{L} = \{Bt : t \in \mathbf{Z}^n\}$.

There are several ways to fix a basis and also several ways to select coordinates. Let us begin with the lattice A_1. Recall that we think of the lattice A_1 as a 1-dimensional lattice in the 1-dimensional Euclidean space

$$E_1 := \{(x_0, x_1) \in \mathbf{R}^2 : x_0 + x_1 = 0\} \cong \mathbf{R}^1$$

seen as a straight line in Figure 3. Thus, we could write $A_1 = \{Bt : t \in \mathbf{Z}\}$ with the 1×1 matrix $B = (\sqrt{2})$. However, we could also take the coordinates from the surrounding $\mathbf{R}^2$ and represent A_1 as $A_1 = \{Bt : t \in \mathbf{Z}\}$ with the 2×1 matrix $B = \binom{1}{-1}$.

Figure 4 reveals that A_2 is the honeycomb lattice formed by the vertices of the tiling of the plane by equilateral triangles whose side length is $\sqrt{2}$. Note anew that we regard A_2 as a lattice in the Euclidean $E_2 \cong \mathbf{R}^2$. We therefore could write

$$A_2 = \left\{ B \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} : t_1, t_2 \in \mathbf{Z} \right\} \quad \text{with} \quad B = \sqrt{2} \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}.$$

Again we prefer taking the coordinates from the surrounding $\mathbf{R}^3$. This gives the alternative representation

$$A_2 = \left\{ B \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} : t_1, t_2 \in \mathbf{Z} \right\} \text{ with } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

We know that A_3 is the fcc lattice. The side length of the cubes is 2. The centers of the lower, left, and front faces of the upper-right cube in Figure 5 form a basis for A_3. In $\mathbf{R}^3$, these centers could be given the coordinates $(1,1,0), (0,1,1), (1,0,1)$, resulting in the representation

$$A_3 = \left\{ B \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} : t_j \in \mathbf{Z} \right\} \text{ with } B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Figure 5 shows that in the surrounding $\mathbf{R}^4$ the coordinates of these centers are $(1,-1,0,0), (1,0,-1,0), (1,0,0,-1)$. This leads to the description

$$A_3 = \left\{ B \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} : t_j \in \mathbf{Z} \right\} \text{ with } B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\{b_1, \ldots, b_n\}$ be a basis of a lattice $\mathcal{L}$ and B be the corresponding basis matrix. The set $D = \{t_1 b_1 + \cdots + t_n b_n : 0 \le t_j < 1\}$ is called the fundamental domain associated with the basis. The matrix $B^\top B = (\langle b_j, b_k \rangle)_{j,k=1}^n$ is referred to as the Gram matrix of the basis. Note that a lattice is essentially specified by solely its Gram matrix. Indeed, given a positive definite symmetric matrix $A = B^\top B$, all factorizations $A = C^\top C$ are provided by $C = UB$ where U is an orthogonal matrix, and hence all lattices with the Gram matrix A result from one of them by orthogonal transformations. This observation will be of importance in connection with Figure 12 in Section 6.

The volume of a fundamental domain is known to be equal to $\sqrt{\det(B^\top B)}$. Different choices of a basis lead to different fundamental domains, but their volume turns out to be independent of the choice of the basis; see Figure 6. This volume is called the determinant of the lattice $\mathcal{L}$ and is denoted by $\det \mathcal{L}$.

Given an n-dimensional lattice $\mathcal{L}$, the packing radius r is defined as the maximal number ϱ such that one can place n-dimensional balls of equal radius ϱ centered at the lattice points without overlap. The goal of sphere packing is cover the largest possible proportion of the ambient space. This proportion, called the packing density $\Delta(\mathcal{L})$ of the lattice, is equal to the volume of one such ball divided by the volume of a fundamental domain of the lattice; see Figure 7. The lattice packing problem consists in finding a lattice of prescribed dimension whose packing density is maximal.

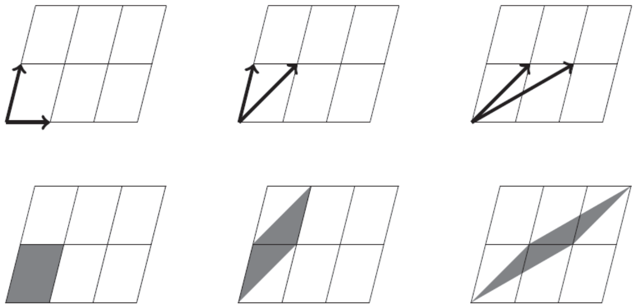

FIGURE 6. Three lattice bases and the corresponding fundamental domains.

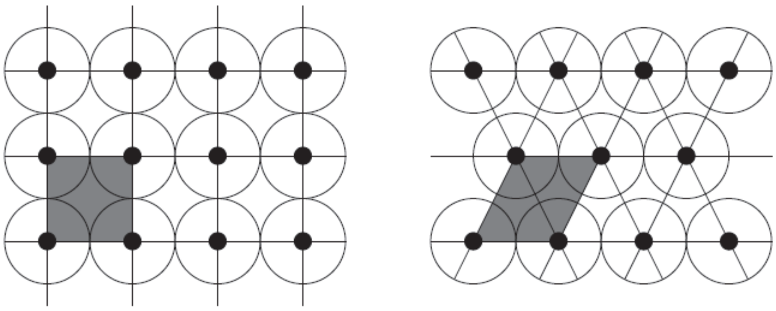

FIGURE 7. Sphere packings and fundamental domains.

Obviously, the packing radius r equals $d(\mathcal{L})/2$ where $d(\mathcal{L})$, the so-called minimal distance of the lattice, is defined by

$$d(\mathcal{L}) = \min\{\|x - y\| : x, y \in \mathcal{L}, x \neq y\} = \min\{\|x\| : x \in \mathcal{L} \setminus \{0\}\}.$$

Thus, the packing density equals

$$\Delta(\mathcal{L}) = \frac{V_n d(\mathcal{L})^n}{2^n \det(\mathcal{L})}, \tag{1}$$

where $V_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the n-dimensional unit ball.

The densest lattice packings are known in dimensions $n \leq 8$ and $n = 24$. The **Minkowski–Hlawka theorem** says that in every dimension $n \geq 2$ there exist lattices $\mathcal{L}_n$ with

$$\Delta(\mathcal{L}_n) \geq \frac{\zeta(n)}{2^{n-1}} > \frac{1}{2^{n-1}},$$

where ζ is the Riemann Zeta function, but unfortunately the known proofs are all non-constructive. It is in particular known that in dimensions $n = 1, 2, 3$ the root lattices A_1, A_2, A_3 yield the densest lattice packings. Trivially, $\Delta(A_1) = 1$.

For $n = 2, 3$, the densities and the Minkowski–Hlawka bounds are

$$\Delta(A_2) = \frac{\pi}{\sqrt{12}} \approx 0.9069, \quad \zeta(2)/2 \approx 0.8224,$$
$$\Delta(A_3) = \Delta(\text{fcc}) = \Delta(D_3) = \frac{\pi}{\sqrt{18}} \approx 0.7404, \quad \zeta(3)/2^2 \approx 0.3005.$$

For $4 \le n \le 8$, the lattices delivering the densest lattice packings are D_4, D_5, E_6, E_7, E_8 with

$$\begin{aligned}
D_4 &= \{(x_1, x_2, x_3, x_4) \in \mathbf{Z}^4 : x_1 + x_2 + x_3 + x_4 \equiv 0 \bmod 2\},\\
D_5 &= \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{Z}^5 : x_1 + x_2 + x_3 + x_4 + x_5 \equiv 0 \bmod 2\},\\
E_8 &= \{(x_1, \dots, x_8) \in \mathbf{Z}^8 : \text{all } x_i \in \mathbf{Z} \text{ or all } x_i \in \mathbf{Z} + \frac{1}{2},\\
&\quad\ x_1 + \dots + x_8 \equiv 0 \bmod 2\},\\
E_7 &= \{(x_1, \dots, x_8) \in E_8 : x_1 + \dots + x_8 = 0\},\\
E_6 &= \{(x_1, \dots, x_8) \in E_8 : x_6 = x_7 = x_8\},
\end{aligned}$$

and in dimension $n = 24$ the champion is the Leech lattice Λ_{24} with

$$\Delta(\Lambda_{24}) = \frac{\pi^{12}}{479\,001\,600} \approx 0.001\,930.$$

(Note that $\Delta(\Lambda_{24})$ is about 10 000 times better than the Minkowksi–Hlawka bound $\zeta(24)/2^{23} \approx 0.000\,000\,119$.) We refer to Conway and Sloane's book [10] for more on this topic.

4. Lattices from finite Abelian groups

In many dimensions below around 1 000, lattices with a packing density greater than the Minkowski–Hlawka bound are known. However, for general dimensions n, so far no one has found lattices whose packing density reaches the Minkowski–Hlawka bound. The best known lattices come from algebraic constructions. We confine ourselves to referring to the books [17, 18]. One such construction uses elliptic curves. An elliptic curve over $\mathbf{R}$ is defined by

$$E = \{(x, y) \in \mathbf{R}^2 : y^2 = x^3 + ax + b\},$$

where $a, b \in \mathbf{R}$ satisfy $4a^3 + 27b^2 \neq 0$. Such a curve, together with a point at infinity, is an Abelian group. Everyone has already seen pictures like those in Figure 8, which show the group operation in E.

An elliptic curve over a finite field $\mathbf{F}_q$, where $q = p^m$ is a prime power, is the set

$$E = \{(x, y) \in \mathbf{F}_q : y^2 = x^3 + ax + b\}.$$

Here $a, b \in \mathbf{F}_q$ and $4a^3 + 27b^2 \neq 0$. Such a curve, together with a point at infinity, is a finite Abelian group. The group operation can be given by translating the geometric construction in Figure 8 into algebraic formulas. Figures 9 and 10 show two examples.

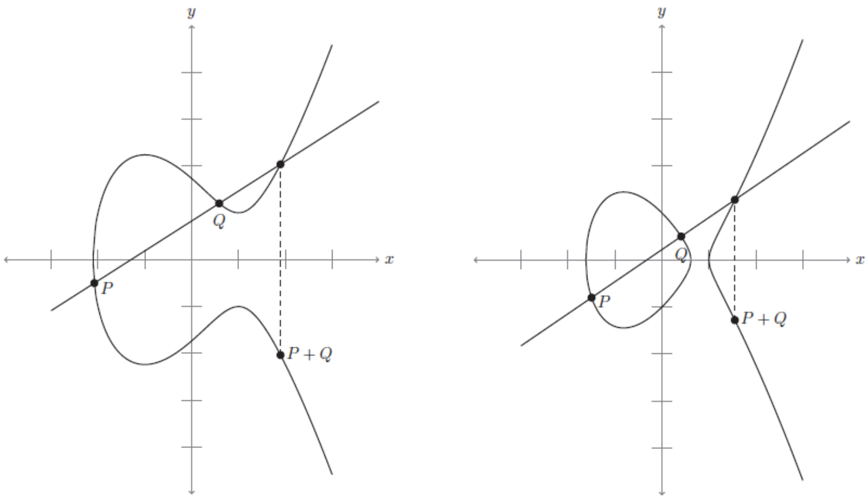

FIGURE 8. The curves $y^2 = x^3 - 3x + 3$ (left) and $y^2 = x^3 - 2x + 1$ (right).

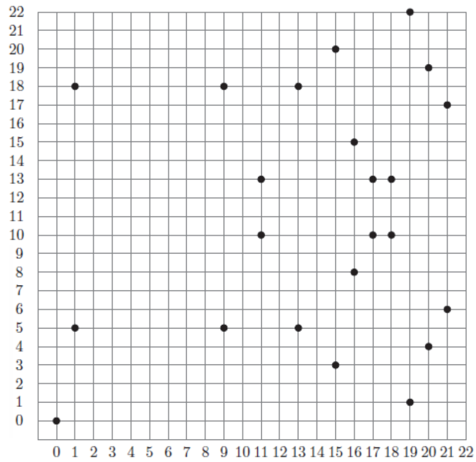

FIGURE 9. The curve $y^2 = x^3 + x$ over $\mathbf{F}_{23}$.

Rück [15] determined all possible finite Abelian groups which are elliptic curves over finite fields. These are always of the form $\mathbf{Z}_{m_1} \times \mathbf{Z}_{m_2}$ with further restrictions on m_1 and m_2. Here and in the following, $\mathbf{Z}_m := \mathbf{Z}/m\mathbf{Z}$.

In [6], we considered lattices that are generated by arbitrary finite Abelian groups. The construction is as follows. Let $G = \{g_0, g_1, \ldots, g_n\}$ be a finite (additively written) Abelian group. We assume that $g_0 = 0$. Note that $|G| = n+1$. The lattice associated with this group is

$$\begin{aligned}\mathcal{L}(G) &= \{(x_0, x_1, \ldots, x_n) \in A_n : x_0 g_0 + x_1 g_1 + \cdots + x_n g_n = 0\} \\ &= \{(x_1, \ldots, x_n, x_0) \in A_n : x_1 g_1 + \cdots + x_n g_n = 0\}.\end{aligned}$$

Equivalently,

$$\begin{aligned}\mathcal{L}(G) = \{&(x_1, \ldots, x_n, x_0) \in \mathbf{Z}^{n+1} : x_1 g_1 + \cdots + x_n g_n = 0 \\ &\text{and } x_1 + \cdots + x_n + x_0 = 0\}.\end{aligned}$$

Let, for example, $n = 2$ and $G = \mathbf{Z}_3 = \{0, 1, 2\}$. Then $\mathcal{L}(\mathbf{Z}_3)$ is the sublattice of A_2 defined by

$$\mathcal{L}(\mathbf{Z}_3) = \{(x_1, x_2, x_0) \in \mathbf{Z}^3 : x_1 + 2x_2 \equiv 0 \bmod 3, \; x_1 + x_2 + x_3 = 0\}.$$

Figure 4 and a little thought reveal that this is just the lattice that results from the honeycomb lattice A_2 after stretching it by the factor $\sqrt{3}$.

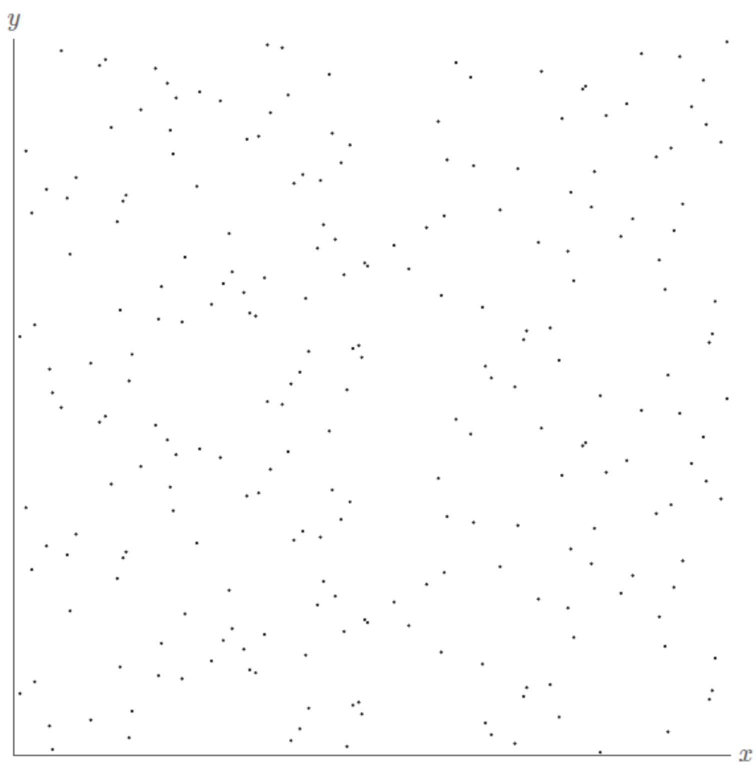

FIGURE 10. The curve $y^2 = x^3 + x + 2$ over $\mathbf{F}_{3^5} = \mathbf{F}_{243}$.

Now let $n = 4$ and let G be the group $G = \mathbf{Z}_4 = \{0, 1, 2, 3\}$. Then $\mathcal{L}(\mathbf{Z}_4)$ is the sublattice of $A_3 = \text{fcc}$ consisting of the points $(x_1, x_2, x_3, x_0) \in \mathbf{Z}^4$ with

$$x_1 + 2x_2 + 3x_4 \equiv 0 \bmod 4 \;\text{ and }\; x_1 + x_2 + x_3 + x_4 = 0.$$

Inspection of Figure 5 shows that these points are just the full dots in Figure 11. Thus, $\mathcal{L}(\mathbf{Z}_4)$ is nothing but $(2\mathbf{Z})^2 \times 4\mathbf{Z}$. As a last example, consider $n = 4$ and

$$G = \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

The lattice $\mathcal{L}(\mathbf{Z}_2 \times \mathbf{Z}_2)$ consists of the points $(x_1, x_2, x_3, x_0) \in \mathbf{Z}^4$ satisfying

$$x_1(0, 1) + x_2(1, 0) + x_3(1, 1) \equiv (0, 0) \bmod 2, \quad x_1 + x_2 + x_3 + x_0 = 0,$$

or equivalently,

$$x_1 + x_3 \equiv 0 \bmod 2, \quad x_2 + x_3 \equiv 0 \bmod 2, \quad x_1 + x_2 + x_3 + x_0 = 0,$$

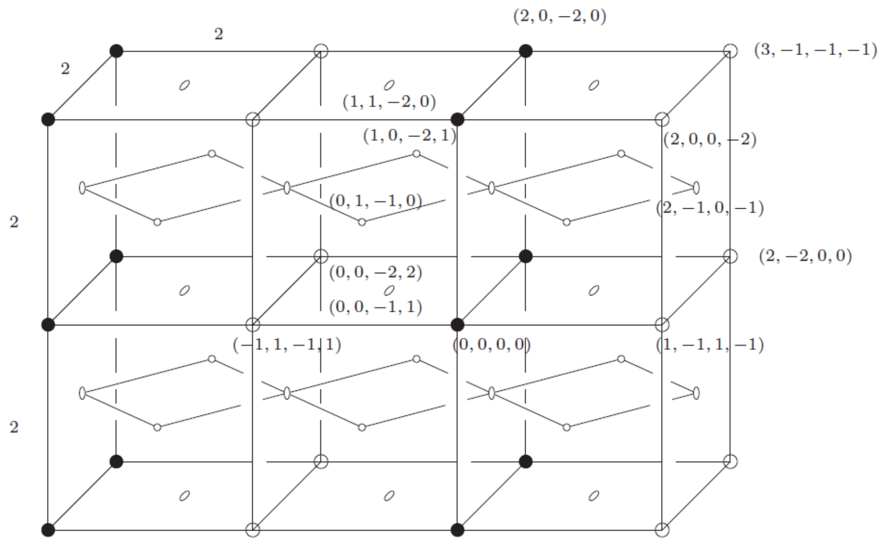

FIGURE 11. The lattice $\mathcal{L}(\mathbf{Z}_4) = (2\mathbf{Z})^2 \times 4\mathbf{Z}$ (full dots at the vertices of the cubes) and the lattice $\mathcal{L}(\mathbf{Z}_2 \times \mathbf{Z}_2) = 2\mathbf{Z}^3$ (full and light dots at the vertices of the cubes).

which is equivalent to the conditions

$$x_1 = x_2 = x_3 \equiv 0 \bmod 2, \quad x_1 + x_2 + x_3 + x_0 = 0.$$

Consequently, $\mathcal{L}(\mathbf{Z}_2 \times \mathbf{Z}_2)$ consists of the full and light dots seen as the vertices of the cubes in Figure 11 and thus equals $2\mathbf{Z}^3$.

5. Minimal distances and determinants

The following result provides us with the minimal distances and the determinants of the lattices considered in the previous section.

Theorem 5.1. *Let G_n be a finite Abelian group of order $|G_n| = n+1$. Then $d(\mathcal{L}(G_2)) = \sqrt{8}$, $d(\mathcal{L}(G_3)) = \sqrt{6}$, and $d(\mathcal{L}(G_n)) = \sqrt{4} = 2$ whenever $n \geq 4$. Moreover,* $\det \mathcal{L}(G_n) = (n+1)^3$ *for $n \geq 2$.*

This was proved by two different methods in the case where G_n is an elliptic curve over a finite field by two of the authors in [13] and by Min Sha in [16]. For general groups this result was established by yet another method in our paper [6]. We find it rather surprising that the minimal distance and the determinant depend only on the order of the group.

With $d(\mathcal{L})$ and $\det \mathcal{L}$ at hand, we can compute the packing density $\Delta(\mathcal{L})$ using formula (1) stated in Section 3. Here is the result.

Corollary 5.2. *If G_n is a finite Abelian group of order $|G_n| = n+1 \geq 4$, then*

$$\Delta(\mathcal{L}(G_n)) = \frac{V_n}{(n+1)^{3/2}}. \tag{2}$$

For the root lattices A_n, it is known that $d(A_n) = \sqrt{2}$ and $\det A_n = (n+1)^{1/2}$. Inserting this in (1) we obtain that

$$\Delta(A_n) = \frac{V_n}{2^{n/2}(n+1)^{1/2}}. \tag{3}$$

Comparing (2) and (3) we see that passage from A_n to $\mathcal{L}(G_n)$ removed the $2^{n/2}$ in the denominator of (3), Thus, for large n, the packing density of the lattices $\mathcal{L}(G_n)$ is significantly larger than that of A_n. We are nevertheless still far away from the Minkowski–Hlawka bound: elementary analysis shows that

$$\frac{\Delta(\mathcal{L}(G_n))}{\zeta(n)/2^{n-1}} = \frac{1}{2^{\frac{n}{2}\log_2 n - n + O(1)}} \to 0 \ \text{as} \ n \to \infty.$$

6. Well-roundedness and bases of minimal vectors

Let $\mathcal{L}$ be an n-dimensional lattice and let $\mathcal{S}(\mathcal{L}) = \{x \in \mathcal{L} : \|x\| = d(\mathcal{L})\}$ be the collection of lattice vectors of minimal length. One says that

(a) $\mathcal{L}$ is well rounded if $\mathcal{S}(\mathcal{L})$ contains n linearly independent vectors,

(b) $\mathcal{L}$ is generated by minimal vectors if every vector in $\mathcal{L}$ is a linear combination with integer coefficients of vectors in $\mathcal{S}(\mathcal{L})$,

(c) $\mathcal{L}$ has a basis of minimal vectors if $\mathcal{S}(\mathcal{L})$ contains a basis for $\mathcal{L}$.

It is easily seen that (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a). independent vectors among them The left lattice in Figure 1 has 4 minimal vectors and a basis of minimal vectors. The two other lattices in Figure 1 have 2 minimal vectors and they are not well rounded. (Note that the middle lattice of Figure 1 is the same as the right lattice in Figure 7, and hence it is not the pure honeycomb lattice, which is constituted of equilateral triangles.) The pure honeycomb lattice has 6 minimal vectors and a basis of minimal vectors. From Figure 11 we infer that $\mathcal{L}(\mathbf{Z}_4)$ has 4 minimal vectors, but as any three of them are linearly dependent, the lattice is not well rounded.

Lattices are full of surprises, and one of them is that the reverse implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are in general not true. That the implication (a) $\Longrightarrow$ (b) is false was already shown by Minkowski. He proved that if $n \le 4$, then well-roundedness implies that the lattice is generated by minimal vectors, but that this is no longer true for $n \ge 5$. His counterexample for $n = 5$ is the lattice $\mathcal{L}$ with the basis matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 \end{pmatrix} =: (e_1\ e_2\ e_3\ e_4\ v).$$

We have

$$e_5 = 2v - e_1 - e_2 - e_3 - e_4 \in \mathcal{L},$$

so

$$S(\mathcal{L}) = \{\pm e_1, \pm e_2, \pm e_3, \pm e_4, \pm e_5\},$$

but no linear combination with integer coefficients of these vectors gives v.

Even more surprising is the fact that the implication $(b) \Longrightarrow (c)$ is true for $n \le 9$ but false for $n \ge 10$. It was Conway and Sloane [9] who were the first to observe this phenomenon. They proved that the implication is false for $n \ge 11$. Figure 12 is a torn-out of [9].[1] Only recently Martinet and Schürmann [14] showed that the implication is also false for $n \ge 10$ but true for $n \le 9$.

In [6] we proved the following, which reveals that this phenomenon does not occur for the lattices $\mathcal{L}(G)$.

Theorem 6.1. *Except for the lattice $\mathcal{L}(\mathbf{Z}_4)$, which is not well rounded, the lattice $\mathcal{L}(G)$ is well rounded for every finite Abelian group G. Moreover, for every finite Abelian group $G \ne \mathbf{Z}_4$, the lattice $\mathcal{L}(G)$ has a basis of minimal vectors.*

Previous results like Theorem 6.1 were established using methods of the theory of function fields in [13, 16] in the case where G is an elliptic curve over a finite field. The proof given in [6] is pure matrix theory, and its strategy is as follows.

[1] In German, this would read “Ausriss aus der Arbeit [9].” The noun “extract” is an acceptable translation of “Ausriss”, but it has not the same beautiful flavor as the German word. We therefore decided to be very literal and to take “torn-out”.

THEOREM 1. *The* 11*-dimensional lattice* Λ *with Gram matrix*

$$\begin{bmatrix} 60 & 5 & 5 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 60 & 5 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 5 & 60 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 5 & 5 & 60 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 5 & 5 & 5 & 60 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 5 & 5 & 5 & 5 & 60 & -12 & -12 & -12 & -12 & -7 \\ -12 & -12 & -12 & -12 & -12 & -12 & 60 & -1 & -1 & -1 & -13 \\ -12 & -12 & -12 & -12 & -12 & -12 & -1 & 60 & -1 & -1 & -13 \\ -12 & -12 & -12 & -12 & -12 & -12 & -1 & -1 & 60 & -1 & -13 \\ -12 & -12 & -12 & -12 & -12 & -12 & -1 & -1 & -1 & 60 & -13 \\ -7 & -7 & -7 & -7 & -7 & -7 & -13 & -13 & -13 & -13 & 96 \end{bmatrix} \tag{1}$$

has minimal norm 60, *is generated by its* 24 *minimal vectors, but no set of* 11 *minimal vectors forms a basis.*

FIGURE 12. A torn-out of [9].

We first construct a basis matrix B for the lattice $\mathcal{L}(G)$. This is easy. For example, if $G = \mathbf{Z}_2 \times \mathbf{Z}_4$, then $B^\top$ and B can be taken to be

$$\underbrace{\begin{pmatrix} 2 & 0 & & & & & & -2 \\ 0 & 4 & & & & & & -4 \\ 0 & -2 & 1 & & & & & 1 \\ 0 & -3 & & 1 & & & & 2 \\ -1 & -1 & & & 1 & & & 1 \\ -1 & -2 & & & & 1 & & 2 \\ -1 & -3 & & & & & 1 & 3 \end{pmatrix}}_{7 \times 8}, \quad \underbrace{\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 4 & -2 & -3 & -1 & -2 & -3 \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ -2 & -4 & 1 & 2 & 1 & 2 & 3 \end{pmatrix}}_{8 \times 7}.$$

The Cauchy–Binet formula gives

$$\det B^\top B = \sum_{k=1}^{8} (\det C_k)^2,$$

where C_k results from B by deleting the kth row. We have $\det C_k = \pm 8$ for all k. Hence $\det B^\top B = 8 \cdot 8^2 = 8^3$. This works for general groups G and results in the following.

Proposition 6.2. *We have* $\det B^\top B = (n+1)^3$ *for general* G *with* $|G| = n+1$.

Due to this proposition, we know that $\det \mathcal{L}(G) = \sqrt{\det B^\top B} = (n+1)^{3/2}$. We then look for n minimal vectors $b_1, \ldots, b_n$, form a matrix M with these vectors as columns, and compute the determinant $\det M^\top M$. If this determinant is equal to $(\det \mathcal{L}(G))^2 = (n+1)^3$ (= square of the volume of a fundamental domain), then $\{b_1, \ldots, b_n\}$ is a basis for the lattice. Neither finding clever $b_1, \ldots, b_n$ nor computing $\det M^\top M$ is easy. In the simplest case where $G = \mathbf{Z}_{n+1} = \{0, 1, \ldots, n\}$ is the cyclic group[2] of order $n+1$, we took $M = M_n$ as the $(n+1) \times n$ analogue of the 7×6 matrix

$$M_6 = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \\ 1 & & & & & 1 \end{pmatrix}.$$

It turns out that

$$M_6^\top M_6 = \begin{pmatrix} 6 & -4 & 1 & & & 1 \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & 1 & -4 & 6 & -4 & 1 \\ & & 1 & -4 & 6 & -4 \\ 1 & & & 1 & -4 & 6 \end{pmatrix} = A_6,$$

that is, we obtain just the matrix we encountered at the beginning of Section 1. To compute $\det A_6 = \det M_6^\top M_6$ we use Cauchy–Binet again:

$$\det M_6^\top M_6 = \sum_{k=1}^{7} (\det D_k)^2,$$

where D_k results from M_6 by deleting the kth row. This leads to computing tridiagonal Toeplitz determinants and eventually yields that

$$\sum_{k=1}^{7} (\det D_k)^2 = \sum_{k=1}^{7} 7^2 = 7 \cdot 7^2 = 7^3.$$

This works anew for general n and proves the following, which was already mentioned in Section 1.

Proposition 6.3. *We have* $\det A_n = \det M_n^\top M_n = (n+1)^3$ *for all* $n \geq 4$.

For general finite Abelian groups, the problem of finding appropriate matrices M and computing the determinants $\det M^\top M$ is more sophisticated, and the Toeplitz structure also gets lost in the more general context. Anyway, at this point we arrived at the situation described in Section 1. We now leave lattice theory and turn over to Toeplitz determinants.

[2] In that case the lattices $\mathcal{L}(G)$ were first studied by E.S. Barnes [1] and are now named after him.

7. Toeplitz matrices

Let a be a (complex-valued) function in L^1 on the complex unit circle $\mathbf{T}$. The Fourier coefficients are defined by

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-ik\theta}\,d\theta \quad (k \in \mathbf{Z}).$$

With these Fourier coefficients, we may form the infinite Toeplitz matrix $T(a)$ and the $n \times n$ Toeplitz matrix $T_n(a)$ as follows:

$$T(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \\ a_1 & a_0 & a_{-1} & \ddots \\ a_2 & a_1 & a_0 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad T_n(a) = \begin{pmatrix} a_0 & \dots & a_{-(n-1)} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \dots & a_0 \end{pmatrix}.$$

The function a is referred to as the symbol of the matrix $T(a)$ and of the sequence $\{T_n(a)\}_{n=1}^{\infty}$ of its principal truncations. Formally we have

$$a(t) = \sum_{k=-\infty}^{\infty} a_k t^k \quad (t = e^{i\theta} \in \mathbf{T}).$$

A class of symbols that is of particular interest in connection with the topic of this survey is given by

$$a(t) = \omega_\alpha(t) := |t-1|^{2\alpha}.$$

These symbols are special so-called pure Fisher–Hartwig symbols because, in 1968, Fisher and Hartwig [12] raised a conjecture on the determinants of $T_n(\omega_\alpha)$. We assume $\operatorname{Re}\alpha > -1/2$ to guarantee that $\omega_\alpha \in L^1(\mathbf{T})$. The cases $\alpha = 1$ and $\alpha = 2$ lead to the symbols

$$\begin{aligned} \omega_1(t) &= |t-1|^2 = (t-1)(t^{-1}-1) = -t^{-1} + 2 - t, \\ \omega_2(t) &= |t-1|^4 = (t-1)^2(t^{-1}-1)^2 = t^{-2}(t-1)^4 \\ &= t^{-2} - 4t^{-1} + 6 - 4t + t^2. \end{aligned}$$

The 4×4 versions of the corresponding Toeplitz matrices are

$$T_4(\omega_1) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad T_4(\omega_2) = \begin{pmatrix} 6 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 6 \end{pmatrix}, \qquad (4)$$

and hence $T_n(\omega_2)$ is nothing but the matrix T_n introduced in Section 1.

We are interested in matrices that arise from pure Toeplitz matrices by perturbations in the corners. The setting is as follows. Fix $m \in \{1, 2, \dots\}$ and let

$E_{11}, E_{12}, E_{21}, E_{22} \in \mathbf{C}^{m\times m}$ be fixed $m \times m$ matrices. For $n \geq 2m$, let E_n be the $n \times n$ matrix

$$E_n = \begin{pmatrix} E_{11} & 0 & E_{12} \\ 0 & 0 & 0 \\ \underbrace{E_{21}}_{m} & \underbrace{0}_{n-2m} & \underbrace{E_{22}}_{m} \end{pmatrix} \in \mathbf{C}^{n\times n}. \tag{5}$$

For example, if $m = 1$ and the four scalars E_{jk} are given by

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then E_n is the $n\times n$ matrix with ones in the upper-right and lower-left corners and zeros elsewhere, and an $n\times n$ Toeplitz matrix perturbed by ones in the upper-right and lower-left corners may therefore be written as $T_n(a) + E_n$.

8. Tame symbols

Suppose $a : \mathbf{T} \to \mathbf{C}$ is a continuous function, the origin does not belong to the range $a(\mathbf{T})$, and the winding number of $a(\mathbf{T})$ about the origin is zero. Such symbols are what we call tame symbols. These assumptions guarantee that $T(a)$ is invertible on ℓ^2. The inverse $T^{-1}(a)$ may again be given by an infinite (but in general not Toeplitz) matrix. Let S_{11} denote the upper-left $m \times m$ block of $T^{-1}(a)$, let $S_{11}^\top$ be the transpose of S_{11}, and put $\widetilde{S}_{11}^\top := J_m S_{11}^\top J_m$, where J_m is the flip matrix that is, the matrix with ones on the counterdiagonal and zeros elsewhere. In [5], we proved that then

$$\lim_{n\to\infty} \frac{\det(T_n(a) + E_n)}{\det T_n(a)} = \det\left[\begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix} + \begin{pmatrix} S_{11} & 0 \\ 0 & \widetilde{S}_{11}^\top \end{pmatrix}\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}\right].$$

Thus, the quotient $\det(T_n(a) + E_n)/\det(T_n(a))$ goes to a completely identified limit. If this limit is zero, then $\det(T_n(a)+E_n)$ grows slower than $\det T_n(a)$, which is a first hint to the drop of growth observed in Section 1.

We remark that if, in addition, $a : \mathbf{T} \to \mathbf{C}$ is smooth enough ($a \in C^{1/2+\varepsilon}$ with $\varepsilon > 0$ will do), then the asymptotic behavior of $\det T_n(a)$ is given by Szegő's strong limit theorem, which says that

$$\det T_n(a) \sim G(a)^n E(a)$$

with

$$G(a) = \exp((\log a)_0), \quad E(a) = \exp\left(\sum_{k=1}^{\infty} k(\log a)_k(\log a)_{-k}\right),$$

where $\log a(t) = \sum_{k=-\infty}^{\infty}(\log a)_k t^k$.

9. Fisher–Hartwig symbols

The proof of the result quoted in the previous section is based on the representation

$$\det(T_n(a) + E_n) = \det T_n(a) \det(I + T_n^{-1}(a)E_n)$$

and the fact that the so-called finite section method is applicable to $T(a)$, which means that

$$T_n^{-1}(a)P_n \to T^{-1}(a) \quad \text{strongly},$$

where $P_n : \ell^2 \to \ell^2$ is the orthogonal projection onto the first n coordinates. (The range of P_n may be identified with $\mathbf{C}^n$.) The basic assumption was that a is continuous and that $0 \notin a(\mathbf{T})$, i.e., that a has no zeros on $\mathbf{T}$. This assumption is not satisfied for $a(t) = \omega_\alpha(t) = |t-1|^{2\alpha}$, because, for $\alpha > 0$, the function has a zero at $t = 1$. For $\alpha < 0$, the function is not even continuous.

Fortunately, there is a nice explicit formula for the inverse $T_n^{-1}(\omega_\alpha)$ due to Roland Duduchava and Steffen Roch. We decline to cite this formula and its history here and refer the interested reader to [4, 7]. This formula is of use twice: first, it almost immediately yields an exact formula for the determinants $\det T_n(\omega_\alpha)$ and secondly, it provides us with explicit expressions for the entries of the inverse matrix $T_n^{-1}(\omega_\alpha)$. As for determinants, the Duduchava–Roch formula implies that

$$T_n^{-1}(\omega_\alpha) = \frac{\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)} D_\alpha T_\alpha D_{2\alpha}^{-1} T_\alpha^\top D_\alpha \tag{6}$$

where D_α and $D_{2\alpha}$ are certain explicitly given diagonal $n \times n$ matrices with binomial coefficients on the diagonal and T_α is a lower-triangular $n \times n$ Toeplitz matrix with ones on the diagonal. Taking the determinant on both sides of (6) we get

$$\frac{1}{\det T_n(\omega_\alpha)} = \frac{\Gamma(1+\alpha)^{2n}}{\Gamma(1+2\alpha)^n} \frac{(\det D_\alpha)^2}{\det D_{2\alpha}} \underbrace{(\det T_\alpha)^2}_{=1},$$

resulting in the following formula, which was established in [7].

Theorem 9.1. *For* $\operatorname{Re}\alpha > -1/2$ *we have*

$$\det T_n(\omega_\alpha) = \frac{G(1+\alpha)^2}{G(1+2\alpha)} \frac{G(n+1)G(n+1+2\alpha)}{G(n+1+\alpha)^2} \sim \frac{G(1+\alpha)^2}{G(1+2\alpha)} n^{\alpha^2}.$$

Here G is the Barnes function.[3] This is an entire function satisfying the identity $G(z+1) = \Gamma(z)G(z)$; note that the Gamma function satisfies the identity $\Gamma(z+1) = z\Gamma(z)$. The values of the Barnes function at the nonnegative integers are $G(0) = G(1) = 1$, $G(m) = (m-2)! \cdots 1!0!$. See [3, 4, 8, 11] for alternative proofs of Theorem 9.1 and for historical notes.

And herewith our result of [5] on corner perturbations of the matrices $T_n(\omega_\alpha)$. It was derived by writing $\det(T_n(\omega_\alpha) + E_n) = \det T_n(\omega_\alpha) \det(I + T_n^{-1}(\omega_\alpha)E_n)$ and using Duduchava–Roch for $T_n^{-1}(\omega_\alpha)$.

[3]The function is named after E.W. Barnes [2], who is not the Barnes we cited in another connection already in footnote 2.

Theorem 9.2. *Let* $\operatorname{Re}\alpha > -1/2$. *If*

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

then

$$\det(T_n(\omega_\alpha) + E_n) \sim \frac{G(1+\alpha)^2}{G(1+2\alpha)} 2\alpha(\alpha+1) n^{\alpha^2-1}.$$

Comparing Theorems 9.1 and 9.2 we see that the corner perturbations (7) indeed lower the growth of the determinants from n^{α^2} to n^{α^2-1}. For $\alpha = 2$, this is exactly what we observed in Section 1.

In fact the exact expressions delivered by the Duduchava–Roch formula for the entries of $T_n^{-1}(\omega_\alpha)$ yield exact formulas for the determinants $\det(T_n(\omega_\alpha)+E_n)$. Here are a few examples. We assume that the corner perturbations are of the form (7). Recall that $T_n(\omega_1)$ and $T_n(\omega_2)$ are the $n\times n$ analogues of the matrices (4). For these matrices,

$$\det T_n(\omega_1) = n+1 \sim n, \quad \det(T_n(\omega_1) + E_n) = 4,$$
$$\det T_n(\omega_2) = \frac{(n+1)(n+2)^2(n+3)}{12} \sim \frac{n^4}{12},$$
$$\det(T_n(\omega_2) + E_n) = (n+1)^3 \sim n^3.$$

The matrix $T_n(\omega_3)$ is the $n \times n$ version of the septadiagonal matrix

$$T_5(\omega_3) = \begin{pmatrix} 20 & -15 & 6 & -1 & 0 \\ -15 & 20 & -15 & 6 & -1 \\ 6 & -15 & 20 & -15 & 6 \\ -1 & 6 & -15 & 20 & -15 \\ 0 & -1 & 6 & -15 & 20 \end{pmatrix}$$

and we can show that

$$\det T_n(\omega_3) = \frac{(n+1)(n+2)^2(n+3)^3(n+4)^2(n+5)}{8640} \sim \frac{n^9}{8640},$$
$$\det(T_n(\omega_3) + E_n) = \frac{(n+1)(n+2)^2(n+3)[(n+2)^2+1][(n+2)^2+2]}{360} \sim \frac{n^8}{360}.$$

Replacing the perturbations (4) by the more general perturbations (5) is not a big problem. Using Duduchava–Roch one gets the beginning entries $x_1, x_2, x_3, \dots$ and the last entries $x_n, x_{n-1}, x_{n-2}, \dots$ of the first column $(x_1, x_2, \dots, x_n)^\top$ of $T_n^{-1}(\omega_\alpha)$ as well as the beginning entries $y_1, y_2, y_3, \dots$ and the last entries y_n, y_{n-1}, $y_{n-2}, \dots$ of the last column $(y_1, y_2, \dots, y_n)^\top$ of $T_n^{-1}(\omega_\alpha)$. To compute the entries close to the corners of $T_n^{-1}(\omega_\alpha)$, one may then employ the Gohberg–Sementsul–Trench formula, which states that if $x_1 \neq 0$, which is satisfied in the cases at hand, then we have

$$T_n^{-1}(a) = \frac{1}{x_1}\begin{pmatrix} x_1 & & \\ \vdots & \ddots & \\ x_n & \dots & x_1 \end{pmatrix}\begin{pmatrix} y_n & \dots & y_1 \\ & \ddots & \vdots \\ & & y_1 \end{pmatrix}$$
$$-\frac{1}{x_1}\begin{pmatrix} y_0 & & \\ \vdots & \ddots & \\ y_{n-1} & \dots & y_0 \end{pmatrix}\begin{pmatrix} x_{n+1} & \dots & x_2 \\ & \ddots & \vdots \\ & & x_{n+1} \end{pmatrix},$$

where $x_{n+1} := 0$ and $y_0 := 0$. See [5] for the details.

The genuine challenge is symbols of the form

$$a(t) = b(t)\prod_{j=1}^{N}|t-t_j|^{2\alpha_j} \quad (t \in \mathbf{T})$$

where $b(t) > 0$ is a sufficiently smooth function and $t_1, \dots, t_N$ are distinct points on $\mathbf{T}$. A particular case of the Fisher–Hartwig conjecture says that

$$\det T_n(a) \sim G(b)^n E(a) n^{\alpha_1^2+\dots+\alpha_N^2}$$

with certain constant nonzero $G(b)$ and $E(a)$. This was proved by Widom [19] in 1973. These symbols satisfy the hypotheses of the following result, which was established in [5].

Theorem 9.3. *Let E_n be as in (5). Suppose $a \in L^1(\mathbf{T})$, $a \geq 0$ a.e. on $\mathbf{T}$, and $\log a \in L^1(\mathbf{T})$. Let $\log a(t) = \sum_{k=-\infty}^{\infty}(\log a)_k t^k$ $(t \in \mathbf{T})$ be the Fourier expansion of a, and define a_+^{-1} for $|z| < 1$ by*

$$a_+^{-1}(z) = \exp\left(-\sum_{k=1}^{\infty}(\log a)_k z^k\right) =: \sum_{k=0}^{\infty}(a_+^{-1})_k t^k.$$

Then $T_n(a)$ is a positive definite Hermitian matrix for every $n \geq 1$ and

$$\lim_{n\to\infty}\frac{\det(T_n(a)+E_n)}{\det T_n(a)} = \det\left[\begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix} + \begin{pmatrix} S_{11} & 0 \\ 0 & \widetilde{S}_{11}^{\top} \end{pmatrix}\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}\right]$$

with explicit expressions for the entries of the $m \times m$ matrix S_{11} in terms of the coefficients $(a_+^{-1})_k$.

We remark that a_+ is just $\exp(-(\log a)_0/2)$ times the outer function whose modulus on $\mathbf{T}$ is $|a|^{1/2}$. Paper [5] contains several examples.

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