ON THE NORM CLOSURE PROBLEM FOR COMPLEX
SYMMETRIC OPERATORS

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ABSTRACT. We prove that the set of all complex symmetric operators on a
separable, infinite-dimensional Hilbert space is not norm closed.

In [2, Sect. 3], it is asked whether the set of all complex symmetric operators
on a separable, infinite-dimensional Hilbert space is norm closed. We answer this
question in the negative. Let

\[ S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, \ldots) \]

denote the unilateral shift on \( \ell^2(\mathbb{N}) \) and let \( \cong \) denote unitary equivalence. Note that

\[ T_n = \frac{n}{n+1} S \oplus \bigoplus_{j=1}^{\infty} \frac{j}{j+1} S \oplus \bigoplus_{j=1}^{\infty} \frac{j}{j+1} S^* \cong \bigoplus_{j=1}^{\infty} \frac{j}{j+1} (S \oplus S^*) \]

is complex symmetric by [1, Ex. 5]. On the other hand, \( T_n \) converges in norm to

\[ T = S \oplus \bigoplus_{j=1}^{\infty} \frac{j}{j+1} S \oplus \bigoplus_{j=1}^{\infty} \frac{j}{j+1} S^* \cong S \oplus \bigoplus_{j=1}^{\infty} \frac{j}{j+1} (S \oplus S^*) . \]

Since \( \|S^k(1,0,0,\ldots)\| = 1 \), there is an \( x \) so that \( \|T^k x\| = 1 \) for \( k \geq 0 \). However,

\[ T^* = S^* \oplus \bigoplus_{j=1}^{\infty} \frac{j}{j+1} (S^* \oplus S) = S^* \oplus (\text{a strict contraction}) \]

possesses no such vector since \( (S^*)^k \) tends strongly to zero. This precludes the
existence of a conjugation \( C \) (i.e., an isometric, conjugate-linear involution) such
that \( T = CT^* C \). Thus \( T \) is not complex symmetric.

\[ \square \]

Remarks: We thank D. Sherman for his helpful suggestions. We also note that
S. Zhu, C.G. Li, and Y.Q. Ji discovered a different approach shortly before us.

References

[3] Sen Zhu, Chun Guang Li, and You Qing Ji. The class of complex symmetric operators is not

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