TWO REMARKS ABOUT NILPOTENT OPERATORS OF ORDER TWO

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Abstract. We present two novel results about Hilbert space operators which are nilpotent of order two. First, we prove that such operators are *indestructible* complex symmetric operators, in the sense that tensoring them with any operator yields a complex symmetric operator. In fact, we prove that this property characterizes nilpotents of order two among all nonzero bounded operators. Second, we establish that every nilpotent of order two is unitarily equivalent to a truncated Toeplitz operator.

1. Introduction

In the following, $\mathcal{H}$ denotes a separable complex Hilbert space and $B(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$. Recall that an operator T in $B(\mathcal{H})$ is called *nilpotent* if $T^n = 0$ for some positive integer n. The least such n is called the *order* of nilpotence of T. This note concerns two rather unusual properties of operators which are nilpotent of order two.

The first result involves complex symmetric operators (see Section 2 for background). It is known that every operator which is nilpotent of order two is a complex symmetric operator (Lemma 1). However, these operators are complex symmetric in a much stronger sense, for the tensor product of a nilpotent of order two with an arbitrary operator always yields a complex symmetric operator. We prove in Section 3 that this property actually characterizes nilpotents of order two among all nonzero bounded operators.

Our second result concerns truncated Toeplitz operators (precise definitions are given in Section 4). To be more specific, we prove that every operator which is nilpotent of order two is unitarily equivalent to a truncated Toeplitz operator having an analytic symbol. This is relevant to a series of open problems, first arising in [1] and developed further in [9], which, in essence, ask whether an arbitrary complex symmetric operator is unitarily equivalent to a truncated Toeplitz operator (or possibly a direct sum of such operators).

We close the paper with several open questions suggested by these results.

2. Complex symmetry

Before proceeding, let us recall a few basic definitions [2–4]. A *conjugation* on a complex Hilbert space $\mathcal{H}$ is a conjugate-linear, isometric involution. We say that an operator T in $B(\mathcal{H})$ is *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that T is *C-symmetric*. The terminology

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reflects the fact that an operator is complex symmetric if and only if it has a self-transpose matrix representation with respect to some orthonormal basis. In fact, any orthonormal basis which is C-*real*, in the sense that each basis vector is fixed by C, yields such a matrix representation.

It is known that if T is nilpotent of order two, then T is a complex symmetric operator. This was first established directly in [8, Thm. 5], using what we now recognize as a somewhat overcomplicated argument. Later on, this result was obtained as a corollary of the more general fact that every binormal operator is complex symmetric [10, Thm. 2, Cor. 4]. A much simpler direct proof is provided below. In what follows, we denote unitary equivalence by $\cong$.

Lemma 1. *If T is nilpotent of order two, then T is a complex symmetric operator.*

Proof. If T in $B(\mathcal{H})$ satisfies $T(Tx) = T^2x = 0$ for every x in $\mathcal{H}$, it follows that $\overline{\operatorname{ran}}\, T \subseteq \ker T = \ker |T|$. Considering the polar decomposition of T, we see that

$$T \cong \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \oplus 0 \tag{1}$$

where A is a positive operator with dense range (the zero direct summand, which acts on $\ker T \ominus \overline{\operatorname{ran}}\, T$, may be absent). Without loss of generality, we may assume that $\ker T = \overline{\operatorname{ran}}\, T$. If J is any conjugation which commutes with A (the existence of such a J follows immediately from the Spectral Theorem), we find that

$$\underbrace{\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}}_{T^*} \underbrace{\begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix}}_{C},$$

whence T is a complex symmetric operator. □

For operators on a finite dimensional space, there is a quite explicit proof. Indeed, the positive semidefinite matrix A is unitarily equivalent to a diagonal matrix $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, whence

$$\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix} \cong \bigoplus_{i=1}^{n} \begin{bmatrix} 0 & 0 \\ \lambda_i & 0 \end{bmatrix} \cong \bigoplus_{i=1}^{n} \frac{\lambda_j}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}.$$

3. Indestructible complex symmetric operators

In the following, $\mathcal{H}$ and $\mathcal{K}$ denote separable complex Hilbert spaces while A and B are bounded operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. Recall that the operator $A \otimes B$ acts on the space $\mathcal{H} \otimes \mathcal{K}$ and satisfies

$$\|A \otimes B\|_{\mathcal{H}\otimes\mathcal{K}} = \|A\|_{\mathcal{H}} \|B\|_{\mathcal{K}}, \tag{2}$$

the subscripts being suppressed in practice. The following relevant lemma is from [3, Sect. 10], where it is stated without proof.

Lemma 2. *The tensor product of complex symmetric operators is complex symmetric.*

Proof. Suppose that A and B are operators and that C and J are conjugations on $\mathcal{H}$ and $\mathcal{K}$, respectively, such that $A = CA^*C$ and $B = JB^*J$. Let u_i and v_j denote C-real and J-real orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$, respectively. Define a conjugation $C \otimes J$ on $\mathcal{H} \otimes \mathcal{K}$ by first setting $(C \otimes J)(u_i \otimes v_j) = u_i \otimes v_j$ on the orthonormal basis $u_i \otimes v_j$ of $\mathcal{H} \otimes \mathcal{K}$ and then extending this to $\mathcal{H} \otimes \mathcal{K}$ by conjugate-linearity and continuity. One can then check that $A \otimes B$ is $(C \otimes J)$-symmetric. □

On the other hand, it is possible for $A \otimes B$ to be complex symmetric even if neither A nor B is complex symmetric. The following lemma provides a simple method for constructing such examples.

Lemma 3. *For each A in $B(\mathcal{H})$ and each conjugation J on $\mathcal{H}$, the operator $T = A \otimes JA^*J$ is complex symmetric.*

Proof. If Φ in $B(\mathcal{H} \otimes \mathcal{H})$ is defined first on simple tensors by $\Phi(x \otimes y) = y \otimes x$ and then extended to $\mathcal{H} \otimes \mathcal{H}$ in the natural way, then $C = \Phi(J \otimes J)$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$ with respect to which $T = CT^*C$. □

Example 4. Suppose $\mathcal{H} = \ell^2$, $A = S$ (the unilateral shift), and J is entry-by-entry complex conjugation on ℓ^2. There are many ways to see that S is not a complex symmetric operator [7, Cor. 7], [3, Prop. 1], [2, Ex. 2.14], [10, Thm. 4]. On the other hand, Lemma 3 implies that $S \otimes S^*$ is complex symmetric. In fact,

$$S \otimes S^* \cong \bigoplus_{n=0}^{\infty} J_n(0),$$

where $J_n(0)$ denotes a $n \times n$ nilpotent Jordan block, which is complex symmetric by [3, Ex. 4]. To see this, note that $S \otimes S^*$ is unitarily equivalent to the operator

$$[Tf](z,w) = z\left(\frac{f(z,w) - f(z,0)}{w}\right)$$

on the Hardy space H^2_2 on the bidisk and observe that $H^2_2 = \bigoplus_{n=0}^{\infty} \mathcal{P}_n$ where $\mathcal{P}_n$ denotes the set of all homogeneous polynomials $p(z,w)$ of degree n. Each subspace $\mathcal{P}_n$ reduces T and $T|_{\mathcal{P}_n} \cong J_n(0)$.

Having briefly explored the interplay between tensor products and complex symmetric operators, we come to the following definition.

Definition. An operator A in $B(\mathcal{H})$ is called an *indestructible* complex symmetric operator if $A \otimes B$ is a complex symmetric operator on $\mathcal{H} \otimes \mathcal{K}$ for all B in $B(\mathcal{K})$.

Let us note that an indestructible complex symmetric operator must indeed be complex symmetric since $A \otimes 1 \cong A$. Clearly, indestructibility is a rather strong property. In fact, from the definition alone, it is not immediately clear whether any nonzero examples exist. As we will see, the nonzero indestructible complex symmetric operators are precisely those operators which are nilpotent of order two.

Theorem 5. *T is an indestructible complex symmetric operator if and only if T is nilpotent of order ≤ 2.*

Proof. If A is nilpotent of order ≤ 2, then $A \otimes B$ is also nilpotent of order ≤ 2. By Lemma 1, $A \otimes B$ is complex symmetric whence A is indestructible.

Before embarking on the remaining implication, let us first remark that if T is C-symmetric, then

$$w(T, T^*) = Cw(T^*, T)C \tag{3}$$

holds for each word $w(x,y)$ in the noncommuting variables x, y. This fact will be useful in what follows.

Now suppose that A is an indestructible complex symmetric operator. For any other operator B and any word $w(x, y)$, we obtain

$$\begin{aligned}\|w(A,A^*)\|\|w(B,B^*)\| &= \|w(A,A^*)\otimes w(B,B^*)\| \\ &= \|w(A\otimes B, A^*\otimes B^*)\| \\ &= \|w(A^*\otimes B^*, A\otimes B)\| && \text{by (3)} \\ &= \|w(A^*,A)\|\|w(B^*,B)\|.\end{aligned}$$

Since A is complex symmetric we apply (3) again to obtain

$$\|w(A,A^*)\|\|w(B,B^*)\| = \|w(A,A^*)\|\|w(B^*,B)\|. \tag{4}$$

Letting $w(x, y) = yx^2$ and

$$B = \begin{bmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix},$$

where α, β are non-negative real numbers, a simple computation reveals that

$$\|w(B,B^*)\| = \alpha^2\beta, \qquad \|w(B^*,B)\| = \alpha\beta^2.$$

If $\alpha \neq \beta$, then (4) implies that $\|w(A,A^*)\| = 0$ so that $(A^*A)A = 0$. Therefore $\operatorname{ran} A \subseteq \ker A^*A = \ker A$ whence $A^2 = 0$, as desired. □

4. Unitary equivalence to a truncated Toeplitz operator

The study of truncated Toeplitz operators has been largely motivated by a seminal paper of Sarason [12]. We briefly recall the basic definitions, referring the reader to the recent survey article [5] for a more thorough introduction.

In the following, H^2 denotes the classical Hardy space on the open unit disk. For each nonconstant inner function u, we consider the corresponding *model space* $\mathcal{K}_u := H^2 \ominus uH^2$. Letting P_u denote the orthogonal projection from L^2 onto $\mathcal{K}_u$, for each φ in L^∞ we define the *truncated Toeplitz operator* $A^u_\varphi : \mathcal{K}_u \to \mathcal{K}_u$ by setting

$$A^u_\varphi f = P_u(\varphi f).$$

Each such operator is C-symmetric with respect to the conjugation $Cf = \overline{fz}u$ on $\mathcal{K}_u$. We say that A^u_φ is an *analytic* truncated Toeplitz operator if the *symbol* φ belongs to H^∞, in which case $A^u_\varphi = \varphi(A^u_z)$ by the H^∞-functional calculus for A^u_z.

A significant amount of evidence has been accumulated which indicates that truncated Toeplitz operators provide concrete models for general complex symmetric operators [1, 9, 13]. In fact, a surprising array of complex symmetric operators can be shown to be unitarily equivalent to truncated Toeplitz operators. These results have led to several open problems and conjectures [1, Question 5.10], [9, Sect. 7]. We refer the reader to [5, Sect. 9] for a thorough discussion of the topic.

By Lemma 1, we know that operators which are nilpotent of order two are complex symmetric. We now go a step further and prove that every such operator is unitary equivalent to an analytic truncated Toeplitz operator.

Before proceeding, we require a few words about Hankel operators. First let us recall that the *Hankel operator* $H_\varphi : H^2 \to H^2_-$ with *symbol* φ in L^∞ is the linear operator defined by $H_\varphi f = P_-(\varphi f)$, where P_- denotes the orthogonal projection from H^2 onto $H^2_- := L^2 \ominus H^2$. A detailed treatise on the subject of Hankel operators is [11]. We refer the reader there for a complete treatment of the subject.

The first result required is the well-known relationship [11, Ch. 1, eq. (2.9)]

$$A_\varphi^u = M_u H_{\overline{u}\varphi}|_{\mathcal{K}_u}, \tag{5}$$

where u is an inner function and φ belongs to H^∞. The next ingredient is [11, Ch. 1, Thm. 2.3].

Lemma 6. *For ψ in L^∞, the following are equivalent:*

(i) $\ker H_\psi$ *is nontrivial,*

(ii) $\operatorname{ran} H_\psi$ *is not dense in H^2_-,*

(iii) $\psi = \overline{u}\varphi$ *for some inner function u and some φ in H^∞.*

Finally, we need the following deep result from [14] (see [11, Ch. 12, Thm. 8.1]):

Lemma 7. *If $A \geq 0$ is an operator on a separable, infinite-dimensional Hilbert space, then the following are equivalent:*

(i) *A is unitarily equivalent to the modulus of a Hankel operator,*

(ii) *A is unitarily equivalent to the modulus of a self-adjoint Hankel operator,*

(iii) *A is not invertible, and* $\ker A$ *is either trivial or infinite-dimensional.*

The following general result may be of independent interest.

Lemma 8. *Any positive operator is unitarily equivalent to the modulus of an analytic truncated Toeplitz operator.*

Proof. Suppose $B \geq 0$ and consider the operator $B' = B \oplus 0$, where 0 acts on an infinite dimensional Hilbert space. By Lemma 7, B' is unitarily equivalent to the modulus $|H_\psi|$ of some Hankel operator $H_\psi : H^2 \to H^2_-$. In light of Lemma 6, it follows that $\psi = \overline{u}\varphi$ for some inner function u and some φ in H^∞. We may assume that u and φ are coprime, since a common inner factor of both would cancel in the evaluation of $\psi = \overline{u}\varphi$.

By [11, Ch.1, Thm 2.4], the restriction

$$\hat{H} : \mathcal{K}_u \to H^2_- \ominus \overline{u}H^2_-$$

of $H_{\overline{u}\varphi}$ to $\mathcal{K}_u$ is injective and has dense range. In other words, $|\hat{H}|$ is unitarily equivalent to $B|_{(\ker B)^\perp}$. The operator

$$W : H^2_- \ominus \overline{u}H^2_- \to \mathcal{K}_u$$

defined by $W = M_u|_{H^2_- \ominus \overline{u}H^2_-}$ is unitary and satisfies $A_\varphi^u = W\hat{H}$ by (5). Therefore

$$|A_\varphi^u| \cong |\hat{H}| \cong B|_{(\ker B)^\perp}.$$

Now let v be an inner function such that $\dim \mathcal{K}_v = \dim \ker B$ (e.g., a Blaschke product with $\dim \ker B$ zeros). Noting that

$$\mathcal{K}_{uv} = K_u \oplus uK_v = vK_u \oplus K_v,$$

we see that the matrix of

$$A_{v\varphi}^{uv} : K_u \oplus uK_v \to vK_u \oplus K_v$$

with respect to the decompositions above is

$$\begin{bmatrix} vA_\varphi^u & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\dim \mathcal{K}_v = \dim \ker B$, we conclude that

$$|A^{uv}_{v\varphi}| \cong |A^u_\varphi| \oplus 0 \cong B|_{(\ker B)^\perp} \oplus B|_{\ker B} = B. \qquad \square$$

Armed with the preceding lemma, we are ready to prove the following theorem.

Theorem 9. *If T is nilpotent of order two, then T is unitarily equivalent to an analytic truncated Toeplitz operator.*

Proof. It follows from (1) that any nilpotent of order two is unitarily equivalent to an operator of the form

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B & 0 & 0 \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}' \oplus \mathcal{H}$, where $\mathcal{H}, \mathcal{H}'$ are Hilbert spaces and $B \geq 0$ acts on $\mathcal{H}$. By Lemma 8, we may assume $\mathcal{H} = \mathcal{K}_u$ and that $B = |A^u_\varphi|$ for some inner function u and φ in H^∞. Let v be an inner function such that $\dim \mathcal{K}_v = \dim \mathcal{H}'$ and let $\omega : \mathcal{H}' \to \mathcal{K}_v$ be unitary. With respect to the decomposition

$$\mathcal{K}_{u^2v} = \mathcal{K}_u \oplus u\mathcal{K}_v \oplus uv\mathcal{K}_u$$

we have

$$A^{u^2v}_{uv\varphi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ uvA^u_\varphi & 0 & 0 \end{bmatrix}.$$

Since A^u_φ is complex symmetric, we can write $A^u_\varphi = V|A^u_\varphi|$ with V unitary [4, Cor. 1]. If W denotes the unitary operator

$$(I_{\mathcal{K}_u} \oplus \omega^* \bar{u} \oplus V^* \bar{u}\bar{v}) : \mathcal{K}_u \oplus u\mathcal{K}_v \oplus uv\mathcal{K}_u \to \mathcal{K}_u \oplus \mathcal{H}' \oplus \mathcal{K}_u,$$

then

$$W(A^{u^2v}_{uv\varphi})W^* = N,$$

which proves the theorem. $\square$

Example 10. If A is a noncompact operator on $\mathcal{H}$, then the operator

$$T = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$ is unitarily equivalent to an analytic truncated Toeplitz operator A^u_φ However, since any truncated Toeplitz operator whose symbol is continuous on the unit circle must be of the form normal plus compact [6, Thm. 1, Cor. 2], φ cannot be continuous.

5. Open questions

We conclude this note with some questions suggested by the preceding work.

Question 1. Formula (3) may be generalized by considering polynomials $p(x, y)$ in two noncommuting variables x, y. Then

$$p(T, T^*) = C\widetilde{p}(T^*, T)C$$

where $\widetilde{p}(x, y)$ is obtained from $p(x, y)$ by conjugating each coefficient. If T is complex symmetric, it follows then that

$$\|p(T, T^*)\| = \|\widetilde{p}(T^*, T)\| \tag{6}$$

holds for every $p(x,y)$. Does the converse hold? That is, if T in $B(\mathcal{H})$ satisfies (6) for every polynomial $p(x,y)$ in two noncommuting variables x,y, does it follow that T is a complex symmetric operator?

Note that considering only *words* in T and T^* is not sufficient to characterize complex symmetric operators. Indeed, if S denotes the unilateral shift, then it is easy to see that $\|w(S,S^*)\| = \|\widetilde{w}(S^*,S)\| = 1$ for any word $w(x,y)$.

The following question stems from the proof of Theorem 9.

Question 2. If T is unitarily equivalent to a truncated Toeplitz operator, then does the operator $T \oplus 0$ have the same property?

Although partial results in this direction appear in [13, Section 6], the preceding question appears troublesome even in low dimensions.

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