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Integral Equations and Operator Theory

Aluthge Transforms of Complex Symmetric Operators

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Abstract. If $T = U|T|$ denotes the polar decomposition of a bounded linear operator T, then the *Aluthge transform* of T is defined to be the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. In this note we study the relationship between the Aluthge transform and the class of complex symmetric operators (T is *complex symmetric* if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \to \mathcal{H}$ so that $T = CT^*C$). In this note we prove that: (1) the Aluthge transform of a complex symmetric operator is complex symmetric, (2) if T is complex symmetric, then $(\widetilde{T})^*$ and $(\widetilde{T^*})$ are unitarily equivalent, (3) if T is complex symmetric, then $\widetilde{T} = T$ if and only if T is normal, (4) $\widetilde{T} = 0$ if and only if $T^2 = 0$, and (5) every operator which satisfies $T^2 = 0$ is necessarily complex symmetric.

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1. Introduction

Throughout this note, $\mathcal{H}$ will denote a separable complex Hilbert space and T a bounded linear operator on $\mathcal{H}$. If $T = U|T|$ denotes the polar decomposition of T, then the *Aluthge transform* of T is defined to be the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This transformation arose in the study of hyponormal operators [1] and has since been studied in many different contexts [3, 4, 5, 6, 8, 9, 10, 32].

Part of the appeal of the Aluthge transform lies in the fact that it respects many of the properties of the original operator. For instance, an operator and its Aluthge transform have the same spectrum [26, Lem. 5]. In fact, much of the fine structure of the spectrum is preserved by the Aluthge transform [10, Thms. 1.3, 1.5]. Another important property is that Lat(T), the lattice of T-invariant

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subspaces of $\mathcal{H}$, is nontrivial if and only if $\mathrm{Lat}(\widetilde{T})$ is nontrivial [10, Thm 1.15] (see also [9, Thm. 2]).

It is our aim in this note to study the relationship between the Aluthge transform and the class of complex symmetric operators. Before proceeding, let us briefly introduce some terminology:

Definition. A *conjugation* is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$, which is both *involutive* ($C^2 = I$) and *isometric*.

Definition. We say that a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is C*-symmetric* if $T = CT^*C$ and *complex symmetric* if there exists a conjugation C with respect to which T is C-symmetric.

The class of complex symmetric operators includes all normal operators, operators defined by Hankel matrices, compressed Toeplitz operators (including finite Toeplitz matrices and the compressed shift), and the Volterra integration operator. We refer the reader to [18, 19] (or [20] for a more expository pace) for further details. Other recent articles concerning complex symmetric operators include [11, 22, 30, 31].

It is not hard to see that T is a complex symmetric operator if and only if T is unitarily equivalent to a *complex symmetric matrix* (a matrix with complex entries which is self-transpose), regarded as an operator acting on an l^2-space of the appropriate dimension [20, Sect. 2.4]. The classical theory of such matrices is discussed in [16, Ch. XI] and [25, Sect. 4.4].

The main results of this note are as follows:

- The Aluthge transform of a complex symmetric operator is complex symmetric (Theorem 1).
- If T is a complex symmetric operator, then $(\widetilde{T})^*$ and $(\widetilde{T^*})$ are unitarily equivalent (Theorem 2). Example 1 shows that this statement is false without the assumption of complex symmetry.
- If T is a complex symmetric operator, then $\widetilde{T} = T$ if and only if T is normal (Theorem 3).
- $\widetilde{T} = 0$ if and only if $T^2 = 0$ (Theorem 4).
- Every operator which satisfies $T^2 = 0$ is necessarily a complex symmetric operator (Theorem 5).

2. The Aluthge Transform Preserves Complex Symmetry

Our first theorem states that the Aluthge transform of a complex symmetric operator is complex symmetric. To prove this, we require a few preliminary remarks concerning the polar decompositions of complex symmetric operators.

Recall that the polar decomposition $T = U|T|$ of an operator $T : \mathcal{H} \to \mathcal{H}$ expresses T uniquely as the product of a positive operator $|T| = \sqrt{T^*T}$ and a partial isometry U which satisfies $\ker U = \ker |T|$ and maps $\mathrm{cl}(\mathrm{ran}\,|T|)$ onto $\mathrm{cl}\,\mathrm{ran}\, T$.

If T is a C-symmetric operator, then the partial isometry U in the polar decomposition $T = U|T|$ factors as the product of C with a so-called *partial conjugation*. Specifically, we say that a conjugate-linear operator J is a partial conjugation if J restricts to a conjugation on $(\ker J)^\perp$ (having values in the same space). In particular, the *linear* operator J^2 is the orthogonal projection onto the closed subspace $\mathrm{ran}\, J = (\ker J)^\perp$. The following lemma is from [19, Thm. 2]:

Lemma 1. *If $T : \mathcal{H} \to \mathcal{H}$ is a bounded C-symmetric operator, then $T = CJ|T|$ where J is a partial conjugation, supported on $\mathrm{cl}(\mathrm{ran}\,|T|)$, which commutes with $|T| = \sqrt{T^*T}$.*

Remark. In fact, we may write $T = CJ|T|$ where J is a conjugation on all of $\mathcal{H}$. Indeed, we need only replace J by the internal orthogonal direct sum $J \oplus J'$ where J' is any partial conjugation supported on $\ker |T|$ (see [19, Cor. 1]). It is important to note that since $\widetilde{T}$ vanishes on $\ker |T|$, this assumption does not cause any complications when discussing the Aluthge transform of T.

Theorem 1. *The Aluthge transform of a complex symmetric operator is complex symmetric. In other words, if $T = CT^*C$ for some conjugation C, then there exists a conjugation J such that $\widetilde{T} = J(\widetilde{T})^*J$.*

Proof. By Lemma 1 and the preceding remark, we may write $T = CJ|T|$ where J is a conjugation on $\mathcal{H}$ which commutes with $|T|$. Since $\widetilde{T} = |T|^{\frac{1}{2}}CJ|T|^{\frac{1}{2}}$ and $(CJ)^* = JC$ [19, Lem. 1], it follows that

$$\begin{aligned} J(\widetilde{T})^*J &= J|T|^{\frac{1}{2}}JC|T|^{\frac{1}{2}}J \\ &= |T|^{\frac{1}{2}}CJ|T|^{\frac{1}{2}} \\ &= \widetilde{T}. \end{aligned}$$

□

We therefore see that the property of being unitarily equivalent to a complex symmetric matrix is preserved by the Aluthge transform. For instance, the Aluthge transform of a Hankel matrix is unitarily equivalent to a complex symmetric matrix.

Remark. In general, we do not expect that T and $\widetilde{T}$ will enjoy complex symmetric matrix representations with respect to the same basis. On the other hand, given orthonormal bases u_n and v_n for $\mathcal{H}$, it is easy to construct an operator T such that T and $\widetilde{T}$ will have complex symmetric matrix representations with respect to the bases u_n and v_n, respectively. Indeed, define conjugations C and J on $\mathcal{H}$ by setting $Cu_n = u_n$ and $Jv_n = v_n$ for all n and then extending by conjugate linearity to all of $\mathcal{H}$. If P is any positive operator which commutes with J, then $T = CJP$ is a C-symmetric operator whose Aluthge transform $\widetilde{T}$ is J-symmetric (by the proof of Theorem 1). In particular, the matrix representations of T and $\widetilde{T}$ with respect to the bases u_n and v_n, respectively, will be complex symmetric (see [20, Lem. 2.7] or [18, Sect. 3.2]).

In light of Theorem 1, it is natural to consider the following question:

Question. If T is complex symmetric and $0 < \epsilon < \frac{1}{2}$, is it necessarily the case that T_ϵ is also complex symmetric?

Here T_ϵ denotes the *generalized Aluthge transform of T of order ϵ*, which is defined to be $T_\epsilon = |T|^\epsilon U|T|^{1-\epsilon}$ for $0 < \epsilon \leq \frac{1}{2}$. This concept originated in the study of p-hyponormal operators [2] and has since been studied by many authors (see for instance [6, 14, 15, 27, 28, 29]).

Remark. A version of Theorem 1 holds for the so-called *Duggal transform* $\widehat{T} = |T|U$ of an operator $T = U|T|$ (see [13] for background). Specifically, if T is C-symmetric, then $\widehat{T}$ is JCJ-symmetric. Here J denotes the conjugation discussed in the remark following Lemma 1.

3. Adjoints and Aluthge Transforms

Our next theorem asserts that the restriction of the Aluthge transform to the class of complex symmetric operators respects the adjoint operation, modulo unitary equivalence:

Theorem 2. *If T is a complex symmetric operator, then*

$$(\widetilde{T})^* \cong \widetilde{(T^*)} \tag{1}$$

where $\cong$ denotes unitary equivalence.

Proof. If T is complex symmetric, then by the remark following Lemma 1, there exist conjugations C and J such that $T = CJ|T|$ and $J|T| = |T|J$. To prove the theorem, it suffices to establish that

$$\widetilde{T} = J(\widetilde{T})^* J, \tag{2}$$

$$\widetilde{T} = C\widetilde{(T^*)}C. \tag{3}$$

Indeed the equality of (2) and (3) will immediately imply (1) since CJ is unitary and $(CJ)^* = JC$ [19, Lem. 1]. As the proof of Theorem 1 yields (2), we need only prove (3).

Since T is C-symmetric, it follows that $C(TT^*)C = T^*T$ and hence

$$C(TT^*)^p C = (T^*T)^p$$

for all $p \geq 0$. In particular, we note that

$$T^* = CTC = C(CJ\sqrt{T^*T})C = J\sqrt{T^*T}C = JC\sqrt{TT^*}$$

whence

$$\begin{aligned} C(\widetilde{T^*})C &= C[(TT^*)^{\frac{1}{4}}JC(TT^*)^{\frac{1}{4}}]C \\ &= (T^*T)^{\frac{1}{4}}CJ(T^*T)^{\frac{1}{4}} \\ &= \widetilde{T}. \end{aligned}$$

This proves (3) and completes the proof. □

The following example demonstrates that (1) does not hold for all operators:

Example 1. Let S denote the unilateral shift and note that the polar decompositions of S and S^* are given by SI and $S^*(SS^*)$, respectively. Since

$$(\widetilde{S})^* = (ISI)^* = S^*$$

and

$$\begin{aligned} (\widetilde{S^*}) &= (SS^*)S^*(SS^*) \\ &= (SS^*)(S^*S)S^* \\ &= S(S^*)^2, \end{aligned}$$

we see that $(\widetilde{S})^*$ and $(\widetilde{S^*})$ are not unitarily equivalent. Indeed, simply note that $\dim\ker(\widetilde{S})^* = 1$ and $\dim\ker(\widetilde{S^*}) = 2$. In light of Theorem 2, this provides yet another proof that the unilateral shift is not a complex symmetric operator (see also [18, Prop. 1], [20, Ex. 2.14], or [17, Cor. 7]).

4. Complex Symmetric Fixed Points are Normal

Recall that an operator T is called *quasinormal* if T commutes with T^*T. Such operators were first considered in [7] and have since become a standard object of study. Although every normal operator is quasinormal, the converse is clearly false as the unilateral shift demonstrates. The relevance of quasinormal operators to the Aluthge transform lies in the fact that $\widetilde{T} = T$ if and only if T is quasinormal [10, Prop. 1.10].

With the additional hypothesis that T is a complex symmetric operator, we can prove that $\widetilde{T} = T$ if and only if T is normal (Theorem 3). This will be a consequence of the following lemma:

Lemma 2. *If T is a C-symmetric operator, then the following are equivalent:*

(i) *T is quasinormal,*

(ii) *C and $|T|$ commute (i.e., $|T|$ is also C-symmetric),*

(iii) *T is normal.*

Proof. Since (iii) $\Rightarrow$ (i), it suffices to prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If T is quasinormal, then the partially isometric factor U in the polar decomposition $T = U|T|$ commutes with $|T|$ [24, Pr. 137]. Writing $U = CJ$ where J commutes with $|T|$ (by Lemma 1), it follows that $C|T|J = CJ|T| = |T|CJ$ whence $C|T| = |T|C$. This establishes (i) $\Rightarrow$ (ii). On the other hand, if (ii) holds, then $TT^* = (CJ|T|)(|T|JC) = C|T|^2C = |T|^2 = T^*T$ whence T is normal. □

Theorem 3. *T is complex symmetric and $\widetilde{T} = T$ if and only if T is normal.*

Proof. Since $\widetilde{T} = T$ if and only if T is quasinormal [10, Prop. 1.10], this follows immediately from Lemma 2 and the fact that all normal operators are complex symmetric (see [20, Ex. 2.8] or [18, Sect. 4.1]). □

5. The Kernel of the Aluthge Transform

Our next theorem (Theorem 4) identifies the kernel of the Aluthge transform as the set of all operators which are nilpotent of order two. The connection between this result and complex symmetric operators lies in the fact (Theorem 5) that all operators which are nilpotent of order two are complex symmetric.

Theorem 4. *$\widetilde{T} = 0$ if and only if T is nilpotent of order two (i.e., $T^2 = 0$).*

Proof. ($\Rightarrow$) Let $T = U|T|$ denote the polar decomposition of T. If $\widetilde{T} = 0$, then

$$T^2 = U|T|U|T| = U|T|^{\frac{1}{2}}\widetilde{T}|T|^{\frac{1}{2}} = 0$$

so that T is nilpotent of order two.

($\Leftarrow$) If $T^2 = 0$, then $U|T|U|T| = 0$ whence $|T|U|T| = 0$ since U^*U is the orthogonal projection onto $\mathrm{cl}(\mathrm{ran}\,|T|)$. In particular, this implies that $|T|^{1/2}\widetilde{T}|T|^{1/2} = 0$. Since $\widetilde{T}$ vanishes on $\ker|T|$, it suffices to show that $\widetilde{T}$ also vanishes on $\mathrm{cl}(\mathrm{ran}\,|T|)$. Suppose toward a contradiction that $y \in \mathrm{ran}\,|T|$ but that $z = \widetilde{T}y \neq 0$. Writing $y = |T|^{\frac{1}{2}}x$ it follows that

$$0 = |T|^{\frac{1}{2}}\widetilde{T}|T|^{\frac{1}{2}}x = |T|^{\frac{1}{2}}\widetilde{T}y = |T|^{\frac{1}{2}}z \neq 0$$

since z is a nonzero vector in $\mathrm{ran}\,|T|$. This contradiction shows that $\widetilde{T}$ vanishes identically on $\mathrm{ran}\,|T|$ and hence on $\mathrm{cl}(\mathrm{ran}\,|T|)$ as well. Thus $\widetilde{T} = 0$. □

6. Nilpotence of Order Two

It turns out that any operator which is nilpotent of order two is necessarily a complex symmetric operator:

Theorem 5. *If T is nilpotent of order two, then T is complex symmetric.*

Proof. Let $T : \mathcal{H} \to \mathcal{H}$ be nilpotent of order two. Since $T(Tx) = T^2x = 0$ for every $x \in \mathcal{H}$, it follows that $\operatorname{cl}(\operatorname{ran} T) \subseteq \ker T = \ker |T|$. We therefore obtain the orthogonal decomposition

$$\mathcal{H} = \underbrace{\operatorname{cl}(\operatorname{ran} T) \oplus [\ker T \ominus \operatorname{cl}(\operatorname{ran} T)]}_{\ker T = \ker |T|} \oplus \operatorname{cl}(\operatorname{ran} |T|) \tag{4}$$

where the term $\ker T \ominus \operatorname{cl}(\operatorname{ran} T)$ is to be disregarded if it is trivial. With respect to the decomposition (4), we can represent T as a 3×3 operator matrix:

$$T = \begin{pmatrix} 0 & 0 & VP \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5}$$

where $P : \operatorname{cl}(\operatorname{ran} |T|) \to \operatorname{cl}(\operatorname{ran} |T|)$ is a positive operator and $V : \operatorname{cl}(\operatorname{ran} |T|) \to \operatorname{cl}(\operatorname{ran} T)$ is an isometry.

Since P is a selfadjoint operator, there exists a conjugation J on $\operatorname{cl}(\operatorname{ran} |T|)$ which commutes with P and hence with all of its spectral projections (see [18, Sect. 4.1], [20, Ex. 2.8], or [22]). If $K : \ker T \ominus \operatorname{cl}(\operatorname{ran} T) \to \ker T \ominus \operatorname{cl}(\operatorname{ran} T)$ is a conjugation, then we claim that the conjugate-linear operator

$$C = \begin{pmatrix} 0 & 0 & VJ \\ 0 & K & 0 \\ JV^* & 0 & 0 \end{pmatrix} \tag{6}$$

is a conjugation on all of $\mathcal{H}$.

To see that C is isometric, note that VJ is isometric on $\operatorname{cl}(\operatorname{ran} |T|)$, K is isometric on $\ker T \ominus \operatorname{cl}(\operatorname{ran} T)$, and JV^* is isometric on $\operatorname{cl}(\operatorname{ran} T)$. This implies that C is isometric on all of $\mathcal{H}$. To see that $C^2 = I$, we note that

$$\begin{aligned} V^*V &= I_{\operatorname{cl}(\operatorname{ran} |T|)} & \qquad VV^* &= I_{\operatorname{cl}(\operatorname{ran} T)} \\ K^2 &= I_{\ker T \ominus \operatorname{cl}(\operatorname{ran} T)} & J^2 &= I_{\operatorname{cl}(\operatorname{ran} |T|)} \end{aligned}$$

and compute

$$\begin{aligned} C^2 &= \begin{pmatrix} 0 & 0 & VJ \\ 0 & K & 0 \\ JV^* & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & VJ \\ 0 & K & 0 \\ JV^* & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} VV^* & 0 & 0 \\ 0 & K^2 & 0 \\ 0 & 0 & JV^*VJ \end{pmatrix} \\ &= I. \end{aligned}$$

Having proved that C is a conjugation on $\mathcal{H}$, we now show that T is C-symmetric:

$$\begin{aligned} CT^*C &= \begin{pmatrix} 0 & 0 & VJ \\ 0 & K & 0 \\ JV^* & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ PV^* & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & VJ \\ 0 & K & 0 \\ JV^* & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & VJ \\ 0 & K & 0 \\ JV^* & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & PJ \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & VJPJ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= T \end{aligned}$$

In particular, we used the fact that J and P commute. □

Remark. W. Wogen and the author have recently proved a generalization of Theorem 5. In particular, they show that every operator which is algebraic of degree two is complex symmetric [21].

Example 2. If $A : \mathcal{H} \to \mathcal{H}$ is *any* bounded linear operator, then $T : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ defined by

$$T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

is nilpotent of order two and thus a complex symmetric operator. In particular, any such operator T has a complex symmetric matrix representation with respect to some orthonormal basis of $\mathcal{H} \oplus \mathcal{H}$.

In the cases $\mathcal{H} = \mathbb{C}^2$ and $\mathcal{H} = \mathbb{C}^3$, we can directly verify that every operator which is nilpotent of order two is unitarily equivalent to a complex symmetric matrix:

Example 3. If T is a 2×2 matrix which satisfies $T^2 = 0$, then by Schur's Theorem on unitary upper-triangularization [25, Thm. 2.3.1] we may assume that T is of the form

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

Being a 2×2 Toeplitz matrix, the preceding will have a complex symmetric matrix representation with respect to any orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ of $\mathbb{C}^2$ whose elements are each held fixed by the conjugation $C(z_1, z_2) = (\overline{z_2}, \overline{z_1})$ (see [20, Sect. 2.2], [12, Sect. 4], or [18, Ex. 10]). Therefore every 2×2 nilpotent matrix (which is necessarily nilpotent of order ≤ 2) is unitarily equivalent to a complex symmetric matrix.

It is worth remarking here that *every* 2×2 matrix, nilpotent or not, is unitarily equivalent to a complex symmetric matrix [18, Ex. 6] (see also [11]).

Example 4. If T is a 3×3 matrix which satisfies $T^2 = 0$, then by Schur's Theorem we may assume that T is of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, since $T^2 = 0$ it follows that $bc = 0$ and thus we may presume that T is of the simpler form

$$\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

where $|a|^2 + |b|^2 = 1$. Following the construction in the proof of Theorem 5, we note that

$$\begin{aligned} \operatorname{ran} T &= \operatorname{span}\{(a, b, 0)\} \\ \ker T \ominus \operatorname{ran} T &= \operatorname{span}\{(-\overline{b}, \overline{a}, 0)\} \\ \operatorname{ran} |T| &= \operatorname{span}\{(0, 0, 1)\} \end{aligned}$$

whence we have the following unitary equivalence:

$$\begin{pmatrix} \overline{a} & \overline{b} & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & -\overline{b} & 0 \\ b & \overline{a} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

Since the matrix on the right hand side of (7) is a 3×3 Toeplitz matrix, it will have a complex symmetric matrix representation with respect to any orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{C}^3$ whose elements are each held fixed by the conjugation $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$. Therefore every 3×3 matrix which is nilpotent of order two is unitarily equivalent to a complex symmetric matrix.

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