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Complex symmetric partial isometries

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Abstract

An operator $T \in B(\mathcal{H})$ is complex symmetric if there exists a conjugate-linear, isometric involution $C : \mathcal{H} \to \mathcal{H}$ so that $T = CT^*C$. We provide a concrete description of all complex symmetric partial isometries. In particular, we prove that any partial isometry on a Hilbert space of dimension $\leqslant 4$ is complex symmetric.

Keywords: Complex symmetric operator; Isometry; Partial isometry

1. Introduction

The aim of this note is to complete the classification of complex symmetric partial isometries which was started in [9]. In particular, we give a concrete necessary and sufficient condition for a partial isometry to be a complex symmetric operator.

Before proceeding any further, let us first recall a few definitions. In the following, $\mathcal{H}$ denotes a separable, complex Hilbert space and $B(\mathcal{H})$ denotes the collection of all bounded linear operators on $\mathcal{H}$.

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Definition. A *conjugation* is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$, which is both *involutive* (i.e., $C^2 = I$) and *isometric* (i.e., $\langle Cx, Cy\rangle = \langle y, x\rangle$).

Definition. We say that $T \in B(\mathcal{H})$ is *C-symmetric* if $T = CT^*C$. We say that T is *complex symmetric* if there exists a conjugation C with respect to which T is C-symmetric.

It is not hard to see that T is a complex symmetric operator if and only if T is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^2-space of the appropriate dimension (see [4, Sect. 2.4] or [7, Prop. 2]).

One can also easily show that if $\dim\ker T \neq \dim\ker T^*$, then T is not a complex symmetric operator. For instance, the unilateral shift is perhaps the most ubiquitous example of a partial isometry which is not complex symmetric (see [7, Prop. 1], [4, Ex. 2.14], [5, Cor. 7]). On the other hand, we have [9, Thm. 4]:

Theorem 1. *Let $T \in B(\mathcal{H})$ be a partial isometry.*

(i) *If* $\dim\ker T = \dim\ker T^* = 1$, *then T is a complex symmetric operator.*
(ii) *If* $\dim\ker T \neq \dim\ker T^*$, *then T is not a complex symmetric operator.*
(iii) *If* $2 \leqslant \dim\ker T = \dim\ker T^* \leqslant \infty$, *then either possibility can* (*and does*) *occur.*

Although these results are the sharpest possible statements that can be made given only the data $(\dim\ker T, \dim\ker T^*)$, they are in some sense unsatisfactory. While it is known that there exist partial isometries in $B(\mathcal{H})$ that are not complex symmetric whenever $\dim\mathcal{H} \geqslant 5$, it turns out that every partial isometry in $B(\mathcal{H})$ is complex symmetric if $\dim\mathcal{H} \leqslant 3$. The authors were unable to settle the issue in the case $\dim\mathcal{H} = 4$. To be more specific, the techniques used in [9] were insufficient to discuss the case $\dim\mathcal{H} = 4$ and $\dim\ker T = 2$. Significant numerical evidence in favor of the assertion that all partial isometries on a four-dimensional Hilbert space are complex symmetric has recently been produced by J. Tener [11]. Let us now describe our results and the resolution of this problem.

Suppose that T is a partial isometry on $\mathcal{H}$ and let

$$\mathcal{H}_1 = (\ker T)^{\perp} = \operatorname{ran} T^* \tag{1}$$

denote the *initial space* of T and $\mathcal{H}_2 = (\mathcal{H}_1)^{\perp} = \ker T$ denote its orthogonal complement (see [10, Pr. 127] or [3, Ch. VIII, Sect. 3] for terminology). With respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we have

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \tag{2}$$

where $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_1 \to \mathcal{H}_2$. Furthermore, the fact that T^*T is the orthogonal projection onto $\mathcal{H}_1$ yields the identity

$$A^*A + B^*B = I,$$

where I denotes the identity operator on $\mathcal{H}_1$. Finally, observe that the operator $A \in B(\mathcal{H}_1)$ is simply the compression of the partial isometry T to its initial space.

The main result of this note is the following concrete description of complex symmetric partial isometries:

Theorem 2. *Let $T \in B(\mathcal{H})$ be a partial isometry. If A denotes the compression of T to its initial space, then T is a complex symmetric operator if and only if A is a complex symmetric operator.*

Due to its somewhat lengthy and computational proof, we defer the proof of the preceding theorem until Section 2. We remark that Theorem 2 remains true if one instead considers the final space of T. Indeed, simply apply the theorem with T^* in place of T and then take adjoints.

Corollary 1. *Every partial isometry of rank $\leqslant 2$ is complex symmetric.*

Proof. Let $T \in B(\mathcal{H})$ be a partial isometry such that $\operatorname{rank} T \leqslant 2$. If $\operatorname{rank} T = 0$, then $T = 0$ and there is nothing to prove. If $\operatorname{rank} T = 1$, then we may appeal to [9, Cor. 5], which asserts that every rank-one operator is complex symmetric. If $\operatorname{rank} T = 2$, then we may write

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

where A is an operator on a two-dimensional space. Since every operator on a two-dimensional Hilbert space is complex symmetric (see [1, Cor. 3], [2, Cor. 3.3], [7, Ex. 6], [9, Cor. 1], or [11, Cor. 3]), the desired conclusion follows immediately from Theorem 2. □

Corollary 2. *Every partial isometry on a Hilbert space of dimension $\leqslant 4$ is complex symmetric.*

Proof. As mentioned earlier, the results of [9] indicate that only the case $\dim \mathcal{H} = 4$ and $\dim \ker T = 2$ requires resolution. The corollary is now an immediate consequence of Theorem 2 and the fact that every operator on a two-dimensional Hilbert space is complex symmetric. □

We conclude this section with the following theorem, which asserts that each C-symmetric partial isometry can be extended to a C-symmetric unitary operator on the whole space (the significance lies in the fact that the corresponding conjugations for these two operators are the same).

Theorem 3. *If T is a C-symmetric partial isometry, then there exists a C-symmetric unitary operator U and an orthogonal projection P such that $T = UP$.*

Proof. Since T is a C-symmetric partial isometry, it follows that $|T| = P$ is an orthogonal projection and that $T = CJP$ where J is a partial conjugation supported on $\operatorname{ran} P$ which commutes with P [8, Sect. 2.2]. We may extend J to a conjugation $\widetilde{J}$ on all of $\mathcal{H}$ by letting $\widetilde{J} = J \oplus J'$ where J' is any conjugation on $\ker P$. The operator $U = C\widetilde{J}$ is the desired C-symmetric unitary operator. □

2. Proof of Theorem 2

This entire section is devoted to the proof of Theorem 2. We first require the following lemma:

Lemma 1. *If $\mathcal{H}, \mathcal{K}$ are separable complex Hilbert spaces, then $T \in B(\mathcal{H})$ is a complex symmetric operator if and only if $T \oplus 0 \in B(\mathcal{H} \oplus \mathcal{K})$ is a complex symmetric operator.*

Proof. If T is a C-symmetric operator on $\mathcal{H}$, then it is easily verified that $T \oplus 0$ is $(C \oplus J)$-symmetric on $\mathcal{H} \oplus \mathcal{K}$ for any conjugation J on $\mathcal{K}$. The other direction is slightly more difficult to prove.

Suppose that $S = T \oplus 0$ is a complex symmetric operator on $\mathcal{H} \oplus \mathcal{K}$. Before proceeding any further, let us remark that it suffices to consider the case where

$$\mathcal{H} = \overline{\operatorname{ran} T + \operatorname{ran} T^*}. \tag{3}$$

Otherwise let $\mathcal{H}_1 = \overline{\operatorname{ran} T + \operatorname{ran} T^*}$ and note that $\mathcal{H}_1$ is a reducing subspace of T. If $\mathcal{H}_2$ denotes the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}$, then with respect to the orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{K}$, the operator S has the form $T' \oplus 0 \oplus 0$, where T' denotes the restriction of T to $\mathcal{H}_1$. By now considering S with respect to the orthogonal decomposition $\mathcal{H} \oplus \mathcal{K} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{K})$, it follows that we need only consider the case where (3) holds.

Suppose now that (3) holds and that S is C-symmetric where C denotes a conjugation on $\mathcal{H} \oplus \mathcal{K}$. Writing the equations $CS = S^*C$ and $CS^* = SC$ in terms of the 2×2 block matrices

$$S = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{4}$$

(the entries C_{ij} of C are conjugate-linear operators), we find that

$$C_{11}T = T^*C_{11}, \tag{5}$$

$$C_{21}T = C_{21}T^* = 0, \tag{6}$$

$$T^*C_{12} = TC_{12} = 0. \tag{7}$$

Since $C_{21}T = C_{21}T^* = 0$, it follows that C_{21} vanishes on $\operatorname{ran} T + \operatorname{ran} T^*$ and hence on $\mathcal{H}$ itself by (3). On the other hand, (7) implies that C_{12} vanishes on the orthogonal complements of $\ker T$ and $\ker T^*$ in $\mathcal{H}$. By (3), this implies that C_{12} vanishes identically.

It follows immediately from (4) that C_{11} and C_{22} must be conjugations on $\mathcal{H}$ and $\mathcal{K}$, respectively, whence T is C_{11}-symmetric by (5). This concludes the proof of the lemma. □

Now let us suppose that T is a partial isometry on $\mathcal{H}$ and let

$$\mathcal{H}_1 = (\ker T)^{\perp} = \operatorname{ran} T^*$$

and $\mathcal{H}_2 = \ker T$. With respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, it follows that

$$T = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

where $A : \mathcal{H}_1 \to \mathcal{H}_1$, $B : \mathcal{H}_1 \to \mathcal{H}_2$, and

$$A^*A + B^*B = I. \tag{8}$$

(⇒) Suppose that T is a complex symmetric operator. For an operator with polar decomposition $T = U|T|$ (i.e., U is a partial isometry satisfying $\ker U = \ker T$ and $|T|$ denotes the positive operator $\sqrt{T^*T}$), the *Aluthge transform* of T is defined to be the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Noting that

$$T^*T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

we find that

$$\widetilde{T} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

By [6, Thm. 1], we know that the Aluthge transform of a complex symmetric operator is complex symmetric. Applying Lemma 1 to $\widetilde{T}$, we conclude that A is complex symmetric, as desired.

($\Leftarrow$) Let us now consider the more difficult implication of Theorem 2, namely that if A is a complex symmetric operator, then T is as well. We claim that it suffices to consider the case where $\overline{\operatorname{ran} B} = \mathcal{H}_2$. In other words, we argue that if

$$\mathcal{K} = \overline{\operatorname{ran} T + \operatorname{ran} T^*},$$

then we may suppose that $\mathcal{K} = \mathcal{H}$. Indeed, $\mathcal{K}$ is a reducing subspace for T and $T = 0$ on $\mathcal{K}^\perp$. By Lemma 1, if $T|_{\mathcal{K}}$ is a complex symmetric operator, then so is T.

Write $B = V|B|$ where $V : \mathcal{H}_1 \to \mathcal{H}_2$ is a partial isometry with initial space $(\ker B)^\perp \subseteq \mathcal{H}_1$ and final space $\mathcal{H}_2$ (since $\overline{\operatorname{ran} B} = \mathcal{H}_2$). In particular, we have the relations

$$V^*B = |B| = B^*V, \quad |B| = \sqrt{I - A^*A}. \tag{9}$$

By hypothesis, the operator $A \in B(\mathcal{H}_1)$ is complex symmetric. Therefore suppose that K is a conjugation on $\mathcal{H}_1$ such that $KA = A^*K$ and observe that the equations

$$\begin{aligned} A\sqrt{I - A^*A} &= \sqrt{I - AA^*}A, \\ A^*\sqrt{I - AA^*} &= \sqrt{I - A^*A}A^*, \\ K\sqrt{I - A^*A} &= \sqrt{I - AA^*}K, \\ K\sqrt{I - AA^*} &= \sqrt{I - A^*A}K, \end{aligned}$$

follow from a standard polynomial approximation argument (i.e., if $p(x) \in \mathbb{R}[x]$, then $Ap(A^*A) = p(AA^*)A$ and $Kp(A^*A) = p(AA^*)K$ hold, so that the desired identities follow upon passage to the norm limit). In particular, it follows from the preceding that

$$(KA)\sqrt{I - A^*A} = \sqrt{I - A^*A}(KA),$$

that is

$$KA|B| = |B|KA, \qquad A^*K|B| = |B|A^*K. \tag{10}$$

Let us now define a conjugate-linear operator C on $\mathcal{H}$ by the formula

$$C = \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix}. \tag{11}$$

Assuming for the moment that C is a conjugation on $\mathcal{H}$, we observe that

$$\underbrace{\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}}_{T} = \underbrace{\begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}}_{|T|}.$$

Since it is clear that J is a partial conjugation which is supported on the range of $|T|$ and which commutes with $|T|$, it follows immediately that T is a C-symmetric operator (see [8, Thm. 2]).

To complete the proof of Theorem 2, we must therefore show that C is a conjugation on $\mathcal{H}$. In other words, we must check that C^2 is the identity operator on $\mathcal{H}$ and that C is isometric. Since these computations are somewhat lengthy, we perform them separately:

Claim. $C^2 = I$.

Proof of Claim. We first expand out C^2 as a 2×2 block matrix:

$$\begin{aligned} C^2 &= \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \\ &= \begin{pmatrix} AKAK + KB^*BK & AKKB^* - KB^*VA^*KV^* \\ BKAK - VA^*KV^*BK & BKKB^* + VA^*KV^*VA^*KV^* \end{pmatrix} \\ &= \begin{pmatrix} AA^* + KB^*BK & AB^* - KB^*VA^*KV^* \\ BA^* - VA^*KV^*BK & BB^* + VA^*KV^*VA^*KV^* \end{pmatrix}. \end{aligned}$$

To obtain the preceding line, we used the fact that K is a conjugation and A is K-symmetric. Letting E_{ij} denote the entries of the preceding block matrix we find that

$$\begin{aligned} E_{11} &= AA^* + KB^*BK \\ &= AA^* + K(I - A^*A)K \\ &= AA^* + (I - AA^*) \\ &= I, \\ E_{12} &= AB^* - KB^*VA^*KV^* \\ &= AB^* - K|B|A^*KV^* \quad \text{by (9)} \\ &= AB^* - KA^*K|B|V^* \quad \text{by (10)} \\ &= AB^* - A|B|V^* \\ &= AB^* - AB^* \quad \text{since } B^* = |B|V \\ &= 0, \\ E_{21} &= BA^* - VA^*KV^*BK \\ &= BA^* - VA^*K|B|K \quad \text{since } V^*B = |B| \\ &= BA^* - V|B|A^*KK \quad \text{by (10)} \end{aligned}$$

$$
\begin{aligned}
&= BA^* - V|B|A^* \\
&= BA^* - BA^* \quad \text{since } B = V|B| \\
&= 0.
\end{aligned}
$$

As for E_{22}, it suffices to show that E_{22} agrees with I (the identity operator on $\mathcal{H}_2$) on the range of B, which is dense in $\mathcal{H}_2$. In other words, we wish to show that $E_{22}Bx = Bx$ for all $x \in \mathcal{H}_2$, which is equivalent to showing that

$$
E_{22}Bx = BB^*Bx + VA^*KV^*VA^*KV^*Bx = Bx \tag{12}
$$

for all $x \in \mathcal{H}_2$. Let us investigate the second term of (12):

$$
\begin{aligned}
VA^*KV^*VA^*KV^*Bx &= VA^*KV^*VA^*K|B|x \quad \text{by (9)} \\
&= VA^*KV^*V|B|A^*Kx \quad \text{by (10)} \\
&= VA^*K|B|A^*Kx \quad \text{since } V^*V = P_{\overline{\operatorname{ran}}\,|B|} \\
&= V|B|A^*KA^*Kx \quad \text{by (10)} \\
&= BA^*KA^*Kx \quad \text{since } B = V|B| \\
&= BA^*Ax \\
&= B(I - B^*B)x \quad \text{since } A^*A + B^*B = I \\
&= Bx - BB^*Bx.
\end{aligned}
$$

Putting this together with (12), we find that $E_{22}Bx = Bx$ for all $x \in \mathcal{H}_2$ whence $E_{22} = I$, as claimed. □

Claim. *C is isometric.*

Proof of Claim. The proof requires three steps:

(i) Show that C is isometric on $\mathcal{H}_1$.
(ii) Show that C is isometric on $B\mathcal{H}_1$, which is dense in $\mathcal{H}_2$.
(iii) Show that $C\mathcal{H}_1 \perp C(B\mathcal{H}_1)$.

For the first portion, observe that

$$
\begin{aligned}
\left\| C \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} AKx \\ BKx \end{pmatrix} \right\|^2 \\
&= \langle AKx, AKx \rangle + \langle BKx, BKx \rangle \\
&= \langle A^*AKx, Kx \rangle + \langle B^*BKx, Kx \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \big\langle (A^*A + B^*B)Kx, Kx \big\rangle \\
&= \langle Kx, Kx \rangle \\
&= \|Kx\|^2 \\
&= \|x\|^2.
\end{aligned}
$$

Thus (i) holds.

Now for (ii):

$$
\begin{aligned}
\left\| C \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ Bx \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} KB^*Bx \\ -VA^*KV^*Bx \end{pmatrix} \right\|^2 \\
&= \|KB^*Bx\|^2 + \|VA^*KV^*Bx\|^2 \\
&= \|B^*Bx\|^2 + \big\|VA^*K|B|x\big\|^2 \\
&= \|B^*Bx\|^2 + \big\|V|B|A^*Kx\big\|^2 \\
&= \|B^*Bx\|^2 + \|BA^*Kx\|^2 \\
&= \|B^*Bx\|^2 + \langle BA^*Kx, BA^*Kx \rangle \\
&= \|B^*Bx\|^2 + \langle B^*BA^*Kx, A^*Kx \rangle \\
&= \|B^*Bx\|^2 + \big\langle (I - A^*A)A^*Kx, A^*Kx \big\rangle \\
&= \|B^*Bx\|^2 + \big\langle A^*K(I - A^*A)x, A^*Kx \big\rangle \\
&= \|B^*Bx\|^2 + \big\langle K(I - A^*A)x, AA^*Kx \big\rangle \\
&= \langle B^*Bx, B^*Bx \rangle + \big\langle KAA^*Kx, (I - A^*A)x \big\rangle \\
&= \big\langle (I - A^*A)x, (I - A^*A)x \big\rangle + \big\langle A^*Ax, (I - A^*A)x \big\rangle \\
&= \big\langle x, (I - A^*A)x \big\rangle - \big\langle A^*Ax, (I - A^*A)x \big\rangle + \big\langle A^*Ax, (I - A^*A)x \big\rangle \\
&= \big\langle x, (I - A^*A)x \big\rangle \\
&= \langle x, B^*Bx \rangle \\
&= \langle Bx, Bx \rangle \\
&= \|Bx\|^2.
\end{aligned}
$$

Thus (ii) holds.

Now for (iii):

$$
\begin{aligned}
\left\langle C \begin{pmatrix} x \\ 0 \end{pmatrix}, C \begin{pmatrix} 0 \\ By \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} AK & KB^* \\ BK & -VA^*KV^* \end{pmatrix} \begin{pmatrix} 0 \\ By \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} AKx \\ BKx \end{pmatrix}, \begin{pmatrix} KB^*By \\ -VA^*KV^*By \end{pmatrix} \right\rangle
\end{aligned}
$$

$$\begin{aligned}
&= \langle AKx, KB^*By\rangle - \langle BKx, VA^*KV^*By\rangle \\
&= \langle B^*By, KAKx\rangle - \big\langle BKx, VA^*K|B|y\big\rangle \\
&= \langle B^*By, A^*x\rangle - \big\langle BKx, V|B|A^*Ky\big\rangle \\
&= \langle AB^*By, x\rangle - \langle BKx, BA^*Ky\rangle \\
&= \langle AB^*By, x\rangle - \langle B^*BKx, A^*Ky\rangle \\
&= \langle AB^*By, x\rangle - \big\langle (I-A^*A)Kx, A^*Ky\big\rangle \\
&= \langle AB^*By, x\rangle - \big\langle K(I-AA^*)x, A^*Ky\big\rangle \\
&= \langle AB^*By, x\rangle - \big\langle KA^*Ky, (I-AA^*)x\big\rangle \\
&= \langle AB^*By, x\rangle - \big\langle Ay, (I-AA^*)x\big\rangle \\
&= \langle AB^*By, x\rangle - \big\langle (I-AA^*)Ay, x\big\rangle \\
&= \langle AB^*By, x\rangle - \big\langle A(I-A^*A)y, x\big\rangle \\
&= \langle AB^*By, x\rangle - \langle AB^*By, x\rangle \\
&= 0.
\end{aligned}$$

By the polarization identity, it follows that

$$\left\langle C\begin{pmatrix} x_1 \\ Bx_2 \end{pmatrix}, C\begin{pmatrix} y_1 \\ By_2 \end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} x_2 \\ By_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ By_1 \end{pmatrix}\right\rangle$$

holds for all $x_1, x_2, y_1, y_2 \in \mathcal{H}_1$ whence C is isometric on $\mathcal{H}$. □

3. Partial isometries and the norm closure problem

Partial isometries on infinite-dimensional spaces often provide examples of note. For instance, one can give a simple example of a partial isometry T satisfying $\dim\ker T = \dim\ker T^* = \infty$ which is not a complex symmetric operator:

Example 1. Let S denote the unilateral shift on $l^2(\mathbb{N})$. Although S is certainly *not* a complex symmetric operator (by (ii) of Theorem 1; see also [4, Ex. 2.14], [7, Prop. 1], or [5, Cor. 7]), part (i) of Theorem 1 ensures that the partial isometry $S \oplus S^*$ *is* complex symmetric. Indeed, take N to be the bilateral shift on $l^2(\mathbb{Z})$, note that $S \oplus S^*$ is unitarily equivalent to $N - Ne_0 \otimes e_0$, and appeal to [9, Thm. 3]. That $S \oplus S^*$ is complex symmetric can also be verified by a direct computation [8, Ex. 5]. On the other hand, the partial isometry $T = S \oplus 0$ on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ is *not* a complex symmetric operator by Lemma 1.

Let $\mathcal{S}(\mathcal{H})$ denote the subset of $B(\mathcal{H})$ consisting of all bounded complex symmetric operators on $\mathcal{H}$. There are several ways to think about $\mathcal{S}(\mathcal{H})$. By definition, we have

$$\mathcal{S}(\mathcal{H}) = \big\{T \in B(\mathcal{H}):\ \exists \text{ a conjugation } C \text{ s.t. } T = CT^*C\big\}.$$

If C is a fixed conjugation on $\mathcal{H}$, then we also have

$$\mathcal{S}(\mathcal{H}) = \{UTU^*\colon\ T = CT^*C,\ U \text{ unitary}\}.$$

Thus if we identify $\mathcal{H}$ with $l^2(\mathbb{N})$ and C denotes the canonical conjugation on $l^2(\mathbb{N})$ (i.e., entry-by-entry complex conjugation), we can think of $\mathcal{S}(\mathcal{H})$ as being the *unitary orbit* of the set of all bounded (infinite) complex symmetric matrices.

The following example shows that the set $\mathcal{S}(\mathcal{H})$ is not closed in the strong operator topology (SOT):

Example 2. We maintain the notation of Example 1. For $n \in \mathbb{N}$, let P_n denote the orthogonal projection onto the span of the basis vectors $\{e_i\colon\ i \geqslant n\}$ of $l^2(\mathbb{N})$. Now observe that each operator $T_n = P_n S \oplus S^*$ is unitarily equivalent to $S \oplus 0_n \oplus S^*$ where 0_n denotes the zero operator on an n-dimensional Hilbert space. Each T_n is complex symmetric since $S \oplus S^*$ is complex symmetric (by Lemma 1). On the other hand, since $P_n S$ is SOT-convergent to 0, it follows that the SOT-limit of the sequence T_n is $0 \oplus S^*$, which is not a complex symmetric operator (by Lemma 1).

The preceding example demonstrates that the set of all complex symmetric operators (on a fixed, infinite-dimensional Hilbert space $\mathcal{H}$) is not SOT-closed. We also remark that the conjugations corresponding to the operators T_n from Example 2 depend on n. In contrast, if we fix a conjugation C, then it is elementary to see that the set of C-symmetric operators is a SOT-closed subspace of $B(\mathcal{H})$.

We conclude with a related question, which we have been unable to resolve:

Question. Is $\mathcal{S}(\mathcal{H})$ norm closed?

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