The Eigenstructure of Complex Symmetric Operators

Stephan Ramon Garcia

Abstract. We discuss several algebraic and analytic aspects of the eigenstructure (i.e. eigenvalues, eigenvectors, and generalized eigenvectors) of complex symmetric operators. In particular, we examine the relationship between the bilinear form $[x, y] = \langle x, Cy\rangle$ induced by a conjugation C on a complex Hilbert space $\mathcal{H}$ and the eigenstructure of a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ which is C-symmetric ($T = CT^*C$).

Mathematics Subject Classification (2000). 47A05, 47A07, 47A15.

Keywords. Complex symmetric operator, bilinear form, Toeplitz matrix, Hankel operator, Riesz idempotent, Riesz basis, generalized eigenvectors.

1. Introduction

In this note, we discuss several algebraic and analytic aspects of the eigenstructure (i.e. eigenvalues, eigenvectors, and generalized eigenvectors) of complex symmetric operators, a particular class of Hilbert space operators discussed in [3, 5, 6, 7]. Before proceeding, let us recall a few definitions.

A *conjugation* on a complex Hilbert space $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ that is involutive ($C^2 = I$) and *isometric*, meaning that $\langle x, y\rangle = \langle Cy, Cx\rangle$ for all x, y in $\mathcal{H}$ (we assume that $\mathcal{H}$ is separable and that our operators are bounded, unless otherwise stated). We say that a linear operator $T : \mathcal{H} \to \mathcal{H}$ is C-*symmetric* if $T = CT^*C$ and *complex symmetric* if it is C-symmetric with respect to some conjugation C. For a fixed conjugation C, there exists an orthonormal basis $(e_n)_{n=1}^{\dim \mathcal{H}}$ of $\mathcal{H}$ such that $Ce_n = e_n$ for all n [6, Lem. 1]. We refer to such a basis as a C-*real* orthonormal basis and note that the matrix representation of a C-symmetric operator with respect to such a basis is symmetric (see [6, Prop. 2] or [5, Sect. 2.4]). In particular, an operator T is complex symmetric if and only if it is unitarily

Work partially supported by National Science Foundation Grant DMS-0638789.

equivalent to a symmetric matrix with complex entries, considered as an operator on an l^2 space of the appropriate dimension.

The class of complex symmetric operators includes all normal operators, operators defined by (finite or infinite) Hankel matrices, compressed Toeplitz operators (including the compressed shift), and the Volterra integration operator (see [3, 5, 6, 7]). Since we are more concerned here with eigenvectors and generalized eigenvectors of operators rather than with the operators which produce them, we could certainly consider *unbounded* complex symmetric operators as well (see [6, 7, 10] for details and references).

When dealing with C-symmetric operators, it turns out that the *bilinear* form

$$[x, y] = \langle x, Cy \rangle \tag{1}$$

induced by C is almost as important as the standard sesquilinear form $\langle \cdot, \cdot \rangle$. We will say that two vectors x and y are *C-orthogonal* if $[x, y] = 0$ (denoted by $x \perp_C y$). We shall also say that two subspaces $\mathcal{E}_1$ and $\mathcal{E}_2$ are C-orthogonal (denoted $\mathcal{E}_1 \perp_C \mathcal{E}_2$) if $[x_1, x_2] = 0$ for every x_1 in $\mathcal{E}_1$ and x_2 in $\mathcal{E}_2$. It is not hard to see that the bilinear form (1) is nondegenerate, in the sense that $[x, y] = 0$ for all y in $\mathcal{H}$ if and only if $x = 0$. Unlike the sesquilinear form $\langle \cdot, \cdot \rangle$, however, the bilinear form $[\cdot, \cdot]$ is not positive since $[e^{i\theta/2}x, e^{i\theta/2}x] = e^{i\theta}[x, x]$ for any θ.

With respect to $[\cdot, \cdot]$, C-symmetric operators somewhat resemble selfadjoint operators. For instance, an operator T is C-symmetric if and only if $[Tx, y] = [x, Ty]$ for all x, y in $\mathcal{H}$. As another example, the eigenvectors of a C-symmetric operator corresponding to distinct eigenvalues are orthogonal with respect to $[\cdot, \cdot]$, even though they are not necessarily orthogonal with respect to the original sesquilinear form $\langle \cdot, \cdot \rangle$.

Lemma 1. *The eigenvectors of a C-symmetric operator T corresponding to distinct eigenvalues are orthogonal with respect to the bilinear form $[\cdot, \cdot]$.*

Proof. The proof is essentially identical to the corresponding proof for selfadjoint operators. If $\lambda_1 \neq \lambda_2$, $Tx_1 = \lambda_1 x_1$, and $Tx_2 = \lambda_2 x_2$, then

$$\lambda_1[x_1, x_2] = [\lambda_1 x_1, x_2] = [Tx_1, x_2] = [x_1, Tx_2] = [x_1, \lambda_2 x_2] = \lambda_2[x_1, x_2].$$

Since $\lambda_1 \neq \lambda_2$, it follows that $[x_1, x_2] = 0$. □

There are some obvious differences between selfadjoint and complex symmetric operators. For instance, a complex symmetric matrix can have any possible Jordan canonical form (see [5, 6] and the references therein) while a selfadjoint matrix must be unitarily diagonalizable. Nevertheless, we will see in Section 2 that the generalized eigenspaces of an arbitrary C-symmetric operator are always C-orthogonal.

Another somewhat superficial resemblance between complex symmetric and selfadjoint operators concerns the relationship between the kernel and range. If T is a C-symmetric operator, then the subspaces $\ker T$ and $\mathrm{cl}(\operatorname{ran} T)$ are C-orthogonal subspaces. Indeed, this follows immediately from the definition of C-symmetry and the fact that $\ker T = (\operatorname{ran} T^*)^\perp$. In this respect, C-symmetric operators resemble selfadjoint operators since the kernel and range of a selfadjoint operator are always orthogonal to each other. On the other hand, it turns out that $\ker T \cap \mathrm{cl}(\mathrm{ran}\, T)$ may be nontrivial for arbitrary complex symmetric operators. The vectors x in this intersection are *isotropic*, meaning that $[x, x] = 0$. The simplest example of this phenomenon occurs in two dimensions:

Example 1. If $T : \mathbb{C}^2 \to \mathbb{C}^2$ denotes the operator induced by a nilpotent 2×2 Jordan block, then T is C-symmetric with respect to $C(z_1, z_2) = (\overline{z_2}, \overline{z_1})$ (see [5, 6]). It is clear that $\ker T = \mathrm{ran}\, T = \mathrm{span}\{\binom{1}{0}\}$ and that $[\binom{1}{0}, \binom{1}{0}] = 0$.

As we will see in Section 4, isotropic eigenvectors of complex symmetric operators play an important role. To be specific, the existence of isotropic eigenvectors is directly related to the multiplicity of the corresponding eigenvalue.

Diagonalizable complex symmetric operators are naturally quite tractable objects of study. We discuss several aspects of the diagonalization of complex symmetric operators in Section 5. We conclude this note with a few basic remarks on Riesz bases of eigenvectors in Section 6.

Acknowledgments: The author would like to thank M. Putinar and D. Sarason for their helpful comments. Some of this work was conducted at the University of California, Santa Barbara.

2. Generalized eigenspaces

In this section, we show that the generalized eigenspaces of a complex symmetric operator (when they exist) are always mutually C-orthogonal. Thus although we do not necessarily have orthogonality with respect to the original hermitian form $\langle \cdot, \cdot \rangle$, we are able to separate the generalized eigenspaces via the bilinear form $[\cdot, \cdot]$. Our first proof is purely algebraic. A somewhat less general, but slicker and more sophisticated approach based on the Riesz functional calculus (which suffices for most cases of interest) is discussed later.

Theorem 1. *If T is a C-symmetric operator and $\lambda_1 \neq \lambda_2$, then*

$$\ker(T - \lambda_1 I)^{m_1} \perp_C \ker(T - \lambda_2 I)^{m_2}$$

for all $m_1, m_2 \geq 0$. In particular, generalized eigenspaces of a C-symmetric operator corresponding to distinct eigenvalues are mutually C-orthogonal.

Proof. Note that the case $m_1 = m_2 = 1$ is handled via Lemma 1. By subtracting a multiple of the identity from T, we may assume that $\lambda_1 = 0$ and that $\lambda_2 = \lambda$ is nonzero. Moreover, it suffices to show that any two subspaces of the form

$$\begin{aligned} \mathcal{E}_1 &= \mathrm{span}\{x, Tx, T^2x, \ldots, T^{m_1}x\} \\ \mathcal{E}_2 &= \mathrm{span}\{y, (T - \lambda I)y, (T - \lambda I)^2 y, \ldots, (T - \lambda I)^{m_2} y\}, \end{aligned}$$

where $T^{m_1}x \neq 0$, $(T - \lambda I)^{m_2} y \neq 0$, and $T^{m_1+1}x = (T - \lambda I)^{m_2+1} y = 0$, are mutually C-orthogonal.

Step 1: We first prove that the eigenvector $(T-\lambda I)^{m_2}y$ is C-orthogonal to $\mathcal{E}_1$ by showing that $[T^jx,(T-\lambda I)^{m_2}y]=0$ for $0\le j\le m_1$. Indeed, we have

$$\begin{aligned}\lambda^{m_1+1-j}[T^jx,(T-\lambda I)^{m_2}y] &= [T^jx,\lambda^{m_1+1-j}(T-\lambda I)^{m_2}y]\\ &= [T^jx,T^{m_1+1-j}(T-\lambda I)^{m_2}y]\\ &= [T^{m_1+1}x,(T-\lambda I)^{m_2}y]\\ &= 0.\end{aligned}$$

Since $\lambda\neq 0$, it follows that $[T^jx,(T-\lambda I)^{m_2}y]=0$ for all $0\le j\le m_1$ as claimed.

Step 2: Suppose now that we have proved $[T^jx,(T-\lambda I)^{m_2-k}g]=0$ for all $0\le j\le m_1$ and some $0\le k\le m_2-1$. Under this assumption we have:

$$\begin{aligned}\lambda[T^jx,(T-\lambda I)^{m_2-k-1}y] &= [T^jx,\lambda(T-\lambda I)^{m_2-k-1}y+(T-\lambda I)^{m_2-k}y]\\ &= [T^jx,(\lambda I+(T-\lambda I))(T-\lambda I)^{m_2-k-1}y]\\ &= [T^jx,T(T-\lambda I)^{m_2-k-1}y]\\ &= [T^{j+1}x,(T-\lambda I)^{m_2-k-1}y].\end{aligned}$$

Iteration ultimately yields

$$\lambda^{m_1+1-j}[T^jx,(T-\lambda I)^{m_2-k-1}y]=[T^{m_1+1}x,(T-\lambda I)^{m_2-k-1}y]=0$$

whence $[T^jx,(T-\lambda I)^{m_2-k-1}y]=0$ for all $0\le j\le m_1$. Thus the theorem is proved by induction. $\square$

A slight modification of this argument shows that generalized eigenspaces can be separated from the "quasinilpotent vectors" as well (see Section 3).

Example 2. If $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ are generalized eigenvectors which correspond to distinct eigenvalues of a complex symmetric matrix acting on $\mathbb{C}^n$, then $\sum_{i=1}^n x_iy_i=[\mathbf{x},\mathbf{y}]=0$.

Example 3. If $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ are generalized eigenvectors which correspond to distinct eigenvalues of an $n\times n$ Toeplitz matrix, then $[\mathbf{x},\mathbf{y}]=\sum_{i=1}^n x_iy_{n-i}=0$ (see [5, 6] for details).

One of the classic techniques of spectral theory for dealing with generalized eigenspaces is the use of contour integrals involving the resolvent. Recall that the resolvent set $\rho(T)$ of a bounded linear operator T is the set of complex numbers z for which the *resolvent* $R(z,T)=(zI-T)^{-1}$ exists as a bounded operator. The spectrum $\sigma(T)$ of T is simply the complement of $\rho(T)$ in $\mathbb{C}$ and the resolvent is an analytic operator valued function on the open set $\rho(T)$.

If T is a bounded linear operator and f is a holomorphic function on a (not necessarily connected) neighborhood Ω of $\sigma(T)$, then the Riesz functional calculus allows us to define an operator $f(T)$ via the Cauchy-type integral formula

$$f(T)=\frac{1}{2\pi i}\int_\Gamma f(z)R(z,T)\,dz \tag{2}$$

where Γ denotes a finite system of rectifiable Jordan curves, oriented in the positive sense, lying in Ω [4, p.568]. The integral in (2) is to be interpreted in the sense of Riemann and hence $f(T)$ is approximable by Riemann sums involving the operator $R(z,T)$.

For each *clopen* (relatively open and closed) subset Δ of $\sigma(T)$, there exists a natural idempotent $P(\Delta)$ defined by the formula

$$P(\Delta) = \frac{1}{2\pi i}\int_\Gamma R(z,T)\,dz \tag{3}$$

where Γ is any rectifiable Jordan curve such that Δ is contained in the interior $\operatorname{int}(\Gamma)$ of Γ and $\sigma(T)\backslash\Delta$ does not intersect $\operatorname{int}(\Gamma)$. We refer to this idempotent as the *Riesz idempotent* corresponding to Δ.

If the spectrum of an operator T decomposes as the disjoint union of two clopen sets, then the corresponding Riesz idempotents are usually neither self-adjoint (i.e. they are not necessarily orthogonal projections), nor are their ranges necessarily orthogonal to each other. Indeed, any diagonalizable but non-normal operator on $\mathbb{C}^2$ shows that this is not the case. Nevertheless, the Riesz idempotents that arise from complex symmetric operators have some nice features.

Theorem 2. *Let T be a C-symmetric operator. If $\sigma(T)$ decomposes as the disjoint union $\sigma(T) = \Delta_1 \cup \Delta_2$ of two clopen sets, then the corresponding Riesz idempotents $P_1 = P(\Delta_1)$ and $P_2 = P(\Delta_2)$ (defined by (3)) are*

(i) *C-symmetric: $P_i = CP_i^*C$ for $i = 1,2$,*

(ii) *C-orthogonal, in the sense that $\operatorname{ran} P_1 \perp_C \operatorname{ran} P_2$ (i.e. $P_1P_2 = P_2P_1 = 0$).*

Proof. For each z in $\rho(T)$, it is easy to see that the resolvent $R(z,T) = (zI - T)^{-1}$ of T is also C-symmetric. Indeed, it can be uniformly approximated by polynomials in T and such polynomials are clearly C-symmetric. Since the Riesz idempotents P_1 and P_2 corresponding to Δ_1 and Δ_2, respectively, are approximated by Riemann sums, it follows that P_1 and P_2 are C-symmetric. In particular, the Riesz idempotents P_1 and P_2 are C-symmetric and satisfy $P_1P_2 = P_2P_1 = 0$, whence their ranges are C-orthogonal. □

We will refer to a C-symmetric idempotent as a *C-projection*. In other words, a bounded linear operator P is a C-projection if and only if $P = CP^*C$ and $P^2 = P$. It is not hard to see that if P is a C-projection, then $\|P\| \geq 1$ and $\operatorname{ran} P$ is closed. Moreover, for any C-projection, we have $\ker P \cap \operatorname{ran} P = \{0\}$. This is not true for all C-symmetric operators, as Example 1 shows.

Example 4. If u is a nonisotropic vector, normalized so that $[u,u] = 1$, then $P_u x = [x,u]u$ is the C-projection onto $\operatorname{span}\{u\}$. On the other hand, if u is isotropic, then there can be no C-projection onto the subspace spanned by u. Indeed, such an operator would have to be of the form $Px = \langle x, v\rangle u$ for some v. If $P = CP^*C$, then it would follow that u and Cv are multiples of each other and hence $Pu = 0$, a contradiction.

If P is a C-projection, then $\ker P$ and $\operatorname{ran} P$ are disjoint C-orthogonal subspaces and hence $I = P + (I - P)$ gives a C-orthogonal decomposition of $\mathcal{H}$ (the C-orthogonality of $\ker P$ and $\operatorname{ran} P$ follows from Lemma 1).

A classical theorem of spectral theory [4, p.579] states that if T is a compact operator, then every nonzero point λ in $\sigma(T)$ is an eigenvalue of finite order $m = m(\lambda)$. For each such λ, the corresponding Riesz idempotent has a nonzero finite dimensional range given by $\operatorname{ran} P_\lambda = \ker(T - \lambda I)^m$. In particular, the nonzero elements of the spectrum of a compact operator correspond to generalized eigenspaces. Using Riesz idempotents, it is possible to give a much shorter proof of Theorem 1 if the complex symmetric operator T is assumed to be compact.

Theorem 3. *The generalized eigenspaces of a compact C-symmetric operator are C-orthogonal.*

Proof. It follows immediately from Theorem 2 and the preceding remarks that the generalized eigenspaces corresponding to nonzero eigenvalues of a compact C-symmetric operator T are mutually C-orthogonal. Since 0 is the only possible accumulation point of the eigenvalues of T, it follows that a generalized eigenvector corresponding to a nonzero eigenvalue is C-orthogonal to any vector in the range of

$$P_\epsilon = \frac{1}{2\pi i} \int_{|z|=\epsilon} R(z,T)\, dz$$

if $\epsilon > 0$ is taken sufficiently small. In particular, $\operatorname{ran} P_\epsilon$ contains the generalized eigenvectors for the eigenvalue 0 (if any exist). □

3. Quasinilpotent Vectors

Recall that a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is called *quasinilpotent* if

$$\lim_{n\to\infty} \| T^n \|^{\frac{1}{n}} = 0.$$

In particular, any nilpotent operator is quasinilpotent and the spectral radius formula implies that a bounded operator is quasinilpotent if and only if $\sigma(T) = 0$.

There are many examples of quasinilpotent complex symmetric operators. For instance, quasinilpotent Hankel operators can be constructed using certain symbols with lacunary Fourier series [9, Section 10.3, p.443-449]. A familiar example of a quasinilpotent complex symmetric operator is the Volterra integration operator [6, 7]. Moreover, the Fredholm alternative indicates that *any* Volterra operator is quasinilpotent [8, Pr. 187] and hence quasinilpotent complex symmetric operators are quite easy to produce.

We say that a vector q in $\mathcal{H}$ is a *quasinilpotent vector* (for T) if

$$\lim_{n\to\infty} \| T^n q \|^{\frac{1}{n}} = 0.$$

If T is compact, then it is not hard to show that the set $\mathcal{Q}$ of quasinilpotent vectors coincides with the orthogonal complement of the span of the generalized eigenspaces of T^*.

Theorem 4. *$\mathcal{Q} \perp_C \ker(T-\lambda I)^m$ for all $m \geq 0$ and $\lambda \neq 0$. In other words, every quasinilpotent vector is C-orthogonal to the generalized eigenspaces of T corresponding to nonzero eigenvalues.*

Proof. We proceed by induction on m. The case $m=0$ is trivial. Now suppose that we have shown that $\mathcal{Q} \perp_C \ker(T-\lambda I)^m$ for some m. Let q denote an arbitrary quasinilpotent vector for T and let x be a unit vector in $\ker(T-\lambda I)^{m+1}$. It follows that the vector $y=(T-\lambda I)x$ belongs to $\ker(T-\lambda I)^m$ and hence

$$\lambda[q,x] = [q,\lambda x] = [q,Tx] - [q,y] = [Tq,x]$$

by the inductive hypothesis. Iteration of the preceding yields $\lambda^n[q,x] = [T^n q, x]$ from which it follows that

$$|\lambda|^n |[q,x]| = |[q,\lambda^n x]| = |[q,T^n x]| = |[T^n q,x]| \leq \| T^n q \|$$

holds for every $n \geq 0$. Taking nth roots shows that $|\lambda| |[q,x]|^{\frac{1}{n}} \leq \| T^n q \|^{\frac{1}{n}}$, which tends to 0. Since $\lambda \neq 0$, it follows that $[q,x]=0$ and hence q is C-orthogonal to $\ker(T-\lambda I)$. □

4. Isotropic eigenvectors and multiplicity

We say that a vector x is *isotropic* if $[x,x]=0$. Although 0 is clearly an isotropic vector, it turns out that nonzero isotropic vectors are nearly unavoidable (see Lemma 2 below). However, isotropic eigenvectors are not mere algebraic inconveniences, for they often have meaningful interpretations. For example, isotropic eigenvectors of complex symmetric matrices are considered in [12] in the context of elastic wave propagation. In that theory, isotropic eigenvectors correspond to circularly polarized waves.

The following simple lemma implies that any subspace of dimension ≥ 2 contains isotropic vectors (see [2, Lem. 2]). In particular, this suggests the relationship between isotropy and multiplicity that we will explore in this section.

Lemma 2. *If C is a conjugation on a complex Hilbert space $\mathcal{H}$, then every subspace of dimension ≥ 2 contains isotropic vectors for the bilinear form $[x,y] = \langle x, Cy \rangle$.*

Proof. Let $\dim \mathcal{H} \geq 2$ and consider the span of two linearly independent vectors x_1 and x_2. If either x_1 or x_2 is isotropic, then we are done. If neither x_1 nor x_2 is isotropic, then we easily obtain C-orthogonal vectors y_1 and y_2 with the same span as x_1 and x_2:

$$y_1 = x_1, \quad y_2 = x_2 - \frac{[x_2,x_1]}{[x_1,x_1]} x_1.$$

In this case, either y_2 is isotropic (and hence we are done) or neither y_1 nor y_2 is isotropic. If this happens, then we may assume that y_1 and y_2 are normalized so that $[y_1,y_1]=[y_2,y_2]=1$. It is then easily verified that the vectors $y_1 \pm i y_2$ are both isotropic. □

The following lemma shows that the existence of an isotropic eigenvector for an isolated eigenvalue is completely determined by the multiplicity of the corresponding eigenvalue.

Theorem 5. *If T is a C-symmetric operator, then an isolated eigenvalue λ of T is simple if and only if T has no isotropic eigenvectors for λ.*

Proof. If λ is an isolated eigenvalue of T, then the Riesz idempotent P corresponding to λ is a C-projection. If λ is a simple eigenvalue, then the eigenspace corresponding to λ is spanned by a single unit vector x. If x is isotropic, then it is C-orthogonal to all of $\mathcal{H}$ since x is C-orthogonal to the range of the complementary C-projection $I - P$. This would imply that x is C-orthogonal to all of $\mathcal{H}$ and hence $x = 0$, a contradiction. Conversely, if λ is not a simple eigenvalue, then there are two cases to consider:

Case 1: If $\dim \ker(T-\lambda I) > 1$, then by Lemma 2, $\ker(T-\lambda I)$ contains an isotropic vector. Thus T has an isotropic eigenvector corresponding to the eigenvalue λ.

Case 2: If $\dim \ker(T - \lambda I) = 1$, then $\ker(T - \lambda I) = \operatorname{span}\{x\}$ for some $x \neq 0$ and $\dim \ker(T - \lambda I)^2 > 1$ since λ is not a simple eigenvalue. We can therefore find a nonzero generalized eigenvector y for λ such that $x = (T - \lambda I)y$. Thus

$$[x, x] = [x, (T - \lambda I)y] = [(T - \lambda I)x, y] = 0$$

and hence x is an isotropic eigenvector. □

We remark that the hypothesis that the eigenvalue λ is isolated is crucial. Indeed, $S \oplus S^*$ (where S is the unilateral shift on l^2) is complex symmetric (see [7]) and has each point in the open unit disk as a simple eigenvalue. The corresponding eigenvectors are all isotropic.

Example 5. If λ is an isolated eigenvalue of multiplicity ≥ 2 for a Hankel matrix (possibly infinite), then there exists an eigenvector $\mathbf{x}$ corresponding to λ so that $\sum_{i=1}^{\dim \mathcal{H}} x_i^2 = [\mathbf{x}, \mathbf{x}] = 0$. Here x_i denotes the ith entry of the vector $\mathbf{x}$.

Example 6. If λ is an eigenvalue of multiplicity ≥ 2 for an $n \times n$ Toeplitz matrix, then there exists an eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ corresponding to λ so that $\sum_{i=1}^{n} x_i x_{n-i} = [\mathbf{x}, \mathbf{x}] = 0$.

Example 7. Let H^2 denote the Hardy space of the open unit disk and let φ denote a nonconstant inner function. If λ is an eigenvalue of multiplicity ≥ 2 for the compression of a Toeplitz operator to $H^2 \ominus \varphi H^2$, then there exists an eigenfunction f in $H^2 \ominus \varphi H^2$ so that

$$\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f^2(z)}{\varphi(z)}\, dz = 0,$$

where $\partial \mathbb{D}$ denotes the unit circle (oriented in the counter-clockwise sense). See [3, 5, 6, 7] for further details and various special cases.

Having seen that isotropic eigenvectors are related to the multiplicity of eigenvalues, we can go a bit further into the decomposition of generalized eigenspaces:

Theorem 6. *If T is a C-symmetric operator, then for each complex number λ there exists an increasing sequence of subspaces $\mathcal{K}_n^{(\lambda)}$ such that*

$$\ker(T-\lambda I)^n = [\ker(T-\lambda I)^n \cap \operatorname{cl}(\operatorname{ran}(T-\lambda I)^n)] \oplus \mathcal{K}_n^{(\lambda)}$$

is both an orthogonal and C-orthogonal direct sum. Furthermore, the subspaces $\ker(T-\lambda I)^n \cap \operatorname{cl}(\operatorname{ran}(T-\lambda I)^n)$ consist entirely of isotropic vectors.

Proof. It suffices to consider the case $\lambda = 0$. For each n, there exists a subspace $\mathcal{K}_n = \mathcal{K}_n^{(0)}$ of $\ker T^n$ (possibly the zero subspace) such that

$$\ker T^n = [\ker T^n \cap \operatorname{cl}(\operatorname{ran} T^n)] \oplus \mathcal{K}_n, \tag{4}$$

where $\oplus$ denotes the usual orthogonal direct sum. Since $\ker T^n$ is C-orthogonal to $\operatorname{cl}(\operatorname{ran} T^n)$ and $\mathcal{K}_n \subseteq \ker T^n$, it follows that (4) is also a C-orthogonal decomposition. In particular, every vector in $\ker T^n \cap \operatorname{cl}(\operatorname{ran} T^n)$ is isotropic. That the sequence $\mathcal{K}_n$ is increasing is clear from the fact that $\ker T^n \subseteq \ker T^{n+1}$ and $\operatorname{ran} T^{n+1} \subseteq \operatorname{ran} T^n$. $\square$

Example 8. Consider the operator $T : \mathbb{C}^5 \to \mathbb{C}^5$ induced by a 5×5 nilpotent Jordan block and let $\{e_1, e_2, e_3, e_4, e_5\}$ denote the standard orthonormal basis for $\mathbb{C}^5$. The relevant subspaces $\mathcal{K}_n^{(0)}$ of the preceding theorem are readily exhibited:

n	$\ker T^n$	$\operatorname{ran} T^n$	$\ker T^n \cap \operatorname{ran} T^n$	$\mathcal{K}_n^{(0)}$
0	$\{0\}$	$\mathbb{C}^5$	$\{0\}$	$\mathbb{C}^5$
1	$\operatorname{span}\{e_1\}$	$\operatorname{span}\{e_1, e_2, e_3, e_4\}$	$\operatorname{span}\{e_1\}$	$\{0\}$
2	$\operatorname{span}\{e_1, e_2\}$	$\operatorname{span}\{e_1, e_2, e_3\}$	$\operatorname{span}\{e_1, e_2\}$	$\{0\}$
3	$\operatorname{span}\{e_1, e_2, e_3\}$	$\operatorname{span}\{e_1, e_2\}$	$\operatorname{span}\{e_1, e_2\}$	$\{e_3\}$
4	$\operatorname{span}\{e_1, e_2, e_3, e_4\}$	$\operatorname{span}\{e_1\}$	$\operatorname{span}\{e_1\}$	$\{e_2, e_3, e_4\}$
5	$\mathbb{C}^5$	$\{0\}$	$\{0\}$	$\mathbb{C}^5$

5. Diagonalization of Complex Symmetric Operators

Suppose that T is a complex symmetric operator which has a complete system of (nonzero) eigenvectors $(u_n)_{n=1}^\infty$. By complete, we mean that the closed linear span of the u_n is all of $\mathcal{H}$. Implicitly, we will assume that $\dim \mathcal{H} = \infty$ since the finite dimensional case is somewhat trivial in comparison.

If the corresponding eigenvalues λ_n are distinct, then Lemma 1 tells us that the $(u_n)_{n=1}^\infty$ are mutually C-orthogonal. We may therefore assume that the system $(u_n)_{n=1}^\infty$ is *C-orthonormal*: $[u_j, u_k] = \delta_{jk}$, where δ_{jk} denotes the Kronecker δ-function. Indeed, if $[u_n, u_n] = 0$ for some n, then $[u_n, x] = 0$ would hold for all x since the system $(u_n)_{n=1}^\infty$ is complete, whence $u_n = 0$.

We consider here the linear extension of the map $u_n \mapsto Cu_n$. Since the u_n are not necessarily orthonormal with respect to the usual hermitian inner product $\langle \cdot, \cdot \rangle$, this map does not immediately extend (as a bounded linear operator) further than the dense linear submanifold spanned by finite linear combinations of the u_n. To be specific, we say that a vector f in $\mathcal{H}$ is *finitely supported* if it is a finite linear combination of the u_n and we denote the linear manifold of finitely supported vectors by $\mathcal{F}$. Due to the C-orthonormality of the u_n, it follows immediately that each such f can be recovered via the *skew Fourier expansion*

$$f = \sum_{n=1}^{\infty} [f, u_n] u_n, \tag{5}$$

where all but finitely many of the *skew Fourier coefficients* $[f, u_n]$ are nonzero. We will let $A_0 : \mathcal{F} \to \mathcal{H}$ denote the linear extension of the map $A_0 u_n = C u_n$ to $\mathcal{F}$. Since $\mathcal{F}$ is a dense linear submanifold of $\mathcal{H}$, it follows that if $A_0 : \mathcal{F} \to \mathcal{H}$ is bounded on $\mathcal{F}$, then A_0 has a unique bounded extension (which we denote by A) to all of $\mathcal{H}$.

It turns out that the presence of the conjugation C ensures that the extension A will have several desirable algebraic properties. In particular, the following lemma shows that if A is bounded, then it is C-orthogonal. Specifically, we say that an operator $U : \mathcal{H} \to \mathcal{H}$ is *C-orthogonal* if $CU^*CU = I$. The terminology comes from the fact that, when represented with respect to a C-real orthonormal basis, the corresponding matrix will be complex orthogonal (i.e. $U^t U = I$ as matrices).

The importance of C-orthogonal operators lies in the fact that they preserve the bilinear form induced by C. To be specific, U is a C-orthogonal operator if and only if $[Ux, Uy] = [x, y]$ for all x, y in $\mathcal{H}$. Unlike unitary operators, C-orthogonal operators can have arbitrarily large norms. In fact, unbounded C-orthogonal operators are considered in [11], where they are called *J-unitary* operators.

Lemma 3. *If A_0 is bounded, then its extension $A : \mathcal{H} \to \mathcal{H}$ is positive and C-orthogonal. If this is the case, then A is invertible with $A^{-1} = CAC \geq 0$ and the operator $B = \sqrt{A}$ is also C-orthogonal.*

Proof. By (5), it follows that $\langle A_0 f, f \rangle = \sum_{n=1}^{\infty} |[f, u_n]|^2 \geq 0$ for all f in $\mathcal{F}$. If A_0 is bounded, then it follows by continuity that A will be positive. The fact that A is C-orthogonal (hence invertible) follows from the fact that $(CA^*C)Au_n = (CA)^2 u_n = u_n$ for all n. Since $(CBC)(CBC) = CAC = A^{-1}$ and $CBC \geq 0$, it follows that CBC is a positive square root of A^{-1}. By the uniqueness of the positive square root of a positive operator, we see that $CBC = B^{-1}$ and hence B is also C-orthogonal. □

We remark that Lemma 3 shows that if the map $u_n \mapsto Cu_n$ is bounded, then its linear extension $A : \mathcal{H} \to \mathcal{H}$ is necessarily invertible. This property distinguishes C-orthonormal systems $(u_n)_{n=1}^{\infty}$ and their duals $(Cu_n)_{n=1}^{\infty}$ from general biorthogonal systems (which do not necessarily arise from conjugations on $\mathcal{H}$). Among other things, Lemma 3 also shows that if A_0 is bounded, then the *skew conjugation* $J\left(\sum_{n=1}^{\infty} c_n u_n\right) = \sum_{n=1}^{\infty} \overline{c_n} u_n$ (defined initially on $\mathcal{F}$) is given by

$$J = CA = CBB = B^{-1}CB.$$

In other words, the skew conjugation J is similar to our original conjugation C via the operator $B = \sqrt{A}$. Another consequence of the boundedness of A_0 is the existence of a natural orthonormal basis for $\mathcal{H}$:

Lemma 4. *If A_0 is bounded, then the vectors $(s_n)_{n=1}^\infty$ defined by $s_n = Bu_n$ (where $B = \sqrt{A}$) satisfy the following:*

(i) *$(s_n)_{n=1}^\infty$ is orthonormal: $\langle s_j, s_k\rangle = \delta_{jk}$ for all j, k,*

(ii) *$(s_n)_{n=1}^\infty$ is C-orthonormal: $[s_j, s_k] = \delta_{jk}$ for all j, k,*

(iii) *$Cs_n = s_n$ for all n.*

Furthermore, $(s_n)_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}$.

Proof. Conditions (i), (ii), and (iii) follow from direct computations:

$$\begin{aligned}\langle s_j, s_k\rangle &= \langle Bu_j, Bu_k\rangle = \langle u_j, Au_k\rangle = \langle u_j, Cu_k\rangle = [u_j, u_k] = \delta_{jk},\\ [s_j, s_k] &= \langle s_j, Cs_k\rangle = \langle Bu_j, CBu_k\rangle = \langle Bu_j, B^{-1}Cu_k\rangle = \langle u_j, Cu_k\rangle = \delta_{jk},\\ Cs_j &= CBu_j = B^{-1}Cu_j = B^{-1}B^2u_j = Bu_j = s_j.\end{aligned}$$

We now show that the system $(s_n)_{n=1}^\infty$ is complete. If f is orthogonal to each s_j, then $\langle Bf, u_j\rangle = \langle f, Bu_j\rangle = \langle f, s_j\rangle = 0$ for all j. Since B is invertible, it follows that $f = 0$ since $(u_n)_{n=1}^\infty$ is complete. □

If the operator A_0 is bounded, then its extension A is a positive, invertible operator whose spectrum is bounded away from zero. Thus $\Theta = -i \log A$ can be defined using the functional calculus for A and the principal branch of the logarithm. Since A is self-adjoint and the principal branch of the logarithm is real on $(0, \infty)$, it follows that Θ is skew-Hermitian: $\Theta^* = -\Theta$. Moreover, since A is a C-orthogonal operator, it follows that Θ is a *C-real* operator: $\overline{\Theta} = \Theta$, where $\overline{\Theta} = C\Theta C$.

Returning to our original C-symmetric operator T, we see that if A_0 is bounded, then T is similar to the diagonal operator $D : \mathcal{H} \to \mathcal{H}$ defined by $Ds_n = \lambda_n s_n$ since $T = B^{-1}DB$. Writing this in terms of the exponential representation $A = \exp(i\Theta)$ and inserting a parameter $t \in [0, 1]$, we obtain a family of operators

$$T_t = e^{-\frac{it}{2}\Theta} D e^{\frac{it}{2}\Theta}$$

which satisfies $T_0 = D$ and $T_1 = T$. This provides a continuous deformation of T to its diagonal model D. We also remark that the fact that Θ is C-real and skew-Hermitian implies that the operators $\exp(\pm\frac{it}{2}\Theta)$ are C-orthogonal for all t. From here, it is easy to show that each intermediate operator T_t is C-symmetric and that the path $t \mapsto T_t$ from $[0, 1]$ to $B(\mathcal{H})$ is norm continuous.

In particular, this framework applies to complex symmetric matrices (e.g. Hankel matrices) or to finite Toeplitz matrices. Moreover, the compressed shift corresponding to an interpolating Blaschke product produces exactly such a system $(u_n)_{n=1}^\infty$ (consisting of certain scalar multiples of reproducing kernels, see [6] for details and references). The boundedness of A_0 is guaranteed by Carleson's interpolation theorem. It would be interesting to concretely identify the operators A, B, and Θ in such an example.

6. Riesz Bases of Eigenvectors

Recall that an arbitrary sequence of vectors $(u_n)_{n=1}^\infty$ is called a *Bessel sequence* if there exists a constant $M > 0$ (called a *Bessel bound*) such that

$$\sum_{n=1}^\infty |\langle x, u_n\rangle|^2 \leq M \| x \|^2$$

for all x in $\mathcal{H}$. Also recall that a sequence $(u_n)_{n=1}^\infty$ is called a *Riesz basis* if it is the image of an orthonormal basis of $\mathcal{H}$ under a bounded, invertible linear operator R. It is well-known (see [1, Prop. 3.6.4]) that $(u_n)_{n=1}^\infty$ is a Riesz basis if and only if there exist there exist constants $M_1, M_2 > 0$ such that

$$M_1^2 \| x \|^2 \leq \sum_{n=1}^\infty |\langle x, u_n\rangle|^2 \leq M_2^2 \| x \|^2.$$

Furthermore, the optimal constants are $M_1 = \| R^{-1} \|^{-1}$ and $M_2 = \| R \|^2$.

Our final theorem consists of a number of equivalent statements concerning C-orthonormal systems. Our interest in such systems stems from the fact that they often arise as eigenvectors of C-symmetric operators.

Theorem 7. *If $(u_n)_{n=1}^\infty$ is a complete C-orthonormal system in $\mathcal{H}$, then the following are equivalent:*

(i) *$(u_n)_{n=1}^\infty$ is a Bessel sequence with Bessel bound M,*

(ii) *$(u_n)_{n=1}^\infty$ is a Riesz basis with lower and upper bounds M^{-1} and M,*

(iii) *The assignment $A_0 u_n = C u_n$ extends to a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ satisfying $\| A \| \leq M$,*

(iv) *There exists $M > 0$ satisfying:*

$$\| \sum_{n=1}^m \overline{c_n} u_n \| \leq M \| \sum_{n=1}^m c_n u_n \|, \tag{6}$$

for every finite sequence $c_1, c_2, \ldots, c_m$.

(v) *The Gram matrix $(\langle u_j, u_k\rangle)_{j,k=1}^\infty$, acting on finitely supported sequences, dominates its transpose:*

$$\big(M^2 \langle u_j, u_k \rangle - \langle u_k, u_j \rangle \big)_{j,k=1}^\infty \geq 0 \tag{7}$$

for some $M > 0$.

(vi) *The Gram matrix $G = (\langle u_j, u_k\rangle)_{j,k=1}^\infty$ is bounded on $l^2(\mathbb{N})$. Furthermore, $\| G \| \leq M$ and G is orthogonal ($G^t G = I$ as matrices).*

(vii) *For each f in $\mathcal{H}$, the skew Fourier expansion $\sum_{n=1}^\infty [f, u_n] u_n$ converges to f in norm and*

$$\frac{1}{M} \| f \|^2 \leq \sum_{n=1}^\infty |[f, u_n]|^2 \leq M \| f \|^2. \tag{8}$$

In all cases, the infimum over all such M equals the norm of A_0.

Proof. The proof consists of a number of parts. We first prove the implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) and then establish the equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v), (iii) $\Leftrightarrow$ (vi), and (ii) $\Leftrightarrow$ (vii).

(i) $\Rightarrow$ (iii) If $(u_n)_{n=1}^{\infty}$ is a Bessel sequence with Bessel bound M, then

$$\sum_{n=1}^{\infty} |[f, u_n]|^2 = \sum_{n=1}^{\infty} |\langle Cf, u_n\rangle|^2 \leq M\| Cf \|^2 = M\| f \|^2$$

holds for all f. It follows that the coordinate map $Lf = ([f, u_n])_{n=1}^{\infty}$ is a bounded linear operator from $\mathcal{H}$ into $l^2(\mathbb{N})$ whose norm satisfies $\| L \| \leq \sqrt{M}$. Since $[f, u_n] = \langle f, Cu_n\rangle$, it is not hard to see that $L^*Lu_n = Cu_n = A_0 u_n$ and thus L^*L agrees with A_0 on the dense submanifold $\mathcal{F}$. This implies that A_0 extends to a bounded linear operator A satisfying $\| A \| = \| L^*L \| \leq M$.

(iii) $\Rightarrow$ (ii) $(u_n)_{n=1}^{\infty}$ is the image of the orthonormal basis $(s_n)_{n=1}^{\infty}$ under the bounded, bijective operator B^{-1} (see the preceding section for terminology). The bounds follow from [1, Prop. 3.6.4].

(ii) $\Rightarrow$ (i) This follows from the well-known fact that a Riesz basis is always a Bessel sequence (See [1, Prop. 3.6.4]).

(iii) $\Leftrightarrow$ (iv) This follows directly from the fact that the *antilinear* operator $J = CA_0$ fixes each u_n. Since $A_0 = CJ$ on $\mathcal{F}$ and C is isometric, the desired result follows.

(iv) $\Leftrightarrow$ (v) Upon squaring both sides of (6) and simplifying, one sees that (6) holds if and only if a (7) holds (with the same M).

(iii) $\Leftrightarrow$ (vi) If A_0 is bounded, then A_0 extends to a bounded, invertible operator $A : \mathcal{H} \to \mathcal{H}$. Indeed, $A^{-1} = CAC$ since these two operators agree on the complete system $(Cu_n)_{n=1}^{\infty}$. The entries in the Gram matrix G are given by $\langle u_j, u_k\rangle = \langle B^{-1}s_j, B^{-1}s_k\rangle = \langle A^{-1}s_j, s_k\rangle$. Hence G is simply the matrix representation for the bounded operator A^{-1} with respect to the C-real basis $(s_n)_{n=1}^{\infty}$. In particular, this implies that G is bounded as an operator on $l^2(\mathbb{N})$. Since $\langle As_j, s_k\rangle = \langle Bs_j, Bs_k\rangle = \langle Cu_j, Cu_k\rangle = \langle u_k, u_j\rangle$ for all j, k, it follows that G^t is simply the matrix representation for A and hence $G^tG = I$. Conversely, if G is bounded, then a straightforward computation shows that A_0 is bounded.

(ii) $\Leftrightarrow$ (vii) Condition (ii) holds if and only if

$$\frac{1}{M}\| f \|^2 \leq \sum_{n=1}^{\infty} |\langle f, u_n\rangle|^2 \leq M\| f \|^2$$

for every f in $\mathcal{H}$. Upon substituting Cf for f and noting that $[f, u_n] = \langle f, Cu_n\rangle = \langle Cf, u_n\rangle$, these inequalities assume the form required by (vii). Once (8) is established, it is clear that each f in $\mathcal{H}$ can be represented as a norm-convergent skew Fourier series. □

We conclude this article with a simple, but illustrative, example:

Example 9. Let $w = \alpha + i\beta$ where α and β are real constants and consider $L^2[0,1]$, endowed with the conjugation $[Cf](x) = \overline{f(1-x)}$. A short computation shows that if w is not an integer multiple of 2π, then the vectors

$$u_n(x) = \exp[i(w + 2\pi n)(x - \tfrac{1}{2})], \quad n \in \mathbb{Z},$$

are eigenfunctions of the C-symmetric operator

$$[Tf](x) = e^{iw/2} \int_0^x f(y)\,dy + e^{-iw/2} \int_x^1 f(y)\,dy$$

(i.e. $T = e^{iw/2}V + e^{-iw/2}V^*$ where V denotes the Volterra operator) and that the system $(u_n)_{n=1}^{\infty}$ is complete and C-orthonormal. One the other hand, one might also say that the u_n are eigenfunctions of the derivative operator with boundary condition $f(1) = e^{iw} f(0)$.

We also see that the map $u_n \mapsto Cu_n$ extends to a bounded operator on all of $L^2[0,1]$. Indeed, this extension is simply the multiplication operator $[Af](x) = e^{2\beta(x-1/2)} f(x)$ whence $B = \sqrt{A}$ is given by

$$[Bf](x) = e^{\beta(x-1/2)} f(x).$$

As expected, the positive operators A and B are both C-orthogonal and the system $(u_n)_{n=1}^{\infty}$ forms a Riesz basis for $L^2[0,1]$. In fact, $(u_n)_{n=1}^{\infty}$ is the image of the C-real orthonormal basis $(s_n)_{n=1}^{\infty}$, defined by $s_n = Bu_n$, under the bounded and invertible operator B^{-1}. The s_n are given by

$$s_n(x) = \exp[i(\alpha + 2\pi n)(x - \tfrac{1}{2})]$$

and they are easily seen to be both orthonormal and C-real (see [5, Lem. 4.3]). Such bases and their relationship to the C-symmetric properties of the Volterra operator and the "compressed shift" corresponding to the atomic inner function $\varphi(z) = \exp[(z+1)/(z-1)]$ are discussed in [5].

References

[1] Christensen, O., *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.

[2] Craven, B.D., *Complex symmetric matrices*, J. Austral. Math. Soc. **10** (1969), 341–354.

[3] Danciger, J., Garcia, S.R., Putinar, M., *Variational principles for symmetric bilinear forms*, Math. Nachr., to appear.

[4] Dunford, N., Schwartz, J.T., *Linear operators, Part I: General Theory*, Wiley, New York, 1988.

[5] Garcia, S.R., *Conjugation and Clark operators*, Contemp. Math. **393** (2006), 67-112.

[6] Garcia, S.R., Putinar, M., *Complex symmetric operators and applications*, Trans. Amer. Math. Soc., **358** (2006), 1285-1315.

[7] Garcia, S.R., Putinar, M., *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc., **359** (2007), 3913-3931.

[8] Halmos, P.R., *A Hilbert Space Problem Book* (Second Edition), Springer-Verlag, 1982.

[9] Peller, V.V., *Hankel Operators and Their Applications*, Springer Monographs in Mathematics, Springer-Verlag, 2003.

[10] Prodan, E., Garcia, S.R., Putinar, M., *Norm estimates of complex symmetric operators applied to quantum systems*, J. Phys. A: Math. Gen. **39** (2006), 389–400.

[11] Riss, U.V., *Extension of the Hilbert space by J-unitary transformations*, Helv. Phys. Acta **71** (1998), 288-313.

[12] Scott, N.H., *A theorem on isotropic null vectors and its application to thermoelasticity*, Proc. Roy. Soc. London Ser. A **440** (1993), no. 1909, 431–442.

Stephan Ramon Garcia
Department of Mathematics
Pomona College
610 North College Avenue
Claremont, CA 91711
e-mail: `Stephan.Garcia@pomona.edu`