Compl. anal. oper. theory *Online First*

DOI 10.1007/s11785-008-0084-8

Complex Analysis and Operator Theory

Hermitian-symmetric Inequalities in Hilbert Space

Stephan Ramon Garcia

Abstract. We establish several conditions which are equivalent to

$$|[Bx, x]| \leq \langle Ax, x \rangle, \quad \forall x \in \mathcal{H},$$

where A is a nonnegative operator and B is a complex symmetric operator on a separable complex Hilbert space $\mathcal{H}$. Along the way, we also prove a new factorization theorem for complex symmetric operators.

Mathematics Subject Classification (2000). 47B99.

Keywords. Complex symmetric operator, complex symmetric matrix, hermitian-symmetric inequalities, hermitian operators, selfadjoint operators, Grunsky inequality, bilinear form.

1. Introduction

This note is concerned with the structure of certain inequalities satisfied by pairs of Hilbert space operators (Theorem 1). Along the way, we also prove a new factorization theorem for complex symmetric operators (Theorem 2).

Throughout this note, $\mathcal{H}$ will denote a separable complex Hilbert space and all operators considered will be bounded. Before describing our results, we first require a few preliminary definitions:

Definition. A *conjugation* is a conjugate-linear operator $C : \mathcal{H} \to \mathcal{H}$, which is both *involutive* ($C^2 = I$) and *isometric* (i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$).

Definition. A bounded linear operator $T \in B(\mathcal{H})$ is *C-symmetric* if $T = CT^*C$ and *complex symmetric* if there exists a conjugation C with respect to which T is C-symmetric.

The class of complex symmetric operators includes all normal operators, operators defined by Hankel matrices, truncated Toeplitz operators (including finite

This work is partially supported by National Science Foundation Grant DMS-0638789.

Toeplitz matrices and the compressed shift) [12,23], the Volterra integration operator [10], certain operators arising in the complex scaling technique for Schrödinger operators [19], Aluthge transforms of complex symmetric operators [9], many partial isometries, and all operators which are algebraic of degree two [14]. Other recent articles concerning complex symmetric operators include [2,15,23,25].

In what follows, it will be useful to have in mind operator analogues of the entry-by-entry complex conjugate and the transpose of a matrix:

Definition. For a fixed conjugation C on $\mathcal{H}$ and an arbitrary $T \in B(\mathcal{H})$ we define the *conjugate* $\overline{T}$ and the *transpose* T^t of T by the formulas

$$\overline{T} := CTC\,, \quad T^t := CT^*C\,.$$

When T is represented as a matrix with respect to a C-real orthonormal basis (i.e., an orthonormal basis e_n such that $Ce_n = e_n$ for all n), then the corresponding matrix representations for $\overline{T}$ and T^t are precisely the entry-by-entry complex conjugate and the transpose of the corresponding matrix for T (see [12, Sect. 2.4] or [10, Prop. 2]). The reader can verify that the usual properties of the transpose and conjugate hold for their analogues above. In particular, note that $(\overline{A})^{1/2} = \overline{A^{1/2}}$ and $A^t = \overline{A}$ hold for any nonnegative operator A.

For the purposes of this note, the symmetric bilinear form

$$[x, y] := \langle x, Cy \rangle \tag{1}$$

induced by a fixed conjugation C plays an important role. Indeed, we are concerned primarily with inequalities of the form

$$|[Bx, x]| \leq \langle Ax, x \rangle \quad \forall x \in \mathcal{H} \tag{2}$$

where B is a C-symmetric operator and A is nonnegative. Following [7], we refer to an inequality of the form (2) as a *hermitian-symmetric inequality.*

Example 1. The prototypical example of a hermitian-symmetric inequality arises in the study of univalent functions. Specifically, the Goluzin-Grunsky inequality [6, Cor. 9, p. 128] asserts that an analytic function f on the open unit disk $\mathbb{D}$, normalized so that $f(0) = 0$ and $f'(0) = 1$, is injective and if and only if for any n distinct $z_1, z_2, \ldots, z_n \in \mathbb{D}$, the hermitian-symmetric inequality

$$\left| \sum_{j,k=1}^{n} x_j x_k \log \left(\frac{z_j z_k}{f(z_j) f(z_k)} \cdot \frac{f(z_j) - f(z_k)}{z_j - z_k} \right) \right| \leq \sum_{j,k=1}^{n} x_j \overline{x_k} \log \frac{1}{1 - z_j \overline{z_k}}$$

holds for all choices of $x_1, x_2, \ldots, x_n \in \mathbb{C}$. This is an inequality of the form (2) where $\mathcal{H} = \mathbb{C}^n$, $x = (x_1, x_2, \ldots, x_n)$, C is the canonical conjugation on $\mathbb{C}^n$, and

$$[A]_{jk} = \log \frac{1}{1 - z_j \overline{z_k}}\,, \quad [B]_{jk} = \log \left(\frac{z_j z_k}{f(z_j) f(z_k)} \cdot \frac{f(z_j) - f(z_k)}{z_j - z_k} \right).$$

Further examples of symmetric bilinear forms arising in function theory can be found in [5].

For the moment, we content ourselves with studying the abstract structure of hermitian-symmetric inequalities. Before proceeding, let observe a few basic properties of the bilinear form (1):

(i) $[\,\cdot\,,\,\cdot\,]$ is not nonnegative since $[e^{i\theta/2}x, e^{i\theta/2}x] = e^{i\theta}[x,x]$ for any $\theta \in \mathbb{R}$. In particular, the bilinear form (1) is *not* the difference of nonnegative sesquilinear forms.

(ii) $[\,\cdot\,,\,\cdot\,]$ is nondegenerate, in the sense that $[x,y] = 0$ for all y in $\mathcal{H}$ if and only if $x = 0$.

(iii) $[\,\cdot\,,\,\cdot\,]$ has isotropic vectors. In other words, there exist nonzero vectors $x \in \mathcal{H}$ such that $[x,x] = 0$. As shown in [13], isotropic vectors arise naturally as eigenvectors of C-symmetric operators which have repeated eigenvalues.

(iv) $[\,\cdot\,,\,\cdot\,]$ satisfies the CSB Inequality: $|[x,y]| \leq \|x\|\|y\|$.

(v) An operator B on $\mathcal{H}$ is C-symmetric if and only if $[Bx,y] = [x,By]$ for all x,y in $\mathcal{H}$.

Due to the fact that complex symmetric operators are generally non-normal (e.g., the Volterra operator) and that the bilinear form (2) is not nonnegative, a significant amount of cancellation can occur in the left hand side of (2). Thus the direct relationship between the operators A and B is not immediately clear. It is our aim in this article to clarify this relationship.

2. The structure of Hermitian-symmetric inequalities

Our first theorem, inspired in part by [7, Thm. 2.1], is the following:

Theorem 1. *If A is a nonnegative operator and B is a C-symmetric operator on $\mathcal{H}$, then the following are equivalent:*

(i) $|[Bx,x]| \leq \langle Ax,x\rangle$ *for all x in $\mathcal{H}$.*

(ii) $2|[Bx,y]| \leq \langle Ax,x\rangle + \langle Ay,y\rangle$ *for all x,y in $\mathcal{H}$.*

(iii) *The operator*

$$\begin{pmatrix} A & B^* \\ B & \overline{A} \end{pmatrix} \tag{3}$$

on $\mathcal{H} \oplus \mathcal{H}$ is nonnegative.

(iv) $B = \overline{A}^{1/2} K A^{1/2}$ *for some C-symmetric contraction K.*

(v) $B = T^t T$ *for some $T \in B(\mathcal{H})$ satisfying $T^*T \leq A$.*

Moreover, if $A > 0$ then we may add

(vi) $BA^{-1}B^* \leq \overline{A}$.

We postpone the somewhat lengthy proof of Theorem 1 until Section 4. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are straightforward generalizations of [7, Thm. 2.1]. The primary difficulty lies in establishing the implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) in the infinite dimensional case. The adaptations required to pass to the infinite dimensional case are nontrivial. Indeed, many theorems concerning complex symmetric matrices or hermitian-symmetric inequalities rely on the singular value decomposition in the guise of the Autonne–Takagi factorization theorem [17, Cor. 4.4.4], which is not applicable in the infinite-dimensional setting. Consequently, we require several lemmas which are presented separately in Section 3.

We also remark that conditions (iv), (v), and (vi) appear to be novel, even in the finite-dimensional case. Furthermore, the proof of the implication (iv) $\Rightarrow$ (v) requires a new factorization theorem for complex symmetric operators (Theorem 2).

Remark. Condition (v) of Theorem 1 must be interpreted carefully. If $B = T^tT$ and the hermitian-symmetric inequality (2) holds, then we cannot immediately conclude that $T^*T \leq A$. To see this, first note that if S is a *C-orthogonal operator* (i.e., $S^tS = SS^t = I$) then clearly $B = (ST)^t(ST)$. It is not hard to see that C-orthogonal operators can be of arbitrarily large norm. Indeed, simply consider elements of the complex orthogonal group $O(n, \mathbb{C})$ of order $n \geq 2$. Moreover, *unbounded* C-orthogonal operators arise in the complex scaling technique for Schrödinger operators [22].

Remark. One might conjecture that the condition $|B| = \sqrt{B^*B} \leq A$ is both necessary and sufficient condition for the hermitian-symmetric inequality (2) to hold. However, this is incorrect. Although it turns out that $|B| \leq A$ is a sufficient condition (see the following corollary), it is not necessary (see Example 2). This reflects the fact, remarked upon previously, that a significant amount of cancellation can occur on the left side of the inequality (2).

Corollary 1. *If B is a C-symmetric operator and $|B| \leq A$, then the hermitian-symmetric inequality* (2) *holds.*

Proof. It suffices to prove that $|[Bx, x]| \leq \langle |B|x, x\rangle$ holds for every $x \in \mathcal{H}$. By [11, Cor. 1], we may write $B = CJ|B|$ where J is a conjugation which commutes with $|B|$ (and hence with $|B|^{\frac{1}{2}}$). Noting that $K = CJ$ is a C-symmetric unitary operator, we find that

$$\overline{|B|}^{\frac{1}{2}} K|B|^{\frac{1}{2}} = (C|B|^{\frac{1}{2}}C)(CJ)|B|^{\frac{1}{2}} = C|B|^{\frac{1}{2}}J|B|^{\frac{1}{2}} = CJ|B| = B\,.$$

Applying Theorem 1 with $A = |B|$, the desired inequality follows. $\square$

On the other hand, the following simple example shows that condition $|B| \leq A$ is not necessary for (2) to hold:

Example 2. Let $\lambda > 0$ and define

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since K is a C-symmetric contraction with respect to the canonical conjugation $C(x_1, x_2) = (\overline{x_1}, \overline{x_2})$ on $\mathbb{C}^2$, condition (iv) of Theorem 1 tells us that the hermitian-symmetric inequality (2) holds where

$$B = \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = K\,.$$

In this case, the inequality (2) reduces to

$$|2x_1 x_2| \leq \lambda |x_1|^2 + \lambda^{-1} |x_2|^2\,,$$

which is simply a special case of the Arithmetic-Geometric Mean Inequality. Since $|B| = I$, we see that $|B| \not\leq A$ since either $\lambda > 1$ or $\lambda^{-1} > 1$. In particular, this shows that the condition $|B| \leq A$ does not follow from the hermitian-symmetric inequality (2). Moreover, this also demonstrates that condition (v) from Theorem 1 does not imply that $|B| \leq A$.

Remark. Even if B is a nonsingular C-symmetric operator, say with bounded inverse, the expression $|[Bx, x]|$ vanishes for infinitely many $x \in \mathcal{H}$. For instance, in the simple case where $B = I$ on $\mathbb{C}^2$ and C denotes the canonical conjugation on $\mathbb{C}^2$, we find that $|[\mathbf{x}, \mathbf{x}]| = x_1^2 + x_2^2$, which vanishes whenever $x_1 = \pm i x_2$. In light of this, one might suspect that a hermitian-symmetric inequality (2) could be constructed for which B is invertible and A has nontrivial kernel. The following corollary asserts that this is not the case:

Corollary 2. *If the hermitian-symmetric inequality* (2) *holds and B is nonsingular, then A is also nonsingular.*

Proof. This follows immediately from condition (iv) of Theorem 1. □

We conclude this section with a remark concerning a further relationship between A and B in (2). It turns out that compact complex symmetric operators enjoy several variational principles in the spirit of Courant's famous minimax theorem. In particular, it was shown in [5] that if B is a compact C-symmetric operator on $\mathcal{H}$ and if $\sigma_0 \geq \sigma_1 \geq \cdots \geq 0$ are the singular values of B (i.e., the eigenvalues of $|B| = \sqrt{B^*B}$), repeated according to multiplicity, then

$$\min_{\operatorname{codim} \mathcal{V} = n} \max_{\substack{x \in \mathcal{V} \\ \|x\| = 1}} |[Bx, x]| = \begin{cases} \sigma_{2n} & \text{if } 0 \leq n < \frac{\dim \mathcal{H}}{2} \\ 0 & \text{otherwise}\,. \end{cases} \tag{4}$$

The notation indicates that $\mathcal{V}$ ranges over all subspaces of $\mathcal{H}$ of codimension n. This theorem, which can be seen as a generalization of Danciger's Minimax Principle [4], is reminiscent of Courant's variational principle for the eigenvalues of a compact self-adjoint operator:

Theorem (Courant). *If A is a compact, self-adjoint operator and $\lambda_0 \geq \lambda_1 \geq \cdots$ are the eigenvalues of A, repeated according to multiplicity, then*

$$\lambda_n = \min_{\operatorname{codim} \mathcal{V} = n} \max_{\substack{x \in \mathcal{V} \\ \|x\| = 1}} \langle Ax, x \rangle \tag{5}$$

holds whenever $0 \leq n < \dim \mathcal{H}$.

It follows from (4) and (5) that if the hermitian-symmetric inequality (2) holds, then we must have

$$\sigma_{2n} \leq \lambda_n$$

for $0 \leq n < \frac{1}{2}\dim\mathcal{H}$. It would be interesting to have a complete understanding of the relationship between the eigenvalues of A and the singular values of B whenever the hermitian-symmetric inequality (2) is satisfied.

3. A factorization theorem

The proof of Theorem 1 requires several preliminary lemmas and a new factorization theorem for complex symmetric operators (Theorem 2). These are discussed below.

Lemma 1. *If $K \in B(\mathcal{H})$, then $\|K\| \leq 1$ if and only if*

$$\begin{pmatrix} I & K \\ K^* & I \end{pmatrix} \geq 0\,. \tag{6}$$

Proof. Observe that $\|K\| \leq 1$ if and only if

$$\left\| \begin{pmatrix} 0 & K \\ K^* & 0 \end{pmatrix} \right\| \leq 1\,. \tag{7}$$

It is then clear that (6) and (7) are equivalent. □

Lemma 2. *If U is a unitary C-symmetric operator and if f is a bounded Borel function on the unit circle $\partial\mathbb{D}$, then the operator $f(U)$ is C-symmetric. In particular, each spectral projection of U commutes with C.*

Proof. If p is a polynomial in two noncommuting variables, then $p(U,U^*)$ can be reduced to a polynomial in U and U^* which contains no cross terms since $U^*U = UU^* = I$. This implies that $p(U,U^*) = p_1(U) + p_2(U^*)$ for some polynomials p_1, p_2 whence $p(U,U^*)$ is C-symmetric by [10, Prop. 1]. In terms of the spectral decomposition

$$U = \int_{\partial\mathbb{D}} \lambda\, dE(\lambda)\,,$$

the equation $U = CU^*C$ then implies that

$$\begin{aligned} \int_{\partial\mathbb{D}} p(\lambda,\overline{\lambda})\, dE(\lambda) &= C \int_{\partial\mathbb{D}} \overline{p(\lambda,\overline{\lambda})}\, dE(\lambda) C \\ &= \int_{\partial\mathbb{D}} p(\lambda,\overline{\lambda})\, dCE(\lambda)C\,. \end{aligned}$$

We therefore see that $E(\Delta) = CE(\Delta)C$ for any Borel subset $\Delta \subseteq \partial\mathbb{D}$. This readily implies that

$$\begin{aligned} f(U) &= \int_{\partial\mathbb{D}} f(\lambda)\, dE(\lambda) \\ &= \int_{\partial\mathbb{D}} f(\lambda)\, dCE(\lambda)C \end{aligned}$$

$$= C\left[\int_{\partial\mathbb{D}} \overline{f(\lambda)}\, dE(\lambda)\right] C$$
$$= Cf(U)^* C$$

for any bounded Borel function f on $\partial\mathbb{D}$. □

In particular, the preceding lemma asserts that each C-symmetric unitary operator U has a C-symmetric unitary square root. However, it should be noted that this square root is not unique. We should also remark that Lemma 2 holds in greater generality. Indeed, if N is a C-symmetric normal operator, then the von Neumann algebra generated by N consists of C-symmetric operators [15, Cor. 3.2].

The following factorization theorem generalizes a well-known property of complex symmetric $n \times n$ matrices [17, Cor. 4.4.6]:

Theorem 2. *B is a C-symmetric operator if and only if there exists $T \in B(\mathcal{H})$ such that $B = T^t T$ (i.e., $B = CT^*CT$). Moreover, T can be chosen so that $T^*T = |B|$.*

Proof. If $B = T^t T$ for some T, then it is clear that $B^t = B$ whence B is C-symmetric. On the other hand, by [11, Cor. 1] we may write $B = U|B|$ where U is a C-symmetric unitary operator. Furthermore, we have $U = CJ$ where J is a conjugation which commutes with $|B|$ and hence with $|B|^{\frac{1}{2}}$. By Lemma 2, there exists a C-symmetric unitary operator W such that $W^2 = U$. Letting $T = W|B|^{\frac{1}{2}}$ we observe that

$$\begin{aligned}
T^t T &= (W|B|^{\frac{1}{2}})^t (W|B|^{\frac{1}{2}}) \\
&= \overline{|B|}^{\frac{1}{2}} W^t W |B|^{\frac{1}{2}} \\
&= \overline{|B|}^{\frac{1}{2}} W^2 |B|^{\frac{1}{2}} \\
&= \overline{|B|}^{\frac{1}{2}} U |B|^{\frac{1}{2}} \\
&= C|B|^{\frac{1}{2}} CU|B|^{\frac{1}{2}} \\
&= C|B|^{\frac{1}{2}} C(CJ)|B|^{\frac{1}{2}} \\
&= C|B|^{\frac{1}{2}} J|B|^{\frac{1}{2}} \\
&= CJ|B|^{\frac{1}{2}} |B|^{\frac{1}{2}} \\
&= CJ|B| \\
&= U|B| \\
&= B\,.
\end{aligned}$$

It is also clear from the definition of T that $T^*T = |B|$. □

Remark. In the finite dimensional setting, the preceding lemma can also be proved by Autonne's theorem (commonly referred to as Takagi's Factorization – see [17, Cor. 4.4.4] and the historical remarks in [18, p. 136]). Indeed, if B is a symmetric matrix with complex entries, then by Autonne's theorem we may write $B = UDU^t$

where D is the diagonal matrix of singular values of B and U is unitary. In this case we may let $T = D^{\frac{1}{2}}U^t$ and verify that T has the desired properties.

Remark. It is also worth mentioning that the operator T produced by Theorem 2 is not in general C-symmetric. Indeed, a 2×2 nilpotent Jordan block B is C-symmetric with respect to the conjugation $C(z_1, z_2) = (\overline{z_2}, \overline{z_1})$ on $\mathbb{C}^2$. The existence of a C-symmetric operator T such that $B = T^tT = TT = T^2$ would imply the existence of a square root for B, which is easily seen to be impossible.

4. Proof of Theorem 1

We are now ready to proceed to the proof of Theorem 1. As mentioned previously, the implications (i) ⇒ (ii) and (ii) ⇒ (iii) are generalizations of [7, Thm. 2.1] and their brief proofs are included for the sake of completeness.

(i) ⇒ (ii) Let $x, y \in \mathcal{H}$ be given. Applying the inequality of condition (i) to $x + y$ and $x - y$, respectively, we obtain

$$\begin{aligned} |[Bx,x] + 2[Bx,y] + [By,y]| &\leq \langle Ax,x\rangle + 2\,\mathrm{Re}\langle Ax,y\rangle + \langle Ay,y\rangle\,,\\ |-[Bx,x] + 2[Bx,y] - [By,y]| &\leq \langle Ax,x\rangle - 2\,\mathrm{Re}\langle Ax,y\rangle + \langle Ay,y\rangle\,. \end{aligned}$$

The preceding inequality now yields

$$\begin{aligned} 4|B[x,y]| &\leq |2[Bx,y] + [Bx,x] + [By,y]| + |2[Bx,y] - [Bx,x] - [By,y]|\\ &\leq 2\langle Ax,x\rangle + 2\langle Ay,y\rangle \end{aligned}$$

from which (ii) follows.

(ii) ⇒ (iii) For any $x, y \in \mathcal{H}$ we have

$$\begin{aligned} \left\langle \begin{pmatrix} A & B^* \\ B & \overline{A} \end{pmatrix} \begin{pmatrix} x \\ Cy \end{pmatrix}, \begin{pmatrix} x \\ Cy \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} Ax + B^*Cy \\ Bx + \overline{A}Cy \end{pmatrix}, \begin{pmatrix} x \\ Cy \end{pmatrix} \right\rangle\\ &= \left\langle \begin{pmatrix} Ax + CBy \\ Bx + CAy \end{pmatrix}, \begin{pmatrix} x \\ Cy \end{pmatrix} \right\rangle\\ &= \langle Ax,x\rangle + \langle CBy,x\rangle + \langle Bx,Cy\rangle + \langle CAy,Cy\rangle\\ &= \langle Ax,x\rangle + \langle B^*Cy,x\rangle + [Bx,y] + \langle y,Ay\rangle\\ &= \langle Ax,x\rangle + \langle Cy,Bx\rangle + [Bx,y] + \langle Ay,y\rangle\\ &= \langle Ax,x\rangle + \overline{[Bx,y]} + [Bx,y] + \langle Ay,y\rangle\\ &= \langle Ax,x\rangle + 2\,\mathrm{Re}[Bx,y] + \langle Ay,y\rangle\\ &\geq \langle Ax,x\rangle - 2|[Bx,y]| + \langle Ay,y\rangle\\ &\geq 0\,. \end{aligned}$$

As x and y range independently over $\mathcal{H}$, the vector (x, Cy) ranges over $\mathcal{H} \oplus \mathcal{H}$ whence we see that (3) is nonnegative.

(iii) $\Rightarrow$ (iv) We prove this first under the hypothesis that A is invertible and then proceed to prove the general case. If A is invertible, then the condition (3) holds if and only if

$$\begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & \overline{A}^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} A & B^* \\ B & \overline{A} \end{pmatrix} \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & \overline{A}^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} I & A^{-\frac{1}{2}} B^* \overline{A}^{-\frac{1}{2}} \\ \overline{A}^{-\frac{1}{2}} B A^{-\frac{1}{2}} & I \end{pmatrix} \geq 0 \,.$$

By Lemma 1, it follows that the operator $K = \overline{A}^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ is a contraction. Furthermore, a short computation reveals that K is C-symmetric:

$$K^t = (\overline{A}^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t = (A^{-\frac{1}{2}})^t B^t (\overline{A}^{-\frac{1}{2}})^t = \overline{A}^{-\frac{1}{2}} B A^{-\frac{1}{2}} = K \,.$$

Now suppose that A is singular and note that for every $\epsilon > 0$, one has

$$\begin{pmatrix} A + \epsilon I & B^* \\ B & \overline{A} + \epsilon I \end{pmatrix} > 0 \,.$$

Since $A + \epsilon I > 0$, it follows from the invertible case that

$$K_\epsilon = \overline{(A + \epsilon I)}^{-\frac{1}{2}} B (A + \epsilon I)^{-\frac{1}{2}}$$

is a C-symmetric contraction such that

$$B = \overline{(A + \epsilon I)}^{\frac{1}{2}} K_\epsilon (A + \epsilon I)^{\frac{1}{2}} \tag{8}$$

holds for every $\epsilon > 0$. Now recall that the closed unit ball of $B(\mathcal{H})$ is compact in the weak operator topology (WOT) [3, Prop. 2.8.3] and extract a sequence $\epsilon_n > 0$ such that $\lim_{n\to\infty} \epsilon_n = 0$ and $K_n = K_{\epsilon_n}$ is WOT convergent to a contraction K (recall that $\mathcal{H}$ is presumed separable). It is straightforward to verify that K is also C-symmetric.

We now claim that

$$\langle Bx, y \rangle = \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y \rangle \tag{9}$$

holds whenever $\|x\| = \|y\| = 1$. If we can establish (9) for such x, y, then the desired formula $B = \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}}$ will follow. To this end, let $A_n = A + \epsilon_n I$ and note that since A is bounded

$$\lim_{n\to\infty} \|A_n^{\frac{1}{2}} - A^{\frac{1}{2}}\| = \lim_{n\to\infty} \sup_{\lambda \in \sigma(A)} |\sqrt{\lambda + \epsilon_n} - \sqrt{\lambda}| = 0 \,,$$

where $\sigma(A)$ denotes the spectrum of A. Using (8) and recalling that x and y are unit vectors we find that

$$\begin{aligned} |\langle Bx, y\rangle - \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y\rangle| &= \lim_{n\to\infty} |\langle Bx, y\rangle - \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y\rangle| \\ &= \lim_{n\to\infty} |\langle \overline{A_n}^{\frac{1}{2}} K_n A_n^{\frac{1}{2}} x, y\rangle - \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y\rangle| \\ &\leq \lim_{n\to\infty} \left|\left\langle (\overline{A_n}^{\frac{1}{2}} - \overline{A}^{\frac{1}{2}}) K_n A_n^{\frac{1}{2}} x, y \right\rangle\right| \\ &\quad + \lim_{n\to\infty} |\langle \overline{A}^{\frac{1}{2}} K_n A_n^{\frac{1}{2}} x, y\rangle - \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y\rangle| \end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n\to\infty} \|A_n^{\frac{1}{2}} - A^{\frac{1}{2}}\| \|K_n\| \|A_n^{\frac{1}{2}}\| \\
&\quad + \lim_{n\to\infty} |\langle \overline{A}^{\frac{1}{2}} K_n A_n^{\frac{1}{2}} x, y\rangle - \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y\rangle| \\
&\leq \|A^{\frac{1}{2}}\| \lim_{n\to\infty} \|A_n^{\frac{1}{2}} - A^{\frac{1}{2}}\| \\
&\quad + \lim_{n\to\infty} |\langle \overline{A}^{\frac{1}{2}} K_n A_n^{\frac{1}{2}} x, y\rangle - \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y\rangle| \\
&= \lim_{n\to\infty} |\langle \overline{A}^{\frac{1}{2}} K_n A_n^{\frac{1}{2}} x, y\rangle - \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y\rangle| \\
&\leq \lim_{n\to\infty} |\langle \overline{A}^{\frac{1}{2}} K_n A_n^{\frac{1}{2}} x, y\rangle - \langle \overline{A}^{\frac{1}{2}} K_n A^{\frac{1}{2}} x, y\rangle| \\
&\quad + \lim_{n\to\infty} |\langle \overline{A}^{\frac{1}{2}} K_n A^{\frac{1}{2}} x, y\rangle - \langle \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} x, y\rangle| \\
&= \lim_{n\to\infty} \left|\left\langle \overline{A}^{\frac{1}{2}} K_n (A_n^{\frac{1}{2}} - A^{\frac{1}{2}}) x, y\right\rangle\right| \\
&\quad + \lim_{n\to\infty} |\langle K_n A^{\frac{1}{2}} x, \overline{A}^{\frac{1}{2}} y\rangle - \langle K A^{\frac{1}{2}} x, \overline{A}^{\frac{1}{2}} y\rangle| \\
&\leq \|A\|^{\frac{1}{2}} \lim_{n\to\infty} \|A_n^{\frac{1}{2}} - A^{\frac{1}{2}}\| \\
&\quad + \lim_{n\to\infty} |\langle K_n A^{\frac{1}{2}} x, \overline{A}^{\frac{1}{2}} y\rangle - \langle K A^{\frac{1}{2}} x, \overline{A}^{\frac{1}{2}} y\rangle| \\
&= \lim_{n\to\infty} \left|\left\langle (K_n - K) A^{\frac{1}{2}} x, \overline{A}^{\frac{1}{2}} y\right\rangle\right| \\
&= 0
\end{aligned}$$

since K_n is WOT convergent to K. This concludes the proof that (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (v) If $B = \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}}$ for some C-symmetric contraction K, then by Theorem 2, we may write $K = K_0^t K_0$ for some contraction K_0. This implies that

$$B = \overline{A}^{\frac{1}{2}} K A^{\frac{1}{2}} = (\overline{A}^{\frac{1}{2}} K_0^t)(K_0 A^{\frac{1}{2}}) = T^t T$$

where $T = K_0 A^{\frac{1}{2}}$. Since K_0 is a contraction, it follows that $\|Tx\| \leq \|A^{\frac{1}{2}} x\|$ for all $x \in \mathcal{H}$ whence $T^*T \leq A$, as claimed.

(v) $\Rightarrow$ (i) If $B = T^t T$ for some $T \in B(\mathcal{H})$ satisfying $T^*T \leq A$, then

$$\begin{aligned}
|[Bx, x]| &= |[T^t T x, x]| \\
&= |[Tx, Tx]| \\
&\leq \|Tx\|^2 \\
&= \langle T^* T x, x\rangle \\
&\leq \langle Ax, x\rangle,
\end{aligned}$$

which yields (i).

To establish (iii) $\Leftrightarrow$ (vi) in the case where $A > 0$, we employ a standard computational trick [1, Thm. 1.3.3]. Suppose that $A > 0$ and observe that (iii) holds if and only if

$$\begin{pmatrix} I & -B^*\overline{A}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B^* \\ B & \overline{A} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\overline{A}^{-1}B & I \end{pmatrix} = \begin{pmatrix} A - B^*\overline{A}^{-1}B & 0 \\ 0 & \overline{A} \end{pmatrix} \geq 0\,.$$

The preceding is equivalent to asserting that $B^*\overline{A}^{-1}B \leq A$ or equivalently that $BA^{-1}B^* \leq \overline{A}$, which is (vi). This concludes the proof of Theorem 1. □

References

[1] R. Bhatia, *Positive Definite Matrices*, Princeton Series in Applied Mathematics, 2007.

[2] N. Chevrot, E. Fricain, D. Timotin, *The characteristic function of a complex symmetric contraction*, Proc. Amer. Math. Soc. **135** (2007), 2877–2886.

[3] J.B. Conway, *A Course in Operator Theory* (second edition), Graduate Studies in Mathematics, 21, Springer-Verlag, New York, 1999.

[4] J. Danciger, *A min-max theorem for complex symmetric matrices*, Linear Algebra Appl. **412** (2006), no. 1, 22–29. MR2180856 (2006g:15015).

[5] J. Danciger, S. Garcia, M. Putinar, *Variational principles for symmetric bilinear forms*, Math. Nachr., Volume 281, Issue 6, 786–802.

[6] P.L. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.

[7] C.H. FitzGerald, R.A. Horn, *On the structure of hermitian-symmetric inequalities*, J. London Math. Soc. (2) **15** (1977), no. 3, 419–430.

[8] K. Friedrichs, *On certain inequalities and characteristic value problems for analytic functions and for functions of two variables*, Trans. Amer. Math. Soc, **41** no.3, 321–364 (1937).

[9] S.R. Garcia, *Aluthge transforms of complex symmetric operators*, Integral Equations Operator Theory **60** (2008), 357–367. MR2392831.

[10] S.R. Garcia, M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), 1285–1315. MR2187654 (2006j:47036).

[11] S.R. Garcia, M. Putinar, *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc. **359** (2007), 3913–3931. MR2302518 (2008b:47005).

[12] S.R. Garcia, *Conjugation and Clark operators*, Contemp. Math. **393** (2006), 67–112. MR2198373 (2007b:47073).

[13] S.R. Garcia, *The eigenstructure of complex symmetric operators*, Operator Theory: Advances and Applications **179** (2008), 169–184. MR2397854.

[14] S.R. Garcia, W. Wogen, *Some new classes of complex symmetric operators*, (preprint).

[15] T.M. Gilbreath, W.R. Wogen, *Remarks on the structure of complex symmetric operators*, Integral Equations Operator Theory, Volume 59, Number 4 / December, 2007.

[16] P.R. Halmos, *A Hilbert Space Problem Book* (Second Edition), Springer-Verlag, New York, 1982.

[17] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1985.

[18] R. A. Horn, C. R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1991.

[19] E. Prodan, S. R. Garcia, M. Putinar, *Norm estimates of complex symmetric operators applied to quantum systems*, J. Phys. A: Math. Gen. 39, no. 2, (2006), 389–400. MR2198968 (2006j:81064).

[20] M. Putinar, H. S. Shapiro, *The Friedrichs Operator of a Planar Domain*, Complex analysis, operators, and related topics, 303–330, Oper. Theory Adv. Appl., 113, Birkhäuser, Basel, 2000.

[21] M. Putinar, H. S. Shapiro, *The Friedrichs Operator of a Planar Domain II*, Recent advances in operator theory and related topics (Szeged, 1999), 519–551, Oper. Theory Adv. Appl., 127, Birkhäuser, Basel, 2001.

[22] U. V. Riss, *Extension of the Hilbert space by J-unitary transformations*, Helv. Phys. Acta **71**(1998), 288–313.

[23] D. E. Sarason, *Algebraic properties of truncated Toeplitz operators*, Operators and Matrices, vol. 1, no. 4 (2007), 491–526.

[24] H. S. Shapiro, *Some inequalities for analytic functions integrable over a plane domain*, Approximation and Function Spaces, Gdansk 1979, North Holland, 645–666 (1981).

[25] J. Tener, *Unitary equivalence to a complex symmetric matrix: An algorithm*, J. Math. Anal. Appl. **341** (2008), 640–648.

Stephan Ramon Garcia
Department of Mathematics
Pomona College
Claremont, California 91711
USA
e-mail: Stephan.Garcia@pomona.edu
URL: http://pages.pomona.edu/~sg064747

Communicated by Leiba Rodman.
Submitted: February 8, 2008.
Revised: July 29, 2008.
Accepted: August 3, 2008.