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Means of unitaries, conjugations, and the Friedrichs operator ☆

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Abstract

If C is a conjugation (an isometric, conjugate-linear involution) on a separable complex Hilbert space $\mathcal{H}$, then $T \in B(\mathcal{H})$ is called C-symmetric if $T = CT^*C$. In this note we prove that each C-symmetric contraction T is the mean of two C-symmetric unitary operators. We discuss several corollaries and an application to the Friedrichs operator of a planar domain.

Keywords: Complex symmetric operator; Complex symmetric matrix; Conjugation; Extreme point; Unilateral shift; Unitary operator; Hankel matrix; Toeplitz matrix; Bergman space; Bergman kernel; Friedrichs operator; Quadrature domain

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1. Introduction

In this note we establish a certain representation theorem for complex symmetric contractions. Several corollaries of this theorem are discussed in Section 2. As an application of our main theorem, we prove in Section 4 that the Friedrichs operator of a planar domain is the mean of two conjugations.

Before stating our main theorem, we require a few preliminaries. A *conjugation* is a conjugate-linear operator C, defined on a separable complex Hilbert space $\mathcal{H}$, which is both involutive ($C^2 = I$) and *isometric*. We say that a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is C-*symmetric* if

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$T = CT^*C$ and *complex symmetric* if there exists a conjugation C with respect to which T is C-symmetric. Moreover, $T \in B(\mathcal{H})$ is complex symmetric if and only if T is unitarily equivalent to a *complex symmetric matrix* (i.e. a self-transpose matrix with complex entries), regarded as an operator acting on an l^2-space of the appropriate dimension [6, Section 2.4].

The class of complex symmetric operators includes all normal operators [6, Example 2.8], operators defined by Hankel matrices, compressed Toeplitz operators (including the compressed shift) [6, Theorem 5.1], and the Volterra integration operator. We refer the reader to [6, Section 2] and [4,5] for further details. Other recent articles concerning complex symmetric operators include [1,3].

Although it is well known that every element in the *open* unit ball of $B(\mathcal{H})$ is the mean of some finite collection of unitary operators [10, Proposition 3.2.23], we can say something much stronger about complex symmetric operators lying in the *closed* unit ball of $B(\mathcal{H})$:

Theorem 1. *If T is a C-symmetric contraction (i.e. $\|T\| \leqslant 1$), then there exist C-symmetric unitary operators U_1 and U_2 so that*

$$T = \frac{1}{2}(U_1 + U_2). \tag{1}$$

Moreover,

(i) *this decomposition is unique (up to the order of the summands) if and only if T is injective,*
(ii) *the unitaries U_1 and U_2 are distinct if and only if T is not unitary.*

We stress that although all unitary operators are complex symmetric (as are all normal operators—see [6, Example 2.8] or [4, Section 4.1]), the preceding theorem asserts that U_1 and U_2 are C-symmetric *with respect to the original C*. We defer the proof of Theorem 1 until Section 3 and instead focus on several corollaries.

2. Some corollaries of Theorem 1

It is well known that there are no restrictions on the Jordan canonical form of an $n \times n$ complex symmetric matrix (see [9, Theorem 4.4.9] or [6, Theorem 2.3] for a discussion in the context of complex symmetric operators). This readily implies the following corollary:

Corollary 1. *The linear span of the $n \times n$ complex symmetric unitary matrices includes members of every similarity class in $M_n(\mathbb{C})$.*

Hankel matrices comprise one notable class of complex symmetric matrices. Indeed, if T is a Hankel matrix, then $T = CT^*C$ where C denotes the canonical conjugation on the underlying l^2-space. From Theorem 1 we immediately conclude:

Corollary 2. *Every contractive Hankel matrix (finite or infinite) is the mean of two complex symmetric unitary matrices.*

A related result holds for finite Toeplitz matrices, for an $n \times n$ Toeplitz matrix is C-symmetric with respect to the conjugation $C(z_1, z_2, \ldots, z_n) = (\overline{z_n}, \overline{z_{n-1}}, \ldots, \overline{z_1})$ on $\mathbb{C}^n$ (see [4, Example 10] or [6, Section 2.2]). Applying Theorem 1 in this case yields:

Corollary 3. *Every finite, contractive Toeplitz matrix is the mean of two unitary matrices which are symmetric with respect to the counter-diagonal.*

For each fixed conjugation C acting on a separable complex Hilbert space $\mathcal{H}$, let $\mathrm{Sym}_C(\mathcal{H})$ denote the closed subspace of $B(\mathcal{H})$ consisting of all C-symmetric operators. The following corollary follows immediately from Theorem 1:

Corollary 4. *The linear span of the set of C-symmetric unitary operators is* $\mathrm{Sym}_C(\mathcal{H})$.

Since the spectral projections of a C-symmetric normal operator are C-symmetric and self-adjoint, they necessarily commute with C. We therefore obtain:

Corollary 5. *The closed linear span of the set of orthogonal projections which commute with C is* $\mathrm{Sym}_C(\mathcal{H})$.

Using Theorem 1 we can immediately identify those complex symmetric operators which are extreme points of the unit ball of $B(\mathcal{H})$:

Corollary 6. *A complex symmetric operator is an extreme point of the unit ball of $B(\mathcal{H})$ if and only if it is unitary.*

Proof. Unitary operators, being normal, are necessarily complex symmetric (see [6, Example 2.8] or [4, Section 4.1]). Since the extreme points of the closed unit ball of $B(\mathcal{H})$ are the maximal partial isometries [8, Problem 136], every unitary operator is both complex symmetric and extreme. On the other hand, if T is a complex symmetric operator which is not unitary, then T is not an extreme point by Theorem 1. $\square$

Although the next result can be proved easily using more elementary means (see [6, Example 2.14] or [4, Proposition 1]), we include two additional short proofs:

Corollary 7. *The unilateral shift is not a complex symmetric operator.*

Proof 1. Let S denote the unilateral shift (realized concretely as an operator on a separable complex Hilbert space $\mathcal{H}$). Being an isometry, S is an extreme point of the unit ball of $B(\mathcal{H})$. On the other hand, S is not unitary and hence cannot be complex symmetric by the preceding corollary. $\square$

Proof 2. Suppose that there exists a conjugation C on $\mathcal{H}$ such that S is C-symmetric. By Theorem 1, $S = \frac{1}{2}U_1 + \frac{1}{2}U_2$ where U_1 and U_2 are distinct C-symmetric unitary operators. However, $\|S - U\| = 2$ holds for any unitary U (i.e. S is on the opposite side of the unit sphere from any unitary operator—see [8, Problem 150]). Therefore $S - U_1 = \frac{1}{2}U_2 - \frac{1}{2}U_1$ from which we obtain the contradiction $2 = \|S - U_1\| = \|\frac{1}{2}U_2 - \frac{1}{2}U_1\| \leqslant 1$. $\square$

3. Proof of the main theorem

To proceed with the proof of Theorem 1, we require a few remarks concerning unitary operators and the polar decompositions of complex symmetric operators. The first result we need is due to Godič and Lucenko [7] (see [6, Section 6.1] for a discussion and several examples):

Lemma 1. *A unitary operator U is C-symmetric if and only if it is of the form $U = CJ$ where J is a conjugation on $\mathcal{H}$ (in this case we have $U^* = JC$).*

Now recall that the polar decomposition $T = U|T|$ of an operator $T : \mathcal{H} \to \mathcal{H}$ expresses T uniquely as the product of a positive operator $|T| = \sqrt{T^*T}$ and a partial isometry U which satisfies $\ker U = \ker|T|$ and maps $\mathrm{cl}(\operatorname{ran}|T|)$ onto $\mathrm{cl}(\operatorname{ran} T)$. If T is a C-symmetric operator, then we can further decompose the partial isometry U as the product of C with a *partial conjugation*. We say that a conjugate-linear operator $J : \mathcal{H} \to \mathcal{H}$ is a partial conjugation if J restricts to a conjugation on $(\ker J)^\perp$ (with values in the same space). In particular, the *linear* operator J^2 is the orthogonal projection onto the closed subspace $\operatorname{ran} J = (\ker J)^\perp$.

Lemma 2. *If $T : \mathcal{H} \to \mathcal{H}$ is a bounded C-symmetric operator, then $T = CJ|T|$ where J is a partial conjugation, supported on* $\mathrm{cl}(\operatorname{ran}|T|)$*, which commutes with $|T| = \sqrt{T^*T}$.*

Proof. The proof can be found in [5] and thus we provide only a sketch. Write $T = U|T|$ and note that $T = CT^*C = C|T|U^*C = (CU^*C)(CU|T|U^*C)$ since U^*U is the orthogonal projection onto $\mathrm{cl}(\operatorname{ran}|T|)$. One shows that $\ker CU^*C = \ker CU|T|U^*C$, notes that CU^*C is a partial isometry and that $CU|T|U^*C$ is positive, then concludes from the uniqueness of the terms in the polar decomposition that $U = CU^*C$ (so that U is C-symmetric) and that the conjugate-linear operator $J = CU = U^*C$ commutes with $|T|$. One then verifies that J is a partial conjugation supported on $\mathrm{cl}(\operatorname{ran}|T|)$. □

Suppose now that T is a C-symmetric contraction. By Lemma 2, we may write $T = CJ|T|$ where J is a partial conjugation supported on $\mathrm{cl}(\operatorname{ran}|T|)$ and commuting with $|T|$. By the remarks preceding Lemma 2, it follows that J is a conjugation on all of $\mathcal{H}$ if and only if $\ker|T|$ is trivial (i.e. T is injective).

Remark. It turns out that we may actually assume that J is a conjugation on all of $\mathcal{H}$ since a partial conjugation J can always be extended to a conjugation $\tilde{J}$ on the entire space $\mathcal{H}$ by forming the internal orthogonal direct sum $\tilde{J} = J \oplus J'$ where J' is any partial conjugation with support $\ker|T| = (\mathrm{cl}(\operatorname{ran}|T|))^\perp$. In particular, we may write $T = U|T|$ where $U = CJ$ is a C-symmetric unitary operator by Lemma 1.

Since $|T|$ is a self-adjoint contraction, it follows that the operators $V_\pm$ defined by

$$V_\pm = |T| \pm i\sqrt{I - |T|^2}$$

are both unitary. Furthermore, Lemma 2 and the preceding remark ensure that J commutes with $|T|$ whence $V_\pm$ are both J-symmetric unitary operators: $V_\pm = JV_\pm^*J$. We may therefore write

$$T = \frac{1}{2}(UV_- + UV_+)$$

where the operators $U_\pm = UV_\pm$ are both unitary.

We claim that the operators $U_\pm$ are both C-symmetric. Indeed, since $U = CJ$, $U^* = JC$, and $V_\pm = JV_\pm^*J$ we see that

$$\begin{aligned} U_\pm &= UV_\pm \\ &= (CJ)\bigl(JV_\pm^*J\bigr) \end{aligned}$$

$$\begin{aligned}
&= \big(CV_{\pm}^*\big)(JC)C \\
&= CV_{\pm}^* U^* C \\
&= C(UV_{\pm})^* C \\
&= CU_{\pm}^* C.
\end{aligned}$$

Thus U_+ and U_- are both C-symmetric unitary operators. In particular, $T = \frac{1}{2}(U_1 + U_2)$ where $U_1 = U_+$ and $U_2 = U_-$ (or vice versa) are C-symmetric unitary operators. It therefore remains to prove remarks (i) and (ii) of the main theorem.

Proof of (i). Suppose that $T = \frac{1}{2}(U_1 + U_2)$ where U_1 and U_2 are C-symmetric unitary operators (in particular this implies that T is a C-symmetric contraction). By Lemma 1, we may write $U_1 = CJ_1$ and $U_2 = CJ_2$ where J_1 and J_2 are conjugations on $\mathcal{H}$. It follows that $2CJ|T| = CJ_1 + CJ_2$ whence

$$2|T| = JJ_1 + JJ_2.$$

Since $W_1 = JJ_1$ and $W_2 = JJ_2$ are unitary operators (by Lemma 1), we may write

$$\begin{aligned}
W_1 &= A_1 + iB_1, \\
W_2 &= A_2 + iB_2
\end{aligned}$$

where A_1, A_2, B_1, B_2 are self-adjoint operators satisfying

$$A_1^2 + B_1^2 = A_2^2 + B_2^2 = I. \tag{2}$$

A short computation reveals that

$$2|T| = A_1 + A_2 \tag{3}$$

and $B_1 = -B_2$, from which it follows from (2) that

$$A_1^2 = A_2^2. \tag{4}$$

From (3) and (4) it follows that

$$\begin{aligned}
2A_1|T| &= A_1^2 + A_1A_2 \\
&= A_2^2 + A_1A_2 \\
&= (A_1 + A_2)A_2 \\
&= 2|T|A_2
\end{aligned}$$

whence $A_1 p(|T|) = p(|T|)A_2$ for any polynomial $p(x)$. If P denotes the orthogonal projection onto $\mathrm{cl}(\mathrm{ran}\,|T|)$, then it follows from a standard limiting argument that $A_1P = PA_2$. Taking adjoints yields $PA_1 = A_2P$ from which we see that

$$A_1x = A_1Px = PA_2x = PA_2Px = PPA_1x = PA_1x = A_2Px = A_2x$$

for any $x \in \mathrm{cl}(\mathrm{ran}\,|T|)$. Thus A_1 and A_2 agree on $\mathrm{cl}(\mathrm{ran}\,|T|)$.

Case 1: If T is injective, then $\ker|T|$ is trivial whence $P = I$ and $A_1 = A_2$. From this it follows that $B_1 = \sqrt{I - A_1^2} = \sqrt{I - A_2^2} = -B_2$ and thus the decomposition of the theorem is unique, up to the order of the terms U_1 and U_2.

Case 2: If T is not injective, then $\ker|T|$ is nontrivial. Thus there are infinitely many possible choices of the partial conjugation J' that appears in the remark following Lemma 2. This in turn leads to infinitely many choices of the conjugation J that appears in the construction of the C-symmetric unitary operators $U_\pm = CJ(|T| \pm i\sqrt{I - |T|^2})$. □

Proof of (ii). This follows immediately from the fact that every unitary is an extreme point of the unit ball of $B(\mathcal{H})$. □

4. Application: The Friedrichs operator

We conclude this note by using Theorem 1 to prove that the Friedrichs operator of a planar domain is the mean of two conjugations. We first introduce the necessary notation and concepts.

Let Ω denote a bounded, connected domain in $\mathbb{C}$ and let $A^2(\Omega)$ denote the *Bergman space* of Ω, the Hilbert subspace of $L^2(\Omega) = L^2(\Omega, dA)$ consisting of all analytic functions in $L^2(\Omega)$. Let $P_\Omega : L^2(\Omega) \to A^2(\Omega)$ denote the *Bergman projection*, the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. The *Friedrichs operator* is defined to be the conjugate-linear operator $F_\Omega : A^2(\Omega) \to A^2(\Omega)$ defined by

$$F_\Omega f = P_\Omega \bar{f}. \tag{5}$$

In terms of the Bergman kernel $K(z,w)$ of Ω, we also have

$$[F_\Omega f](z) = \int_\Omega K(z,w)\overline{f(w)}\,dA(w), \quad z \in \Omega.$$

This operator originated in work related to planar elasticity [2, Section 5] and more recently surfaced in the study of quadrature domains [11,12]. The significance of F_Ω lies in the fact that it can reveal certain geometric and algebraic properties of the domain Ω (see [11,12]).

Our next theorem asserts that F_Ω is the mean of two conjugations on $A^2(\Omega)$:

Theorem 2. *If Ω is a domain in $\mathbb{C}$, then there exist conjugations J_1, J_2 on $A^2(\Omega)$ such that $F_\Omega = \frac{1}{2}(J_1 + J_2)$. Moreover, if F_Ω is injective, then the conjugations J_1 and J_2 are uniquely determined, up to their ordering.*

Proof. Since P_Ω is a projection, it follows immediately from (5) that $\|F_\Omega\| \leqslant 1$ and hence F_Ω is a conjugate-linear contraction on $A^2(\Omega)$. If C is any conjugation on $A^2(\Omega)$, then the linear operator $T = CF_\Omega$ is C-symmetric since

$$\langle f, CTg\rangle = \langle f, F_\Omega g\rangle = \langle f, P_\Omega \bar{g}\rangle = \langle f, \bar{g}\rangle_{L^2(\Omega)} = \langle g, \bar{f}\rangle_{L^2(\Omega)}$$

and

$$\bigl\langle f, T^*Cg\bigr\rangle = \langle Tf, Cg\rangle = \langle CF_\Omega f, Cg\rangle = \langle g, F_\Omega f\rangle = \langle g, P_\Omega \bar{f}\rangle = \langle g, \bar{f}\rangle_{L^2(\Omega)}$$

hold for all $f, g \in A^2(\Omega)$. Since C is isometric, it follows that T is a C-symmetric contraction. By Theorem 1, there exist C-symmetric unitary operators U_1 and U_2 such that $T = \frac{1}{2}(U_1 + U_2)$. By Lemma 1, we may write $U_1 = CJ_1$ and $U_2 = CJ_2$ where J_1 and J_2 are conjugations on $A^2(\Omega)$. It follows that $F_\Omega = \frac{1}{2}(J_1 + J_2)$.

By (i) of Theorem 1, the unitary operators U_1 and U_2 are uniquely determined (up to their order) if and only if T (and thus F_Ω) is injective. Moreover, it is clear that the particular choice of C does not affect the J_1 and J_2 that are produced. □

References

[1] N. Chevrot, E. Fricain, D. Timotin, The characteristic function of a complex symmetric contraction, Proc. Amer. Math. Soc., in press.

[2] K. Friedrichs, On certain inequalities for analytic functions and for functions of two variables, Trans. Amer. Math. Soc. 41 (1937) 321–364.

[3] T.M. Gilbreath, W.R. Wogen, Remarks on the structure of complex symmetric operators, preprint.

[4] S.R. Garcia, M. Putinar, Complex symmetric operators and applications, Trans. Amer. Math. Soc. 358 (2006) 1285–1315.

[5] S.R. Garcia, M. Putinar, Complex symmetric operators and applications II, Trans. Amer. Math. Soc., in press.

[6] S.R. Garcia, Conjugation and Clark Operators, Contemp. Math., vol. 393, 2006, pp. 67–112.

[7] V.I. Godič, I.E. Lucenko, On the representation of a unitary operator as a product of two involutions, Uspekhi Mat. Nauk 20 (1965) 64–65.

[8] P.R. Halmos, A Hilbert Space Problem Book, second ed., Springer-Verlag, New York, 1982.

[9] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, Cambridge, 1985.

[10] G.K. Pedersen, Analysis Now, Grad. Texts in Math., vol. 118, Springer-Verlag, New York, 1989.

[11] M. Putinar, H.S. Shapiro, The Friedrichs operator of a planar domain, in: Complex Analysis, Operators, and Related Topics, in: Oper. Theory Adv. Appl., vol. 113, Birkhäuser, Basel, 2000, pp. 303–330.

[12] M. Putinar, H.S. Shapiro, The Friedrichs operator of a planar domain II, in: Recent Advances in Operator Theory and Related Topics, Szeged, 1999, in: Oper. Theory Adv. Appl., vol. 127, Birkhäuser, Basel, 2001, pp. 519–551.