THE NORM OF A TRUNCATED TOEPLITZ OPERATOR

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ABSTRACT. We prove several lower bounds for the norm of a truncated Toeplitz operator and obtain a curious relationship between the H^2 and H^∞ norms of functions in model spaces.

1. INTRODUCTION

In this paper, we continue the discussion initiated in [6] concerning the norm of a truncated Toeplitz operator. In the following, let H^2 denote the classical Hardy space of the open unit disk $\mathbb{D}$ and $K_\Theta := H^2 \cap (\Theta H^2)^\perp$, where Θ is an inner function, denote one of the so-called Jordan model spaces [2, 4, 7]. If H^∞ is the set of all bounded analytic functions on $\mathbb{D}$, the space $K_\Theta^\infty := H^\infty \cap K_\Theta$ is norm dense in K_Θ (see [2, p. 83] or [9, Lem. 2.3]). If P_Θ is the orthogonal projection from $L^2 := L^2(\partial\mathbb{D}, \frac{|d\zeta|}{2\pi})$ onto K_Θ and $\varphi \in L^2$, then the operator

$$A_\varphi f := P_\Theta(\varphi f), \quad f \in K_\Theta^\infty,$$

is densely defined on K_Θ and is called a *truncated Toeplitz operator*. Various aspects of these operators were studied in [3, 5, 6, 9, 10].

If $\|\cdot\|$ is the norm on L^2, we let

$$\|A_\varphi\| := \sup\{\|A_\varphi f\| : f \in K_\Theta^\infty, \|f\| = 1\} \tag{1}$$

and note that this quantity is finite if and only if A_φ extends to a bounded operator on K_Θ. For $\varphi \in L^\infty$, the set of bounded measurable functions on $\partial\mathbb{D}$, we have the basic estimates

$$0 \leq \|A_\varphi\| \leq \|\varphi\|_\infty.$$

However, it is known that equality can hold, in nontrivial ways, in either of the inequalities above and hence finding the norm of a truncated Toeplitz operator can be difficult. Furthermore, it turns out that there are many unbounded symbols $\varphi \in L^2$ which yield bounded operators A_φ. Unlike the situation for classical Toeplitz operators on H^2, for a given $\varphi \in L^2$, there many $\psi \in L^2$ for which $A_\varphi = A_\psi$ [9, Thm. 3.1].

For a given symbol $\varphi \in L^2$ and inner function Θ, lower bounds on the quantity (1) are useful in answering the following nontrivial questions:

(i) is A_φ unbounded?

(ii) if $\varphi \in L^\infty$, is A_φ norm-attaining (i.e., is $\|A_\varphi\| = \|\varphi\|_\infty$)?

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When Θ is a finite Blaschke product and $\varphi \in H^\infty$, the paper [6] computes $\|A_\varphi\|$ and gives necessary and sufficient conditions as to when $\|A_\varphi\| = \|\varphi\|_\infty$. The purpose of this short note is to give a few lower bounds on $\|A_\varphi\|$ for general inner functions Θ and general $\varphi \in L^2$. Along the way, we obtain a curious relationship (Corollary 5) between the H^2 and H^∞ norms of functions in K_Θ^∞.

2. Lower bounds derived from Poisson's formula

For $\varphi \in L^2$, let

$$(\mathfrak{P}\varphi)(z) := \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2} \varphi(\zeta) \frac{|d\zeta|}{2\pi}, \quad z \in \mathbb{D}, \tag{2}$$

be the standard Poisson extension of φ to $\mathbb{D}$. For $\varphi \in C(\partial\mathbb{D})$, the continuous functions on $\partial\mathbb{D}$, recall that $\mathfrak{P}\varphi$ solves the classical Dirichlet problem with boundary data φ. Also note that

$$k_\lambda(z) := \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda} z}, \quad \lambda, z \in \mathbb{D},$$

is the reproducing kernel for K_Θ [9].

Our first result provides a general lower bound for $\|A_\varphi\|$ which yields a number of useful corollaries:

Theorem 1. *If $\varphi \in L^2$, then*

$$\sup_{\lambda \in \mathbb{D}} \frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2} \left| \int_{\partial\mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi} \right| \leq \|A_\varphi\|. \tag{3}$$

In other words,

$$\sup_{\lambda \in \mathbb{D}} \left| \int_{\partial\mathbb{D}} \varphi(z) d\nu_\lambda(z) \right| \leq \|A_\varphi\|$$

where

$$d\nu_\lambda(z) := \frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2} \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi}$$

is a family of probability measures on $\partial\mathbb{D}$ indexed by $\lambda \in \mathbb{D}$.

Proof. For $\lambda \in \mathbb{D}$ we have

$$\|k_\lambda\| = \sqrt{\frac{1 - |\Theta(\lambda)|^2}{1 - |\lambda|^2}}, \tag{4}$$

from which it follows that

$$\begin{aligned}
\|A_\varphi\| &\geq \frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2} |\langle A_\varphi k_\lambda, k_\lambda \rangle| \\
&= \frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2} |\langle P_\Theta \varphi k_\lambda, k_\lambda \rangle| \\
&= \frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2} |\langle \varphi k_\lambda, k_\lambda \rangle| \\
&= \frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2} \left| \int_{\partial\mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi} \right|.
\end{aligned}$$

That the measures $d\nu_\lambda$ are indeed probability measures follows from (4). $\square$

Now observe that if $\Theta(\lambda) = 0$, the argument in the supremum on the left hand side of (3) becomes the absolute value of the expression in (2). This immediately yields the following corollary:

Corollary 1. *If $\varphi \in L^2$, then*

$$\sup_{\lambda \in \Theta^{-1}(\{0\})} |(\mathfrak{P}\varphi)(\lambda)| \leq \|A_\varphi\|, \tag{5}$$

where the supremum is to be regarded as 0 if $\Theta^{-1}(\{0\}) = \varnothing$.

Under the right circumstances, the preceding corollary can be used to prove that certain truncated Toeplitz operators are norm-attaining:

Corollary 2. *Let Θ be an inner function having zeros which accumulate at every point of $\partial\mathbb{D}$. If $\varphi \in C(\partial\mathbb{D})$ then $\|A_\varphi\| = \|\varphi\|_\infty$.*

Proof. Let $\zeta \in \partial\mathbb{D}$ be such that $|\varphi(\zeta)| = \|\varphi\|_\infty$. By hypothesis, there exists a sequence λ_n of zeros of Θ which converge to ζ. By continuity, we conclude that

$$\|\varphi\|_\infty = \lim_{n\to\infty} |(\mathfrak{P}\varphi)(\lambda_n)| \leq \|A_\varphi\| \leq \|\varphi\|_\infty$$

whence $\|A_\varphi\| = \|\varphi\|_\infty$. □

The preceding corollary stands in contrast to the finite Blaschke product setting. Indeed, if Θ is a finite Blaschke product and $\varphi \in H^\infty$, then it is known that $\|A_\varphi\| = \|\varphi\|_\infty$ if and only if φ is the scalar multiple of the inner factor of some function from K_Θ [6, Thm. 2].

At the expense of wordiness, the hypothesis of Corollary 2 can be considerably weakened. A cursory examination of the proof indicates that we only need ζ to be a limit point of the zeros of Θ, $\varphi \in L^\infty$ to be continuous on an open arc containing ζ, and $|\varphi(\zeta)| = \|\varphi\|_\infty$.

Theorem 1 yields yet another lower bound for $\|A_\varphi\|$. Recall that an inner function Θ has a finite angular derivative at $\zeta \in \partial\mathbb{D}$ if Θ has a non-tangential limit $\Theta(\zeta)$ of modulus one at ζ and Θ' has a finite non-tangential limit $\Theta'(\zeta)$ at ζ. This is equivalent to asserting that

$$\frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \tag{6}$$

has the non-tangential limit $\Theta'(\zeta)$ at ζ. If Θ has a finite angular derivative at ζ, then the function in (6) belongs to H^2 and

$$\lim_{r\to 1^-} \int_{\partial\mathbb{D}} \left| \frac{\Theta(z) - \Theta(r\zeta)}{z - r\zeta} \right|^2 \frac{|dz|}{2\pi} = \int_{\partial\mathbb{D}} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi}.$$

Furthermore, the above is equal to

$$\lim_{r\to 1^-} \frac{1 - |\Theta(r\zeta)|^2}{1 - r^2} = |\Theta'(\zeta)| > 0.$$

See [1, 8] for further details on angular derivatives. Theorem 1 along with the preceding discussion and Fatou's lemma yield the following lower estimate for $\|A_\varphi\|$.

Corollary 3. *For an inner function Θ, let D_Θ be the set of $\zeta \in \partial\mathbb{D}$ for which Θ has a finite angular derivative $\Theta'(\zeta)$ at ζ. If $\varphi \in L^\infty$ or if $\varphi \in L^2$ with $\varphi \geq 0$, then*

$$\sup_{\zeta \in D_\Theta} \frac{1}{|\Theta'(\zeta)|} \left| \int_{\partial\mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi} \right| \leq \|A_\varphi\|.$$

In other words,

$$\sup_{\zeta \in D_\Theta} \left| \int_{\partial \mathbb{D}} \varphi(z) d\nu_\lambda(z) \right| \leq \|A_\varphi\|,$$

where

$$d\nu_\lambda(z) := \frac{1}{|\Theta'(\zeta)|} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi}$$

is a family of probability measures on $\partial\mathbb{D}$ indexed by $\zeta \in D_\Theta$.

3. Lower bounds and projections

Our next several results concern lower bounds on $\|A_\varphi\|$ involving the orthogonal projection $P_\Theta : L^2 \to K_\Theta$.

Theorem 2. *If Θ is an inner function and $\varphi \in L^2$, then*

$$\frac{\|P_\Theta(\varphi) - \overline{\Theta(0)} P_\Theta(\Theta\varphi)\|}{(1 - |\Theta(0)|^2)^{\frac{1}{2}}} \leq \|A_\varphi\|.$$

Proof. First observe that $\|k_0\| = (1 - |\Theta(0)|^2)^{\frac{1}{2}}$. Next we see that if $\varphi \in L^2$ and $g \in K_\Theta$ is any unit vector, then

$$\begin{aligned} (1 - |\Theta(0)|^2)^{\frac{1}{2}} \|A_\varphi\| &\geq |\langle A_\varphi k_0, g \rangle| \\ &= |\langle P_\Theta(\varphi k_0), g \rangle| \\ &= |\langle P_\Theta(\varphi) - \overline{\Theta(0)} P_\Theta(\Theta\varphi), g \rangle|. \end{aligned}$$

Setting

$$g = \frac{P_\Theta(\varphi) - \overline{\Theta(0)} P_\Theta(\Theta\varphi)}{\|P_\Theta(\varphi) - \overline{\Theta(0)} P_\Theta(\Theta\varphi)\|}$$

yields the desired inequality. □

In light of the fact that $P_\Theta(\Theta\varphi) = 0$ whenever $\varphi \in H^2$, Theorem 2 leads us immediately to the following corollary:

Corollary 4. *If Θ is inner and $\varphi \in H^2$, then*

$$\frac{\|P_\Theta(\varphi)\|}{(1 - |\Theta(0)|^2)^{1/2}} \leq \|A_\varphi\|. \tag{7}$$

It turns out that (7) has a rather interesting function-theoretic implication. Let us first note that for $\varphi \in H^\infty$, we can expect no better inequality than

$$\|\varphi\| \leq \|\varphi\|_\infty$$

(with equality holding if and only if φ is a scalar multiple of an inner function). However, if φ belongs to K_Θ^∞, then a stronger inequality holds.

Corollary 5. *If Θ is an inner function, then*

$$\|\varphi\| \leq (1 - |\Theta(0)|^2)^{\frac{1}{2}} \|\varphi\|_\infty \tag{8}$$

holds for all $\varphi \in K_\Theta^\infty$. If Θ is a finite Blaschke product, then equality holds if and only if φ is a scalar multiple of an inner function from K_Θ.

Proof. First observe that the inequality

$$\|\varphi\| \leq (1 - |\Theta(0)|^2)^{\frac{1}{2}} \|\varphi\|_\infty$$

follows from Corollary 4 and the fact that $P_\Theta \varphi = \varphi$ whenever $\varphi \in K_\Theta$. Now suppose that Θ is a finite Blaschke product and assume that equality holds in the preceding for some $\varphi \in K_\Theta^\infty$. In light of (7), it follows that $\|A_\varphi\| = \|\varphi\|_\infty$. From [6, Thm. 2] we see that φ must be a scalar multiple of the inner *part* of a function from K_Θ. But since $\varphi \in K_\Theta^\infty$, then φ must be a scalar multiple of an inner function from K_Θ. □

When Θ is a finite Blaschke product, then K_Θ is a finite dimensional subspace of H^2 consisting of bounded functions [3, 5, 9]. By elementary functional analysis, there are $c_1, c_2 > 0$ so that

$$c_1 \|\varphi\| \leq \|\varphi\|_\infty \leq c_2 \|\varphi\|$$

for all $\varphi \in K_\Theta$. This prompts the following question:

Question. What are the optimal constants c_1, c_2 in the above inequality?

4. Lower bounds from the decomposition of K_Θ

A result of Sarason [9, Thm. 3.1] says, for $\varphi \in L^2$, that

$$A_\varphi \equiv 0 \Leftrightarrow \varphi \in \Theta H^2 + \overline{\Theta H^2}. \tag{9}$$

It follows that the most general truncated Toeplitz operator on K_Θ is of the form $A_{\psi + \overline{\chi}}$ where $\psi, \chi \in K_\Theta$. We can refine this observation a bit further and provide another canonical decomposition for the symbol of a truncated Toeplitz operator.

Lemma 1. *Each bounded truncated Toeplitz operator on K_Θ is generated by a symbol of the form*

$$\varphi = \underbrace{\psi}_{\in H^2} + \underbrace{\chi\overline{\Theta}}_{\in \overline{zH^2}} \tag{10}$$

where $\psi, \chi \in K_\Theta$.

Before getting to the proof, we should remind the reader of a technical detail. It follows from the identity $K_\Theta = H^2 \cap \Theta\overline{zH^2}$ (see [2, p. 82]) that

$$C : K_\Theta \to K_\Theta, \quad Cf := \overline{zf}\Theta,$$

is an isometric, conjugate-linear, involution. In fact, when A_φ is a bounded operator we have the identity $CA_\varphi C = A_\varphi^*$ [9, Lemma 2.1].

Proof of Lemma 1. If T is a bounded truncated Toeplitz operator on K_Θ, then there exists some $\varphi \in L^2$ such that $T = A_\varphi$. We claim that this φ can be chosen to have the special form (10). First let us write $\varphi = f + \overline{zg}$ where $f, g \in H^2$. Using the orthogonal decomposition $H^2 = K_\Theta \oplus \Theta H^2$, it follows that φ may be further decomposed as

$$\varphi = (f_1 + \Theta f_2) + \overline{z(g_1 + \Theta g_2)}$$

where $f_1, g_1 \in K_\Theta$ and $f_2, g_2 \in H^2$. By (9), the symbols Θf_2 and $\overline{\Theta(zg_2)}$ yield the zero truncated Toeplitz operator on K_Θ. Therefore we may assume that

$$\varphi = f + \overline{zg}$$

for some $f, g \in K_\Theta$. Since $Cg = \overline{gz}\Theta$, we have $\overline{zg} = (Cg)\overline{\Theta}$ and hence (10) holds with $\psi = f$ and $\chi = Cg$. □

Corollary 6. *Let Θ be an inner function. If $\psi_1, \psi_2 \in K_\Theta$ and $\varphi = \psi_1 + \psi_2\overline{\Theta}$, then*

$$\frac{\|\psi_1 - \overline{\Theta(0)}\psi_2\|}{(1 - |\Theta(0)|^2)^{\frac{1}{2}}} \leq \|A_\varphi\|.$$

Proof. If $\varphi = \psi_1 + \psi_2\overline{\Theta}$, then, since $\psi_1, \psi_2 \in K_\Theta$ and $\psi_2\overline{\Theta} \in \overline{zH^2}$, we have

$$P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta\varphi) = \psi_1 - \overline{\Theta(0)}\psi_2.$$

The result now follows from Theorem 2. □

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