Real Outer Functions

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ABSTRACT. A product representation is obtained for outer functions in the unit disk whose boundary functions are real valued. Our treatment is based on work of A.B. Aleksandrov and A.G. Poltoratski.

1. INTRODUCTION

In the realm of functions of bounded characteristic, inner functions and outer functions play unique roles. In many respects functions of these two kinds exhibit opposite behaviors, yet there are subtle connections between them. We are concerned here with one such connection.

We shall be working with holomorphic and harmonic functions in the unit disk, $\mathbb{D}$, and with their boundary functions on the unit circle, $\partial\mathbb{D}$. Most often we shall not distinguish between a function in $\mathbb{D}$ and its boundary function; the context will make our meaning clear. We endow $\partial\mathbb{D}$ with normalized Lebesgue measure, and we denote the measure of a set E by $|E|$.

An inner function is a function in H^∞ whose boundary function has unit modulus almost everywhere. An outer function is one which, on $\partial\mathbb{D}$, has the form $f = e^{u+i\tilde{u}+ic}$ where c is a real constant, u is a real-valued function in L^1 (of $\partial\mathbb{D}$), and $\tilde{u}$ is the conjugate function of u. Thus, the function f is given in $\mathbb{D}$ by

$$f(z) = e^{ic}\exp\left[\frac{1}{2\pi}\int_{\partial\mathbb{D}}\frac{e^{i\theta}+z}{e^{i\theta}-z}u(e^{i\theta})\,d\theta\right];$$

the expression in square brackets is the Herglotz integral of u.

We let RO denote the class of outer functions whose boundary functions are real valued. Thus, the function $f = e^{u+i\tilde{u}+ic}$ lies in RO if and only if the values of $\tilde{u}+c$ lie in $\pi\mathbb{Z}$. We let PO denote the subclass of functions in RO whose boundary functions are positive, in other words, the squares of functions in RO.

Interest in PO has been spurred by recent attempts to understand the structure of so-called rigid functions in the space H^1. A rigid function in H^1 is one that is determined by its argument on $\partial\mathbb{D}$, in the sense that the only other functions

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in H^1 having the same argument as it on $\partial\mathbb{D}$ are the positive scalar multiples of itself. All nonouter functions in H^1 are nonrigid, and one can show that an outer function is nonrigid if and only if there is a nonconstant function in PO that multiplies it into H^1. At present there exists no simple structural characterization of rigid functions.

In connection with rigid functions, functions in RO have been considered by H. Helson [2] and A.G. Poltoratski [5]. Helson showed how to represent each function in RO by means of a pair of inner functions, which led to a corresponding characterization of rigid functions. Poltoratski introduced basic building blocks of the form $i((1+\varphi)/(1-\varphi))$, with φ an inner function. He observed that every function in RO whose argument is bounded is a finite product of such functions, to within a real scalar multiple, and he used certain of these products to construct new kinds of nonrigid functions. Earlier, J. Inoue [3] had used an infinite product of these functions in order to refute a conjectured characterization of rigid functions; a variant of his construction can be found in [5]

In this note we shall extend Poltoratski's basic insight by showing that a function in RO with an unbounded argument can be expressed as an infinite product involving his basic building blocks. We shall review in Section 2 how the building blocks arise (slightly altering the notation for them used above). In establishing the convergence of our infinite products, we rely on A.B. Aleksandrov's Cauchy A-integral representation [1]. Actually, to obtain local uniform as opposed to just pointwise convergence, we need an extension of Aleksandrov's basic result. Our main lemma, a kind of locally uniform Herglotz A-integral representation, is stated in Section 5, preceded by some introductory material. The proof of the lemma, which follows Aleksandrov's approach, is given in Section 7. Section 6 contains the statement and proof of our product representation, and a discussion of certain issues related to it. Section 4 briefly mentions finite products, and Section 3 contains a few remarks on Helson's representation. The concluding Section 8 pertains to the absolute convergence of our infinite products, something about which our understanding is fragmentary.

2. BUILDING BLOCKS

As noted in Section 1, with slightly different notation, the general function f in RO has the form $f = \exp[\pi(u + i\tilde{u} + ic)]$, where u is a real function in L^1, c is a real constant, and the values of the function $v = \tilde{u} + c$ on $\partial\mathbb{D}$ lie in $\mathbb{Z}$. If the conjugate function $\tilde{u}$ happens also to be in L^1, we can rewrite f as $f = |f(0)|\exp[\pi(-\tilde{v} + iv)]$. The simplest case is where v takes only the values 0 and 1, which brings us to our basic building blocks.

Definition 2.1. For E a measurable subset of $\partial\mathbb{D}$, we let $f_E = \exp[\pi(-\tilde{\chi}_E + i\chi_E)]$.

We note that in the two extreme cases $E = \emptyset$ and $E = \partial\mathbb{D}$, the function f_E is constant (1 and -1, respectively). Otherwise f_E is a nonconstant outer function whose argument on $\partial\mathbb{D}$ takes only the values 0 and π, the most general such function to within a positive multiplicative constant. We have $f_E(0) = e^{\pi i|E|}$.

The preceding discussion and the ensuing one paraphrase a portion of Poltoratski's paper [5]. Poltoratski noted that the functions f_E have simple expressions in terms of inner functions. It will be convenient to employ the linear-fractional map $T(z) = i((1 - iz)/(1 + iz))$, whose inverse is given by $T^{-1}(z) = i((z - i)/(z + i))$. One easily sees that T^{-1} maps the upper half-plane to the unit disk and the real axis to the unit circle. In $\mathbb{D}$, the argument of the function f_E lies between 0 and π, so f_E maps $\mathbb{D}$ to the upper half-plane. Hence $T^{-1}(f_E)$ maps $\mathbb{D}$ to $\mathbb{D}$, and its boundary values have unit modulus because f_E is real on $\partial\mathbb{D}$. In other words, $T^{-1}(f_E)$ is an inner function; we denote it by φ_E. Thus $f_E = T(\varphi_E) = i((1 - i\varphi_E)/(1 + i\varphi_E))$.

The map T^{-1} sends the positive real axis to the right half of $\partial\mathbb{D}$ and the negative real axis to the left half of $\partial\mathbb{D}$. Thus φ_E takes values in the left half of $\partial\mathbb{D}$ on E and in the right half of $\partial\mathbb{D}$ on $\partial\mathbb{D}\backslash E$. The set E is thus recaptured as the inverse image under φ_E of the left half of $\partial\mathbb{D}$. A calculation shows that φ_E is real at the origin; precisely,

$$\varphi_E(0) = \tan\left[\frac{\pi}{2}\left(\frac{1}{2} - |E|\right)\right].$$

Consider, on the other hand, any nonconstant inner function φ, and let $f = T(\varphi)$. The map T sends the unit disk to the upper half-plane and the unit circle to the real axis, so f is a function in RO whose argument on $\partial\mathbb{D}$ takes only the values 0 and π. From the way the functions f_E were defined, we see that f is one of them if and only if it is unimodular at the origin, which happens if and only if $\varphi(0)$ is real. The inner functions φ_E are thus precisely the inner functions that are real at the origin.

Example 2.2. Let α and β be real numbers such that $\beta < \alpha < \beta + 2\pi$, and let $E_{\beta,\alpha} = \{e^{i\theta} : \beta < \theta < \alpha\}$, one of the two subarcs of $\partial\mathbb{D}$ with end points $e^{i\beta}$ and $e^{i\alpha}$. The function $f_{\beta,\alpha} = f_{E_{\beta,\alpha}}$ can be expressed by means of the Herglotz integral of $\pi\chi_{E_{\beta,\alpha}}$:

$$f_{\beta,\alpha}(z) = \exp\left[\frac{i}{2}\int_\beta^\alpha \frac{e^{i\theta} + z}{e^{i\theta} - z}\,d\theta\right].$$

After some routine calculations one finds that

$$f_{\beta,\alpha}(z) = e^{-i((\alpha-\beta)/2)}\left(\frac{e^{i\alpha} - z}{e^{i\beta} - z}\right).$$

With further effort one can write down an explicit expression for the corresponding inner function $\varphi_{E_{\beta,\alpha}}$, but we shall refrain from that; it is the conformal automorphism of $\mathbb{D}$ that maps $E_{\beta,\alpha}$ onto the left half of $\partial\mathbb{D}$ and is real at the origin. Its zero is at the point $e^{(i/2)(\alpha+\beta)}\tan[\frac{1}{4}(\pi - (\alpha - \beta))]$.

Example 2.3. Let φ_1 and φ_2 be nonconstant inner functions, and let $f_1 = T(\varphi_1)$ and $f_2 = T(\varphi_2)$. The function $f_1 + f_2$ is a nonconstant function in RO whose argument on $\partial\mathbb{D}$ assumes only the values 0 and π. Hence there exists an inner function φ such that $f_1 + f_2 = T(\varphi)$. A computation shows that φ is given by the formula

$$\varphi = \frac{3i\varphi_1\varphi_2 + \varphi_1 + \varphi_2 + i}{3 + i\varphi_1 + i\varphi_2 + \varphi_1\varphi_2}.$$

The unusual presence of the number 3 in the denominator of the preceding equation is easily explained. The equation $3 + i\varphi_1 + i\varphi_2 + \varphi_1\varphi_2 = (1 + i\varphi_1) + (1 + i\varphi_2) + (1 + \varphi_1\varphi_2)$ exhibits the denominator as the sum of three outer functions which assume values in the right half-plane. In particular, the denominator $3 + i\varphi_1 + i\varphi_2 + \varphi_1\varphi_2$ is outer and thus φ is precisely the inner factor of the numerator $3i\varphi_1\varphi_2 + \varphi_1 + \varphi_2 + i$.

3. HELSON'S REPRESENTATION

Before taking up product representations of functions in RO, we mention briefly the representation of Helson [2]. Suppose to start that f is any function of bounded characteristic whose boundary function is real valued. Then $T^{-1}(f)$ is a function of bounded characteristic (being a rational function of a function of bounded characteristic) and it has a unimodular boundary function, so it is the ratio of two inner functions. We accordingly write $T^{-1}(f) = \psi_1/\psi_2$ where ψ_1 and ψ_2 are relatively prime inner functions; they are unique to within reciprocal multiplicative constants of unit modulus. This gives

$$f = T\left(\frac{\psi_1}{\psi_2}\right) = i\left(\frac{\psi_2 - i\psi_1}{\psi_2 + i\psi_1}\right).$$

The function f is in the Smirnov class if and only if $\psi_2 + i\psi_1$ is outer, and it is an outer function if and only if $\psi_2 - i\psi_1$ and $\psi_2 + i\psi_1$ are both outer. As is easily seen, the last requirement is equivalent to the requirement that $\psi_1^2 + \psi_2^2$ be outer.

In the converse direction, one easily checks that if ψ_1 and ψ_2 are inner functions such that $\psi_1^2 + \psi_2^2$ is outer, then the expression above determines a function f in RO. This is Helson's representation of functions in RO (with slightly altered notation).

In a handful of cases one can display the functions ψ_1 and ψ_2 explicitly.

Example 3.1. Let $f = f_E$ (E as in Section 2).
In this case it is obvious that $\psi_2 = 1$ and $\psi_1 = \varphi_E$.

Example 3.2. Let $f = f_E^2$.
We have

$$f = -\frac{(1 - i\varphi_E)^2}{(1 + i\varphi_E)^2},$$

so

$$T^{-1}(f) = i\left[\frac{-(1-i\varphi_E)^2 - i(1+i\varphi_E)^2}{-(1-i\varphi_E)^2 + i(1+i\varphi_E)^2}\right].$$

The polynomial $-(1-iz)^2 - i(1+iz)^2$ has roots $\sqrt{2}-1$ and $-(\sqrt{2}+1)$, and the polynomial $-(1-iz)^2 + i(1+iz)^2$ has roots $-(\sqrt{2}-1)$ and $\sqrt{2}+1$, enabling us to write

$$\begin{aligned} T^{-1}(f) &= i\left(\frac{1+i}{1-i}\right)\left[\frac{(\varphi_E - \sqrt{2}+1)(\varphi_E+\sqrt{2}+1)}{(\varphi_E+\sqrt{2}-1)(\varphi_E-\sqrt{2}-1)}\right] \\ &= \frac{\varphi_E - \sqrt{2}+1}{1-(\sqrt{2}-1)\varphi_E} \Big/ \frac{\varphi_E+\sqrt{2}-1}{1+(\sqrt{2}-1)\varphi_E}. \end{aligned}$$

The numerator in the last expression is ψ_1 and the denominator is ψ_2 for this case.

Example 3.3. Let $f = f_E^3$.
Reasoning similar to that in Example 3.2 enables one to show in this case that

$$\psi_1 = \left(\frac{\varphi_E - \frac{1}{\sqrt{3}}}{1 - \frac{\varphi_E}{\sqrt{3}}}\right)\left(\frac{\varphi_E + \frac{1}{\sqrt{3}}}{1 + \frac{\varphi_E}{\sqrt{3}}}\right), \quad \psi_2 = \varphi_E.$$

With additional effort one can similarly treat the case $f = f_E^n$, where n is any positive integer.

4. Finite Products

Continuing to follow Poltoratski [5], we note that any function f in RO whose argument is bounded can be expressed to within a positive multiplicative constant as a finite product of functions of the form $T(\varphi_E)$. In fact, such an f can be written as $f = |f(0)|\exp[\pi(-\tilde{v}+iv)]$, where v is nonnegative, bounded, and integer valued. For each positive integer n let $E_n = \{v \geq n\}$. Then $E_n \supset E_{n+1}$ for all n, and $E_n = \emptyset$ eventually, say for $n > n_0$. We have $v = \sum_{n=1}^{n_0} \chi_{E_n}$, so

$$\exp[\pi(-\tilde{v}+iv)] = \prod_{n=1}^{n_0} \exp[\pi(-\tilde{\chi}_{E_n} + i\chi_{E_n})] = \prod_{n=1}^{n_0} f_{E_n}.$$

Letting $\varphi_n = \varphi_{E_n}$, we get

$$f = |f(0)|\prod_{n=1}^{n_0} T(\varphi_n) = |f(0)|\prod_{n=1}^{n_0} i\left(\frac{1-i\varphi_n}{1+i\varphi_n}\right).$$

5. THE CAUCHY A-INTEGRAL

We now state Aleksandrov's result from [1] on Cauchy A-integral representations, and the variant of it we shall be using. The distribution function of a measurable function h on $\partial\mathbb{D}$ will be denoted by λ_h; in other words $\lambda_h(t) = |\{|h| > t\}|$ for $t > 0$. We recall that h is said to belong to the space $L_0^{1,\infty}$ if $\lambda_h(t) = o(\frac{1}{t})$ as $t \to \infty$. In particular, functions in L^1 and conjugates of functions in L^1 are in $L_0^{1,\infty}$.

The function h is said to be A-integrable if it belongs to $L_0^{1,\infty}$ and

$$\lim_{A\to+\infty} \int_{\{|h|\le A\}} h(e^{i\theta})\, d\theta$$

exists. The preceding limit is then called the A-integral of h over $\partial\mathbb{D}$ and denoted by $(A)\int_{\partial\mathbb{D}} h(e^{i\theta})d\theta$.

A holomorphic function in $\mathbb{D}$ is said to belong to the space $H_0^{1,\infty}$ if it is in the Smirnov class and its boundary function is in $L_0^{1,\infty}$. Aleksandrov's result is that such a function is the Cauchy A-integral of its boundary function: if h is in $H_0^{1,\infty}$ then, for z in $\mathbb{D}$,

$$h(z) = \frac{1}{2\pi}(A)\int_{\partial\mathbb{D}} \frac{h(e^{i\theta})}{1 - e^{-i\theta}z}\, d\theta.$$

Implicit in Aleksandrov's proof of this is an estimate of the rate of convergence of the A-integral on the right side. We shall need such an estimate to establish the local uniform convergence of our product representation for functions in RO. We have the following lemma.

Lemma 5.1. *Let $h = u + iv$ be a function in $H_0^{1,\infty}$, with $u(0) = 0$. For $A > 0$ let*

$$v_A = \begin{cases} v & \text{where } |v| \le A \\ A & \text{where } v > A \\ -A & \text{where } v < -A. \end{cases}$$

Then

$$h(z) = \lim_{A\to+\infty} \frac{i}{2\pi} \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z} v_A(e^{i\theta})\, d\theta,$$

the limit existing uniformly on compact subsets of $\mathbb{D}$.

The proof is deferred to Section 7.

6. INFINITE PRODUCTS

We consider now a function $f = \exp[\pi(u + iv)]$ in RO such that v is unbounded. For n a positive integer we define

$$E_n^+ = \{v \ge n\}, \quad E_n^- = \{v \le -n\},$$

and

$$f_n^+ = f_{E_n^+}, \quad \varphi_n^+ = \varphi_{E_n^+}, \quad f_n^- = f_{E_n^-}, \quad \varphi_n^- = \varphi_{E_n^-}.$$

Theorem 6.1.

$$\begin{aligned} f &= |f(0)| \prod_{n=1}^{\infty} \frac{T(\varphi_n^+)}{T(\varphi_n^-)} \\ &= |f(0)| \prod_{n=1}^{\infty} \frac{(1 - i\varphi_n^+)(1 + i\varphi_n^-)}{(1 + i\varphi_n^+)(1 - i\varphi_n^-)}, \end{aligned}$$

with the product converging uniformly on compact subsets of $\mathbb{D}$.

Proof. For the proof we can assume with no loss of generality that $|f(0)| = 1$. Applying Lemma 5.1 to the function $h = u + iv$, we conclude that

$$h = \sum_{n=1}^{\infty} (-\tilde{\chi}_{E_n^+} + i\chi_{E_n^+} + \tilde{\chi}_{E_n^-} - i\chi_{E_n^-}),$$

the harmonic extension of the series converging uniformly on compact subsets of $\mathbb{D}$. Therefore, since $f_E = \exp[\pi(-\tilde{\chi}_E + i\chi_E)]$, we have

$$f = e^{\pi h} = \prod_{n=1}^{\infty} f_n^+ / f_n^-,$$

the product converging uniformly on compact subsets of $\mathbb{D}$, which is the desired result. □

A couple of questions arise about the product in the theorem. First, must the two products $\prod_{n=1}^{\infty} f_n^+$ and $\prod_{n=1}^{\infty} f_n^-$ converge separately? We shall see that the answer is no. Second, if the two separate products do converge, must they represent functions in RO? Again, we shall see that the answer is no.

We recall that an infinite product $\prod_{n=1}^{\infty} c_n$ ($c_n \in \mathbb{C}$) is said to converge absolutely if the c_n are all nonzero and $\sum_{n=1}^{\infty} |\operatorname{Log} c_n| < \infty$ (Log denotes the principal branch of log), or if finitely many of the c_n are 0 and the preceding condition holds after the zero terms have been deleted. An equivalent condition is $\sum_{n=1}^{\infty} |1 - c_n| < \infty$.

Lemma 6.2. *Let $c_1, c_2, \dots$ be complex numbers, all different from i. Then the product $\prod_{n=1}^{\infty} c_n$ converges absolutely if and only if the product $\prod_{n=1}^{\infty} T(c_n)$ converges absolutely.*

Proof. This follows immediately from the equality

$$1 - T(c_n) = \frac{(1 - i)(1 - c_n)}{1 + ic_n}.$$

□

Proposition 6.3. *Let $E_1, E_2, \dots$ be measurable subsets of $\partial\mathbb{D}$, $f_n = f_{E_n}$, $\varphi_n = \varphi_{E_n}$. The following conditions are equivalent.*

(i) *The product $\prod_{n=1}^{\infty} \varphi_n(0)$ converges.*
(ii) *The product $\prod_{n=1}^{\infty} \varphi_n$ converges locally uniformly in $\mathbb{D}$.*
(iii) *The product $\prod_{n=1}^{\infty} f_n(0)$ converges.*
(iv) *The product $\prod_{n=1}^{\infty} f_n$ converges locally uniformly in $\mathbb{D}$.*

The convergence in all cases is absolute.

Proof. Because the values $\varphi_n(0)$ lie in $(-1, 1)$, convergence of the product in (i) implies $\varphi_n(0) \nearrow 1$, from which it is clear that the convergence must be absolute. Similarly, since $f_n(0) = e^{\pi i |E_n|}$, the product in (iii) converges if and only if $\sum_{n=1}^{\infty} |E_n| < \infty$, which means the convergence must be absolute.

The implication (i) ⇒ (ii) is well known. Namely, for z in $\mathbb{D}$ Schwarz's lemma gives

$$|\varphi(z) - \varphi(0)| \le |z| \left| 1 - \overline{\varphi(0)}\varphi(z) \right|,$$

which can be rewritten

$$|(1 - \varphi(0)) - (1 - \varphi(z))| \le |z| \left| (1 - \varphi(z)) + \varphi(z)(1 - \overline{\varphi(0)}) \right|.$$

From this one readily deduces the inequality

$$|1 - \varphi(z)| \le \left(\frac{1 + |z|}{1 - |z|} \right) |1 - \varphi(0)|,$$

which gives the implication (i) ⇒ (ii), with absolute convergence.

The implication (i) ⇒ (iii) follows by Lemma 6.2, as does the implication (ii) ⇒ (iv) (in view of the absoluteness of the convergence in (i) and (ii)). Finally, if (iii) holds then so does (i), by Lemma 6.2, hence so does (ii), hence so does (iv). □

We return now to our function $f = \exp[\pi(u + iv)]$ from the theorem. In the infinite product representing f, the nth factor has the value $\exp[\pi i(|E_n^+| - |E_n^-|)]$ at the origin. From the convergence of the product at 0 we conclude that the series $\sum_{n=1}^{\infty} (|E_n^+| - |E_n^-|)$ converges.

Now suppose that v is not integrable, which means that $\sum_{n=1}^{\infty} (|E_n^+| + |E_n^-|) = \infty$. Then both series $\sum_{n=1}^{\infty} |E_n^+|$ and $\sum_{n=1}^{\infty} |E_n^-|$ diverge, and it follows that both products $\prod_{n=1}^{\infty} f_n^+$ and $\prod_{n=1}^{\infty} f_n^-$ diverge at 0. This answers the first question raised above, except for producing an example with v not in L^1. To get such an example, take any real L^1 function u_0 whose conjugate $\tilde{u}_0$ is not in L^1. Take an integer-valued function v such that the difference $w = \tilde{u}_0 - v$ is bounded. Since w is bounded, its conjugate $\tilde{w}$ is in L^1, so $u = u_0 + \tilde{w}$ is in L^1. We have $\tilde{u} = \tilde{u}_0 - w + \text{const.} = v + \text{const.}$, so $\exp[\pi(u + iv)]$ is in RO, with v not in L^1.

Suppose on the other hand that v is in L^1 but not in $L \log L$. Then at least one of the two functions $v_+ = \max\{v, 0\}$ and $v_- = -\min\{v, 0\}$ is not in $L \log L$, so, by a theorem of M. Riesz [4, p.97], at least one of the functions $\tilde{v}_+$ and $\tilde{v}_-$ is not in L^1. But $\tilde{v}_+ - \tilde{v}_- = -u + \text{const.}$ is in L^1, so in fact neither $\tilde{v}_+$ nor $\tilde{v}_-$ is in L^1. (Then, by a Theorem of A. Zygmund [4, p.96], both v_+ and v_- fail to be in $L \log L$.) Since v_+ and v_- are however in L^1, the preceding proposition tells us that both products $\prod_{n=1}^{\infty} f_n^+$ and $\prod_{n=1}^{\infty} f_n^-$ converge. The respective limits are $\exp[\pi(-\tilde{v}_+ + iv_+)]$ and $\exp[\pi(-\tilde{v}_- + iv_-)]$, which fail to be of bounded characteristic since the logarithms of their absolute values are not integrable on $\partial\mathbb{D}$.

To produce an example where v is in L^1 but not in $L \log L$, one can proceed as in the preceding example, starting this time with a function u_0 in L^1 such that $\tilde{u}_0$ is in L^1 but not in $L \log L$, then perturbing $\tilde{u}_0$ by a bounded function to get an integer-valued function. A standard way to produce such a u_0 is to take a function g in H^1 such that $|g|$ is not in $L \log L$, an easy matter. Either $\operatorname{Re} g$ or $\operatorname{Im} g$, possibly both, can serve as u_0. This answers the second question raised above.

7. Proof of Lemma 5.1

The basic ingredients of the proof are from [1]. We are given a function $h = u + iv$ in $H_0^{1,\infty}$ with $u(0) = 0$. We can also assume, with no loss of generality, that $v(0) = 0$, so that we have $h(0) = 0$. We define

$$\rho_h(t) = t\lambda_h(t) \quad (t > 0),$$

$$\sigma_h(A) = \sup_{t \geq A} \rho(t) \quad (A > 0).$$

Thus, both ρ_h and σ_h tend to 0 at ∞.

We shall let m denote normalized Lebesgue measure on $\partial\mathbb{D}$.

Step 1. $$\left| \int_{|h| \leq A} h \, dm \right| \leq 2\sqrt{\sigma_h(0)\sigma_h(A)} + \rho_h(A).$$

Proof. Fix $A > 0$. Following Aleksandrov, we introduce the outer function g that is positive at the origin and whose absolute value equals 1 on the subset $\{|h| \leq A\}$ of $\partial\mathbb{D}$ and equals $A/|h|$ on the subset $\{|h| > A\}$. We let $w = \log|g|$, so that $g = e^{w + i\tilde{w}}$. We have

$$\int_{|h| \leq A} h \, dm = \int_{\partial\mathbb{D}} hg \, dm + \int_{|h| \leq A} h(1 - g) \, dm - \int_{|h| > A} hg \, dm.$$

The function hg (which is bounded) vanishes at the origin, so the first integral on the right side equals 0. We denote the second integral by $I(A)$ and the third one by $J(A)$.

To estimate $I(A)$ we use the Cauchy-Schwarz inequality:

$$|I(A)|^2 \leq \left(\int_{|h| \leq A} |h|^2 \, dm \right) \left(\int_{|h| \leq A} |1 - g|^2 \, dm \right).$$

We have

$$\int_{|h|\le A} |h|^2\,dm = 2\int_0^A t\lambda_h(t)\,dt \le 2\sigma_h(0)A.$$

On the set $\{|h| \le A\}$ we have $w = 0$, so that $|1-g| = |1-e^{i\tilde{w}}| \le |\tilde{w}|$. Thus

$$\begin{aligned}\int_{|h|\le A} |1-g|^2\,dm &\le \int_{\partial\mathbb{D}} |\tilde{w}|^2\,dm\\ &\le \int_{\partial\mathbb{D}} |w|^2\,dm\\ &= \int_{|h|>A} \left(\log\frac{|h|}{A}\right)^2 dm\\ &= -\int_A^\infty \left(\log\frac{t}{A}\right)^2 d\lambda_h(t).\end{aligned}$$

An integration by parts shows that the right side equals

$$\begin{aligned}2\int_A^\infty \frac{\lambda_h(t)\log(t/A)}{t}\,dt &\le 2\sigma_h(A)\int_A^\infty \frac{\log(t/A)}{t^2}\,dt\\ &= \frac{2\sigma_h(A)}{A}\int_1^\infty \frac{\log s}{s^2}\,ds\\ &= \frac{2\sigma_h(A)}{A}.\end{aligned}$$

We obtain

$$|I(A)| \le 2\sqrt{\sigma_h(0)\sigma_h(A)}.$$

As for $J(A)$, we have simply

$$|J(A)| \le \int_{h\ge A} |hg|\,dm = A\lambda_h(A) = \rho_h(A).$$

This completes the proof of Step 1. □

Step 2.

$$\lim_{A\to+\infty} \frac{1}{2\pi}\int_{|h|\le A} \frac{h(e^{i\theta})}{1-e^{-i\theta}z}\,d\theta = h(z)$$

uniformly on compact subsets of $\mathbb{D}$.

Proof. For z in $\mathbb{D}$ define the function h_z by

$$h_z(\zeta) = \frac{\zeta(h(\zeta)-h(z))}{\zeta - z},$$

and note that

$$\int_{|h|\le A} h_z\,dm = \frac{1}{2\pi}\int_{|h|\le A}\frac{h(e^{i\theta})}{1-e^{-i\theta}z}\,d\theta - \frac{1}{2\pi}\int_{|h|\le A}\frac{h(z)}{1-e^{-i\theta}z}\,d\theta.$$

By Step 1 we have

$$\left|\int_{|h_z|\le A} h_z\,dm\right| \le 2\sqrt{\sigma_{h_z}(0)\sigma_{h_z}(A)} + \rho_{h_z}(A). \tag{7.1}$$

We show first that the right side here tends to 0 uniformly on compact subsets of $\mathbb{D}$ as $A \to +\infty$.

Fix $r \in (0,1)$, and let M_r be the maximum of $|h(z)|$ for $|z| \le r$. For $|z| \le r$ we have

$$|h_z| \le \frac{|h| + |h(z)|}{1-|z|} \le \frac{|h| + M_r}{1-r}, \tag{7.2}$$

implying that

$$\lambda_{h_z}(t) \le \lambda_h((1-r)t - M_r).$$

So, for $|z| \le r$ and $t > 2M_r/(1-r)$, we have

$$t\lambda_{h_z}(t) \le \frac{2}{1-r}((1-r)t - M_r)\lambda_h((1-r)t - M_r),$$

implying that

$$\rho_{h_z}(A) \le \sigma_{h_z}(A) \le \frac{2}{1-r}\sigma_h\left(\frac{(1-r)A}{2}\right) \quad (A \ge ((2M_r)/(1-r))$$

and

$$\sigma_{h_z}(0) \le \frac{2}{1-r}\max\{M_r, \sigma_h(M_r)\}.$$

These estimates show that the quantity on the right side of (7.1) tends to 0 uniformly on the disk $|z| \le r$ as $A \to +\infty$, the desired conclusion.

Next, for $A > M_r/(1-r)$ let $A_r = (1-r)A - M_r$. We show that the difference between the two integrals

$$\int_{|h_z|\le A} h_z\,dm \quad \text{and} \quad \int_{|h|\le A_r} h_z\,dm \tag{7.3}$$

tends to 0 uniformly in the disk $|z| \le r$ as $A \to +\infty$. For $|z| \le r$ we have by (7.2) the set inclusion $\{|h| \le A_r\} \subset \{|h_z| \le A\}$, implying that the difference between the two integrals in (7.3) is bounded in absolute value by $(1-r)^{-1}A\lambda_h(A_r)$, which in turn is bounded by

$$\left(\frac{A}{(1-r)A_r}\right)\rho_h(A_r).$$

As $A \to +\infty$ the ratio A/A_r stays bounded, so the preceding quantity tends to 0, giving the desired conclusion.

We now know that $1/(2\pi)\int_{|h|\le A} h_z\,dm \to 0$ uniformly on $|z| \le r$ for each r in $(0,1)$. Finally, the difference between $h(z)$ and

$$\frac{1}{2\pi}\int_{|h|\le A} \frac{h(z)}{1-e^{-i\theta}z}\,d\theta$$

is bounded in absolute value by $(1-|z|)^{-1}|h(z)|\lambda_h(A)$ and so tends to 0 uniformly for $|z| \le r < 1$. This completes the proof of Step 2. □

Step 3.

$$\lim_{A\to+\infty}\frac{1}{2\pi}\int_{|h|\le A}\frac{h(e^{i\theta})}{1-e^{i\theta}\overline{z}}\,d\theta = 0$$

uniformly on compact subsets of $\mathbb{D}$.

Proof. One obtains this by applying Step 1 to the function $(h(\zeta))/(1-\overline{z}\zeta)$ and arguing as in the proof of Step 2. The details are basically the same, even a mite simpler. □

Conclusion of the Proof. From Steps 2 and 3 we have

$$\begin{aligned} h(z) &= \lim_{A\to+\infty}\frac{1}{2\pi}\int_{|h|\le A}\frac{h(e^{i\theta})-\overline{h(e^{i\theta})}}{1-e^{-i\theta}z}\,d\theta \\ &= \lim_{A\to+\infty}\frac{i}{2\pi}\int_{|h|\le A}\frac{2v(e^{i\theta})}{1-e^{-i\theta}z}\,d\theta \end{aligned}$$

uniformly on compact subsets of $\mathbb{D}$. Also, since $v(0)=0$, we know from Step 1 that

$$\lim_{A\to+\infty}\frac{i}{2\pi}\int_{|h|\le A} v(e^{i\theta})\,d\theta = 0.$$

Subtracting this from the preceding equality, we conclude that

$$\lim_{A\to+\infty}\frac{i}{2\pi}\int_{|h|\le A}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)v(e^{i\theta})\,d\theta = h(z)$$

uniformly on compact subsets of $\mathbb{D}$. Because

$$\{|h|\le A\}\subset\{|v|\le A\},$$

the difference between

$$\frac{1}{2\pi}\int_{|h|\le A}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)v(e^{i\theta})\,d\theta \quad \text{and} \quad \frac{1}{2\pi}\int_{|v|\le A}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)v(e^{i\theta})\,d\theta$$

is bounded in absolute value by $((1+|z|)/(1-|z|))A\lambda_h(A)$. Finally, the difference between

$$\frac{1}{2\pi}\int_{|v|\le A}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)v(e^{i\theta})\,d\theta \quad \text{and} \quad \frac{1}{2\pi}\int_{\partial\mathbb{D}}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)v_A(e^{i\theta})\,d\theta$$

is bounded in absolute value by $((1+|z|)/(1-|z|))A\lambda_v(A)$. This completes the proof of the lemma. □

8. Absolute Convergence

We retain all the notations in Section 6 and consider the possible absolute convergence of the infinite product in the theorem. Absolute convergence is not a by-product of the proof of the theorem, and we are uncertain about the general situation. Proposition 6.3 tells us that the product does converge absolutely if

$$\sum_{n=1}^{\infty}(|E_n^+|+|E_n^-|)<\infty, \tag{8.1}$$

in other words, if the function v is integrable. This holds, in particular, if v is semibounded, because the convergence of the product at 0 guarantees the convergence of the series

$$\sum_{n=1}^{\infty}(|E_n^+|-|E_n^-|)\,. \tag{8.2}$$

The value at 0 of the nth term in the product is $\exp[\pi i(|E_n^+|-|E_n^-|)]$, so the product converges absolutely at 0 if and only if

$$\sum_{n=1}^{\infty}\big||E_n^+|-|E_n^-|\big|<\infty. \tag{8.3}$$

We have been unable to produce an example where (8.3) fails. Proposition 8.1 below states that the question whether it can fail is equivalent to a general question about conjugates of integrable functions. The distribution function λ_{v_+} of the positive part of v takes the value $|E_{n+1}^+|$ on the interval $(n,n+1]$, while λ_{v_-} equals $|E_{n+1}^-|$ on that interval. The convergence of the series (8.2) is equivalent to the convergence of

$$\int_0^{\infty}(\lambda_{v_+}(t)-\lambda_{v_-}(t))\,dt$$

as an improper Riemann integral, while (8.3) is equivalent to the absolute integrability of $\lambda_{v_+} - \lambda_{v_-}$ on $[0,\infty)$.

Proposition 8.1. *The following statements are equivalent.*

(i) *For every function f in RO, the infinite product in the theorem converges absolutely at 0.*

(ii) *If w is the conjugate of a real-valued function in L^1, then $\lambda_{w_+} - \lambda_{w_-}$ is absolutely integrable on $[0,\infty)$.*

Proof. A simple argument shows that, for any real-valued measurable function w on $\partial\mathbb{D}$, the absolute integrability of $\lambda_{w_+} - \lambda_{w_-}$ on $[0,\infty)$ is equivalent to the same condition for any bounded perturbation of w. As explained in the preceding discussion, statement (i) is equivalent to statement (ii) restricted to the case where w is integer valued to within an additive constant. As explained in Section 6, any w as in (ii) is a bounded perturbation of an integer-valued such w. These observations combine to yield the proposition. □

The next result will enable us to produce a function f in RO with a nonintegrable argument such that the product in the theorem converges absolutely in $\mathbb{D}$.

Proposition 8.2. *Let $f = \exp[\pi(u + iv)]$ be a function in RO such that v is antisymmetric with respect to the real axis (i.e., $v(\overline{z}) = -v(z)$), positive on the upper half of $\partial\mathbb{D}$, and nonincreasing there with respect to θ. Then the product in the theorem converges absolutely.*

Proof. Under the given conditions, the set E_n^+ is an arc in the upper half of $\partial\mathbb{D}$ with one endpoint at 1 and the other at, say, $e^{i\alpha_n}$. The set E_n^- is the reflection of E_n^+ with respect to the real axis. For the nth term in the product we have (see Example 2.2 in Section 2)

$$\frac{f_n^+(z)}{f_n^-(z)} = \frac{(z - e^{i\alpha_n})(z - e^{-i\alpha_n})}{(1-z)^2}.$$

Hence

$$\begin{aligned}\frac{f_n^+(z)}{f_n^-(z)} - 1 &= \frac{2z(1-\cos\alpha_n)}{(1-z)^2}\\ &= \frac{4z\sin^2\dfrac{\alpha_n}{2}}{(1-z)^2}\\ &= O\left(\frac{\alpha_n^2}{(1-z)^2}\right).\end{aligned}$$

Since $\alpha_n = o(1/n)$, it follows that

$$\sum_{n=1}^{\infty} \left| \frac{f_n^+(z)}{f_n^-(z)} - 1 \right| < \infty$$

(with uniform convergence on compact subsets of $\mathbb{D}$), giving the absolute convergence of the product. □

We now show there is a function satisfying the conditions of the proposition and having a nonintegrable argument. For $0 < p < 1$ define the function g_p on $\mathbb{D}$ by $g_p = ((1+z)/(1-z))^p$ (a conformal map of $\mathbb{D}$ onto the sector $|\arg z| < (\pi p)/2$). On the top half of $\partial\mathbb{D}$ we have

$$g_p(e^{i\theta}) = e^{\pi i p/2} \left| \cot \frac{\theta}{2} \right|^p .$$

The function g_p is conjugate-symmetric with respect to the real axis. Its imaginary part is positive on the upper half of $\partial\mathbb{D}$ and decreasing there with respect to θ. Letting

$$A_p = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \left| \cot \frac{\theta}{2} \right|^p d\theta,$$

we have $\|\operatorname{Re} g_p\|_1 = A_p \cos(\pi p)/2$ and $\|\operatorname{Im} g_p\|_1 = A_p \sin(\pi p)/2$. In particular, both $\|\operatorname{Im} g_p\|_1 / \|\operatorname{Re} g_p\|_1$ and A_p tend to ∞ as $p \to 1$. We can thus find a sequence $(c_n)_1^\infty$ of positive numbers and a sequence $(p_n)_1^\infty$ of numbers in $(0,1)$ tending to 1 such that $\sum_{n=1}^\infty c_n \|\operatorname{Re} g_{p_n}\|_1 < \infty$ and $\sum_{n=1}^\infty c_n \|\operatorname{Im} g_{p_n}\|_1 = \infty$. The series $\sum_{n=1}^\infty c_n \operatorname{Re} g_{p_n}$ then converges in L^1, say to the function u_0. Since the terms of the series are everywhere positive, the convergence is also pointwise. For $0 < q < 1$ the series $\sum_{n=1}^\infty c_n \operatorname{Im} g_{p_n}$ converges in L^q, because the conjugation operator is of type $(1,q)$. This convergence is pointwise and monotone on $\partial\mathbb{D}$ since the terms of the series are either all positive (on the upper half of $\partial\mathbb{D}$) or all negative (on the lower half). The sum of the series is therefore $\tilde{u}_0$, and is not in L^1.

The outer function $\exp[\pi(u_0 + i\tilde{u}_0)]$ has the properties required in Proposition 8.2, except for one, namely, it is not in RO (i.e., $\tilde{u}_0$ is not integer-valued). This is easily remedied as in Section 6. For n a positive integer let $E_n^+ = \{\tilde{u}_0 \geq n\}$ and $E_n^- = \{\tilde{u}_0 \leq -n\}$, and let $v = \sum_{n=1}^\infty \chi_{E_n^+} - \sum_{n=1}^\infty \chi_{E_n^-}$. The function $w = \tilde{u}_0 - v$ is then bounded, so $u = u_0 + \tilde{w}$ is in L^1, and $\tilde{u} = v$. The function $f = \exp[\pi(u + iv)]$ meets all the requirements of Proposition 8.2, and v is not in L^1.

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