Complex Symmetric Operators

Matrices, operators, functions

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Introduction

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Approach: Accessibility over Technicality

Preliminaries
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Example: $C(x_1, x_2, \ldots, x_n) = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$ on $\mathbb{C}^n$.

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We are therefore not interested in conjugations themselves, but operators that interact with them . . .

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Recall that the adjoint of T is the unique operator (denoted T^*) which satisfies

$$\langle T\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, T^*\mathbf{y} \rangle$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Its definition does not refer "conjugate transposes" of matrix representations . . .

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Such operators are the basic building blocks of normal operators (i.e. $T^*T = TT^*$) . . .

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Lemma: T is a CSO if and only if T has a *symmetric* matrix with respect to some *orthonormal* basis.

In other words, T is a CSO if and only if T is unitarily similar to a symmetric matrix.

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Example: If $T = T^t \in M_n(\mathbb{C})$, then $T = CT^*C$ where

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A few other places where such matrices arise are . . .

Linear Partial Differential Operators

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $a_{jk}(\mathbf{x})$ are $\mathbb{C}$-valued, and

$$[Tf](\mathbf{x}) = \sum_{j,k=1}^{n} a_{jk}(\mathbf{x}) \, \frac{\partial^2 f(\mathbf{x})}{\partial x_j \, \partial x_k},$$

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and hence we may assume that the coefficient matrix is symmetric.

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Hankel matrices arise in *function theory*, *control theory*, *moment problems*, and many other areas.

Grunsky / Goluzin Inequalities

Injective holomorphic functions can be characterized in terms of complex symmetric matrices . . .

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If f is *holomorphic* on $\mathbb{D}$, *the open unit disk in* $\mathbb{C}$*, . . .*

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holds for all $\mathbf{w} = (w_1, w_2, \ldots, w_n)$.

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holds for all $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. *Don't worry if this looks intimidating. The main point is that . . .*

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holds for all $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. This is actually a majorization of *symmetric bilinear* forms by *positive sesquilinear* forms:

$$|\langle A\mathbf{w}, \overline{\mathbf{w}} \rangle| \leq \langle B\mathbf{w}, \mathbf{w} \rangle$$

where $A = A^t$ and $B = B^* \geq 0$.

CSOs: Toeplitz Matrices

An $n \times n$ Toeplitz matrix T satisfies $T = CT^*C$ where

$$C(z_1, z_2, \ldots, z_n) = (\overline{z_n}, \overline{z_{n-1}}, \cdots, \overline{z_1}).$$

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An $n \times n$ Toeplitz matrix T satisfies $T = CT^*C$ where

$$C(z_1, z_2, \ldots, z_n) = (\overline{z_n}, \overline{z_{n-1}}, \cdots, \overline{z_1}).$$

This reflects the symmetry of a Toeplitz matrix with respect to the second diagonal . . .

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Applying this to direct sums of Jordan blocks yields a surprisingly little-known "folk theorem" . . .

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Theorem: Any square matrix is *similar* (over $\mathbb{C}$) to a complex symmetric matrix.

This implies that a CSM can have any possible Jordan canonical form. The conditions $T = T^t$ and $T = T^*$ are therefore quite different!

CSOs: Jordan Blocks

The *backward shift* $B : l^2 \to l^2$ has little symmetry.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \vdots \end{pmatrix}$$

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Restricting it to an invariant subspace reveals some symmetry. . .

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The restriction of the backward shift to an invariant subspace is a Jordan block – hence complex symmetric.

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Example: *Jordan operators*, the infinite dimensional generalizations of Jordan blocks, are CSOs. So are *compressed Toeplitz operators* and *Aleksandrov-Clark operators*.

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Example: All complex symmetric contractions of defect $(1,1)$ and $(2,2)$ are now known (SG; Chevrot, Fricain, Timotin).

Example: Many non-normal *integral* and *differential operators* are CSOs.

CSOs: A Non-Selfadjoint Integral Operator

The *Volterra operator*

$$[Tf](x) = \int_0^x f(y)\,dy$$

on $L^2[0,1]$

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Recall that this means $T = CT^*C\ldots$

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In terms of integral kernels, we may also write. . .

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This simple example illustrates some key ideas . . .

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- Measure-theoretic symmetry: $x \mapsto 1-x$.

Theorems and Applications

CSMs: Takagi’s Factorization

Recall that the singular values of a matrix T *are the (necessarily non-negative) eigenvalues of the positive, self-adjoint matrix* $|T| = \sqrt{T^*T}$*.*

CSMs: Takagi's Factorization

Theorem: If $T \in M_n(\mathbb{C})$ and $T = T^t$, then

$$T = UDU^t$$

where U is *unitary* and $D = \operatorname{diag}(\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$. Here $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{n-1} \geq 0$ are the *singular values* of T.

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Takagi's Theorem was rediscovered many times and in many different contexts. . .

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- *In fact, Takagi was not the first...*

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- **L. Autonne**, *singular value decomposition*, 1915

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Theorem: (Godič, Lucenko) If $U : \mathcal{H} \to \mathcal{H}$ is *unitary*, then there are *conjugations* C and J so that $U = CJ$.

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The fine structure of unitary operators somehow arises from the manner in which two copies of the same simple symmetry are put together . . .

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Example: Explicitly computable for Aleksandrov-Clark operators from function theory.

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The following recent generalization of the Godič-Lucenko theorem applies to all complex symmetric operators . . .

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- J can be extended to a conjugation on all of $\mathcal{H}$.
- $U = CJ$ can be extended to a *unitary* on $\mathcal{H}$.
- U is also C-symmetric: $\quad U = CU^*C$.

Antilinear Eigenproblems

The operator norm of T is defined by

$$|\!|\!|\, T \,|\!|\!| = \sup_{\|\,\mathbf{x}\,\|=1} \|\, T\mathbf{x} \,\|.$$

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Theorem: If T is a compact C-symmetric operator, then the *operator norm* of T is

$$\|T\| = \max\{\sigma \geq 0 : (\exists \mathbf{x} \neq \mathbf{0})(T\mathbf{x} = \sigma C\mathbf{x})\}.$$

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The preceding theorem is a special case of . . .

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Theorem: If T is a compact C-symmetric operator, then the singular values

$$\sigma_0 \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq 0$$

of T are precisely the non-negative solutions σ to the *antilinear eigenproblem* $(\mathbf{x} \neq \mathbf{0})$

$$T\mathbf{x} = \sigma C\mathbf{x}.$$

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Recall that the *Volterra operator*

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The antilinear eigenproblem $Tf = \sigma Cf$ *is just*

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Differentiating twice and back-substituting . . .

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Since $\|\!|\, T \,|\!\|$ is the largest singular value of T . . .

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Since $\| T \|$ is the largest singular value of T . . .

$$\| T \| = \frac{2}{\pi}.$$

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$$[Ce_n](x) = (-i)^n \sqrt{2} \sin[(n + \tfrac{1}{2})\pi x].$$

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Now recall that

$$[Tf](x) = \int_0^x f(y)\,dy = \int_0^1 K(x,y) f(y)\,dy$$

where $K(x,y)$ is the characteristic function of

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A double Fourier expansion for $K(x,y)$ drops out of these simple computations . . .

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In terms of e_n and Ce_n, we have the *Schmidt expansion*

$$K(x,y) = \sum_{n=0}^{\infty} \frac{2}{(n+\frac{1}{2})\pi} \sin[\pi(n+\tfrac{1}{2})x] \cos[\pi(n+\tfrac{1}{2})y].$$

A Complex Symmetric AAK Theorem

Theorem: (Adamyan, Arov, Krein) If T is a *compact Hankel operator*, then

$$\sigma_n = \inf_{\substack{\operatorname{rank} R \leq n, \\ R \text{ Hankel}}} \| T - R \|.$$

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Almost the same theorem is true for complex symmetric operators. . .

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Theorem: If T is a *compact C-symmetric operator*, then

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A Complex Symmetric Minimax Theorem

Theorem: (Courant) If $T \in M_n(\mathbb{C})$ is *self-adjoint*, then the eigenvalues $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ of T satisfy

$$\lambda_j = \min_{\operatorname{codim} \mathcal{V} = j} \max_{\substack{\mathbf{x} \in \mathcal{V} \\ \|\mathbf{x}\| = 1}} \mathbf{x}^* T \mathbf{x}.$$

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A new and useful variant exists for complex symmetric matrices. . .

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Theorem: (Danciger) If $T \in M_n(\mathbb{C})$ is *symmetric*, then

$$\min_{\operatorname{codim} \mathcal{V} = k} \quad \max_{\substack{\mathbf{x} \in \mathcal{V} \\ \|\,\mathbf{x}\,\| = 1}} \left| \mathbf{x}^t T \mathbf{x} \right| \quad = \quad \begin{cases} \sigma_{2k} & 0 \leq k < \frac{n}{2} \\ 0 & \frac{n}{2} \leq k \leq n. \end{cases}$$

A General Minimax Theorem

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Using the $T = CJ|T|$ factorization of complex symmetric operators, the previous theorem can be considerably extended . . .

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Theorem: If $\sigma_0 \geq \sigma_1 \geq \cdots \geq 0$ denotes the *singular values* of a *compact, symmetric bilinear form* $B(\mathbf{x}, \mathbf{y})$ on $\mathcal{H} \times \mathcal{H}$ (where $\dim \mathcal{H} = \infty$), then

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The following bilinear form arises in classical planar elasticity theory. It does not have a “natural” representing operator . . .

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Example: If $\Omega \subset \mathbb{C}$ is bounded domain with $\partial\Omega$ of class $C^{1+\epsilon}$ for some $\epsilon > 0$, then the theorem applies to

$$B(f,g) = \int_\Omega fg \, dA(z)$$

on the *Bergman space* $A^2(\Omega, dA)$ – the space of *holomorphic* functions in $L^2(\Omega, dA)$.

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A normalization $f(a) = 0$ for some $a \in \Omega$ means that f is orthogonal to the *reproducing kernel* at a. In other words, $f = u + iv$ is restricted to lie in a subspace of codimension one.

The theorem implies that there exists an optimal subspace $\mathcal{V} \subset A^2(\Omega, dA)$ such that $\operatorname{codim} \mathcal{V} = 1$ and

$$\max_{\substack{f \in \mathcal{V} \\ \| f \| = 1}} \left| \int_\Omega f^2 \, dA(z) \right| \quad \leq \quad \sigma_2 .$$

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for all f in $\mathcal{V}$. *Adjusting the phase of f and writing $f = u + iv$ we obtain . . .*

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References

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