Witnesses and local cohomology

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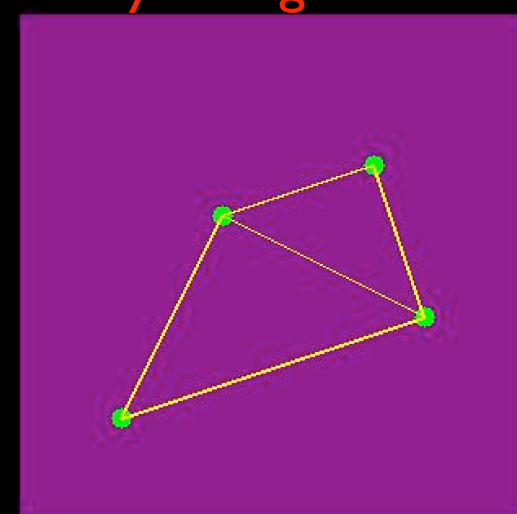
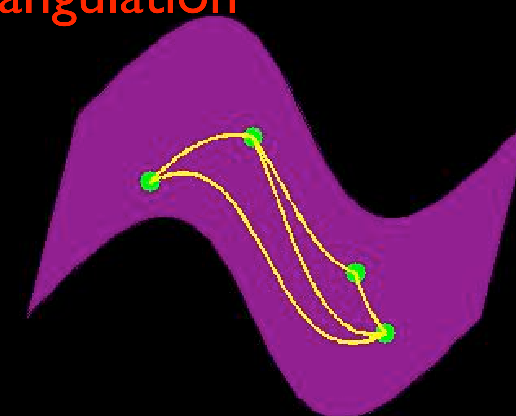
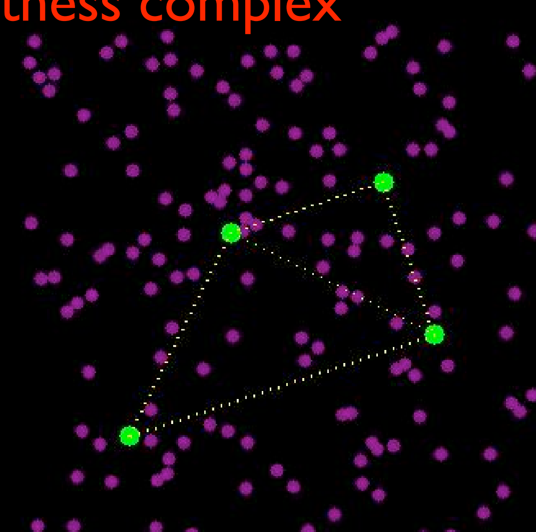
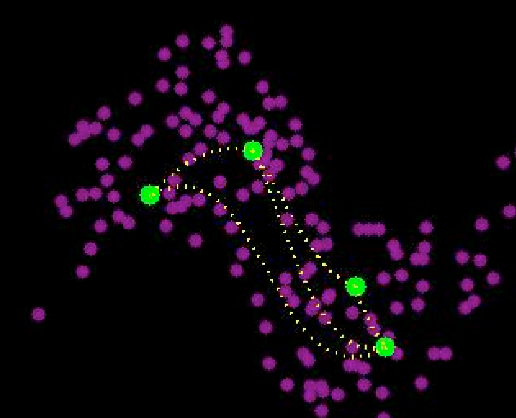
Reconstruction problem

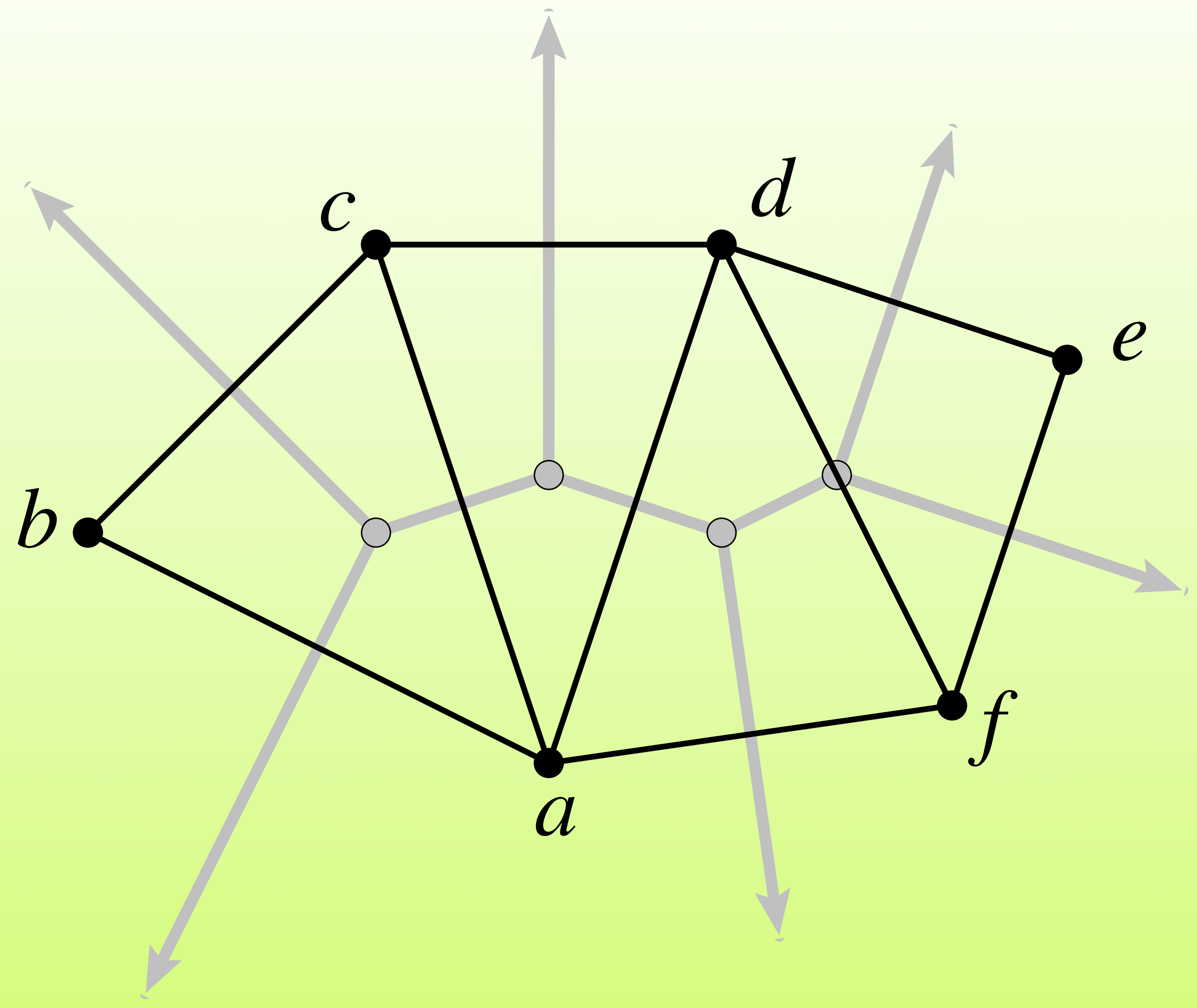
- Finite point-set B sampled from unknown manifold X
- Recover the homotopy type/topology of X from B

- Pick a small subset $A \subset B$
- Build a simplicial complex with vertices A
- Use B as an approximation to X

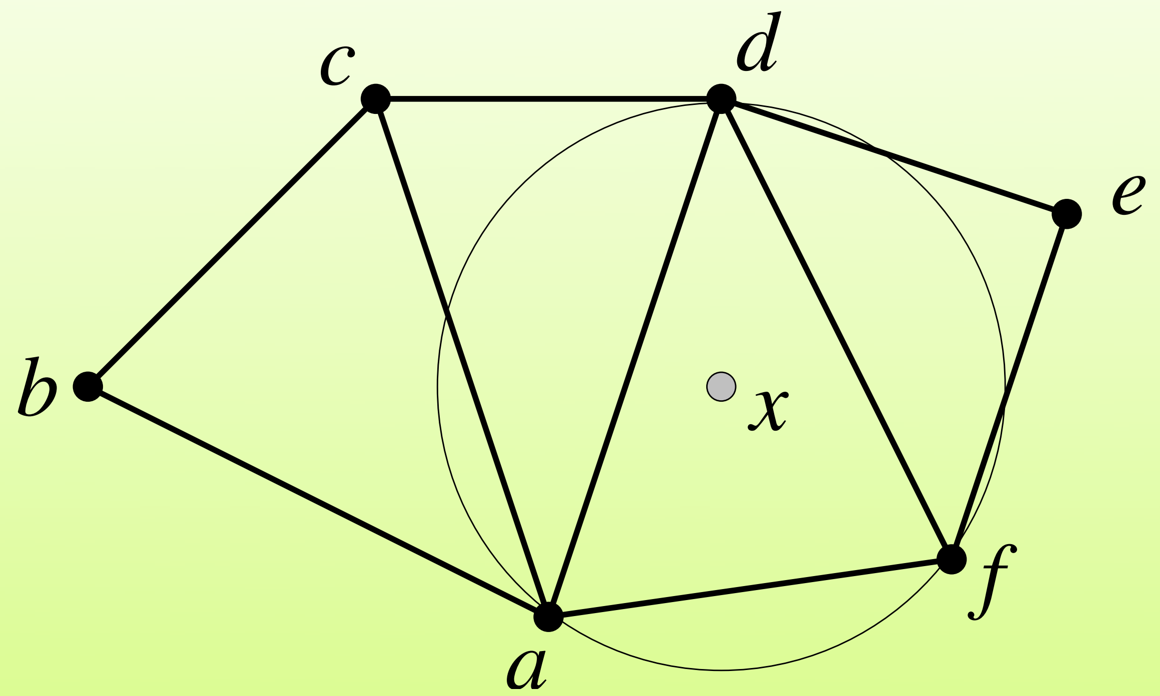
Witness complex paradigm(1)

	flat	curved
continuous	Delaunay triangulation 	restricted Delaunay triangulation 
point cloud	witness complex 	witness complex 

Delaunay triangulation



Strong witnesses



Strong witnesses

- $x \in \mathbf{R}^n$ is a strong witness for $S \subseteq A$

  $\Leftrightarrow d(x,a) \leq d(x,b)$ for all $a \in S$, $b \in A$

  x is necessarily equidistant from the vertices of S

Witness complexes

- strong witness complex

  S ∈ Del(A,X) ⇔ S has a strong witness in X

- Del(A,$\mathbf{R}^n$) = Delaunay triangulation
- Del(A,X) = restricted Delaunay triangulation

Witness complex paradigm(1)

X —— Čech nerve theorem —— Del(A,X)

Del(A,X) | almost surely unequal | Del(A,B)

Del(A,B) ←······ Strong witnesses exist with probability zero

Green means "plausibly equal"

Red means "clearly unequal"

Strong and weak witnesses

- $x \in \mathbf{R}^n$ is a strong witness for $S \subseteq A$

  $\Leftrightarrow d(x,a) \leq d(x,b)$ for all $a \in S$, $b \in A$

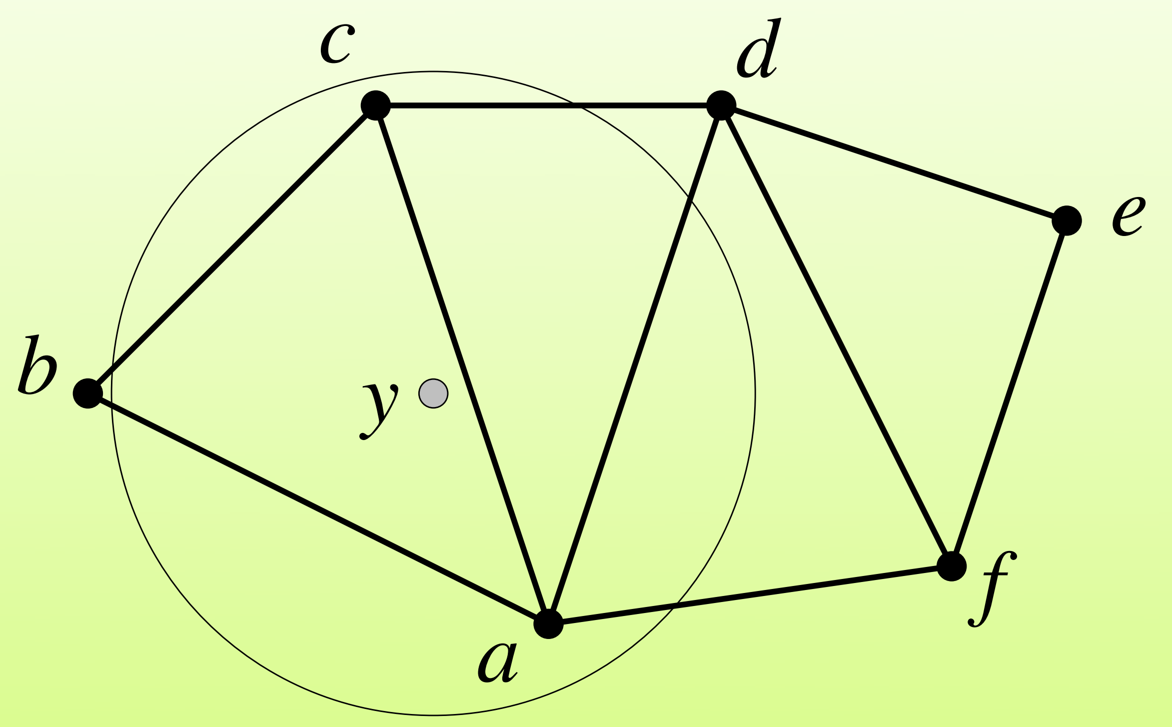
  x is necessarily equidistant from the vertices of S

- $x \in \mathbf{R}^n$ is a weak witness for $S \subseteq A$

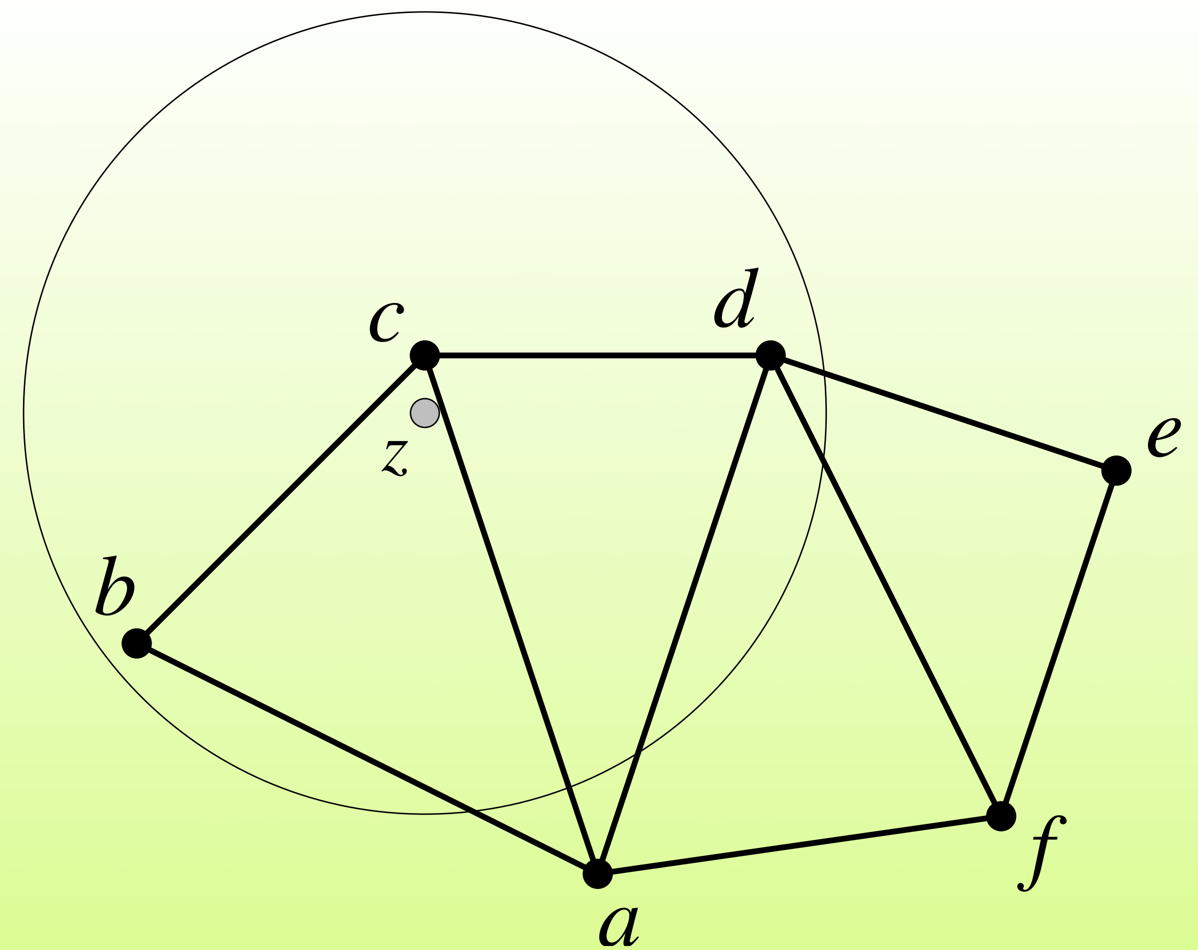
  $\Leftrightarrow d(x,a) \leq d(x,b)$ for all $a \in S$, $b \in A - S$

  x need not be equidistant from the vertices of S

Weak witnesses



Weak witnesses



Witness complexes

- strong witness complex

  $S \in \mathrm{Del}(A,X) \Leftrightarrow$ S has a strong witness in X

- $\mathrm{Del}(A,\mathbf{R}^n)$ = Delaunay triangulation
- $\mathrm{Del}(A,X)$ = restricted Delaunay triangulation

- weak witness complex

  $S \in \mathrm{Del}^w(A,X) \Leftrightarrow$ every $T \subseteq S$ has a weak witness in X

The weak witnesses theorem

$$\mathrm{Del}(A,\mathbf{R}^n) = \mathrm{Del}^w(A,\mathbf{R}^n)$$

$S \subseteq A$ has a strong witness

$\Leftrightarrow$

every $T \subseteq S$ has a weak witness

$\Rightarrow$ trivial

$\Leftarrow$ construct strong witness in convex hull of weak witnesses

Witness complex paradigm(2)

X — Čech nerve theorem — Del(A,X) — weak witnesses theorem — $\mathrm{Del}^w(A,X)$

Del(A,X) — almost surely unequal — Del(A,B)

$\mathrm{Del}^w(A,X)$ — $\mathrm{Del}^w(A,B)$

Green means "plausibly equal"

Red means "clearly unequal"

The weak witnesses theorem

$$\mathrm{Del}(A,\mathbf{R}^n) = \mathrm{Del}^w(A,\mathbf{R}^n)$$

$S \subseteq A$ has a strong witness

$\Leftrightarrow$

every $T \subseteq S$ has a weak witness

$\Rightarrow$ trivial

$\Leftarrow$ construct strong witness in convex hull of weak witnesses

Voronoi convexity

- Under the following assumptions...
  - topological space X, set A
  - $\forall x,y \in X, \exists$ connected $\gamma(x,y) \ni x,y$
    convex hull of {x,y}
  - $\forall a \in A, \exists$ continuous function $d(a,x)$
  - $\forall x,y \in X$, the Voronoi half-space

$$R(a,b) = \{x \in X \mid d(a,x) \leq d(b,x)\}$$

    is γ-convex (i.e. closed under γ)

- ...it follows that $\mathrm{Del}(A,X) = \mathrm{Del}^w(A,X)$

Examples

- Voronoi convexity is satisfied by:
  - $A \subset X = \mathbf{R}^n$, $d(a,x) = |a-x|$ = geodesic metric
  - $A \subset X = \frac{1}{2}\mathbf{S}^n$ (hemisphere), $d(a,x)$ = geodesic metric
  - $A \subset X = \mathbf{H}^n$ (hyperbolic space), $d(a,x)$ = geodesic metric
  - $A \subset X = \mathbf{T}$ (tree), $d(a,x)$ = geodesic metric
  - $A \subset X = \mathbf{R}^{p,q}$, $d(a,x) = |a-x|^2$ = Minkowski square norm
  - $A \subset X = \mathbf{R}^{p,q}$, $d(a,x) = a*x$ = Minkowski inner product
  - $X = \mathbf{R}^n, A \subset \mathbf{R}^n \times \mathbf{R}$, $d((a,c),x) = a.x - c$ linear inequalities

Laguerre diagrams

Weight schemes which satisfy Voronoi convexity

- Euclidean

$$D(a, w(a), x) = \tfrac{1}{2}|a-x|^2 - w(a)$$

- Spherical (restrict to hemisphere)

$$D(a, w(a), x) = -e^{w(a)} \cos(\theta(a,x))$$

- Hyperbolic

$$D(a, w(a), x) = e^{-w(a)} \cosh(u(a,x))$$

(Spherical Laguerre diagrams due to Sugihara, 2002)

Tolerancϵ

- $x \in \mathbf{R}^n$ is an + strong ϵ-witness for $S \subseteq A$

  $\Leftrightarrow d(x,a) \leq d(x,b)+\epsilon$, for all $a \in S$, $b \in A$

- $x \in \mathbf{R}^n$ is an weak ϵ-witness for $S \subseteq A$

  $\Leftrightarrow d(x,a) \leq d(x,b)+\epsilon$, for all $a \in S$, $b \in A - S$

ϵ-witness complexes

- strong ϵ-witness complex

  $S \in \mathrm{Del}(A,X;\epsilon) \Leftrightarrow$ S has a strong ϵ-witness in X

- weak ϵ-witness complex

  $S \in \mathrm{Del}^w(A,X;\epsilon) \Leftrightarrow$ every $T \subseteq S$ has a weak ϵ-witness in X

Filtered complexes: use persistent homology!
(Edelsbrunner, Letscher, Zomorodian 2000)

Weak ϵ-witnesses theorem

- Under the following assumptions...

  (strong Voronoi convexity)

  - topological space X, set A
  - $\forall x,y \in X$, $\exists$ connected $\gamma(x,y) \ni x,y$
  - $\forall a \in A$, $\exists$ continuous function $L(a,x)$
  - $\forall x,y \in X$, the ϵ-Voronoi half-space

    $$R_\epsilon(a,b) = \{x \in X \mid L(a,x) \leq L(b,x)+\epsilon\}$$

    is γ-convex (i.e. closed under γ)

- ...it follows that $\mathrm{Del}(A,X;\epsilon) = \mathrm{Del}^w(A,X;\epsilon)$

Examples

The weak ϵ-witnesses theorem applies in the following cases:

- Euclidean

  $L(a, x) = \frac{1}{2}|a-x|^2$

- Spherical (restrict to subset of diameter less than 90°)

  $L(a,x) = \log \sec(\theta(a,x))$

- Hyperbolic

  $L(a, x) = \log \cosh(u(a,x))$

Witness complex paradigm(3)

X —— Čech nerve theorem —— $\mathrm{Del}(A,X;\epsilon)$ —— weak ϵ-witnesses theorem —— $\mathrm{Del}^w(A,X;\epsilon)$

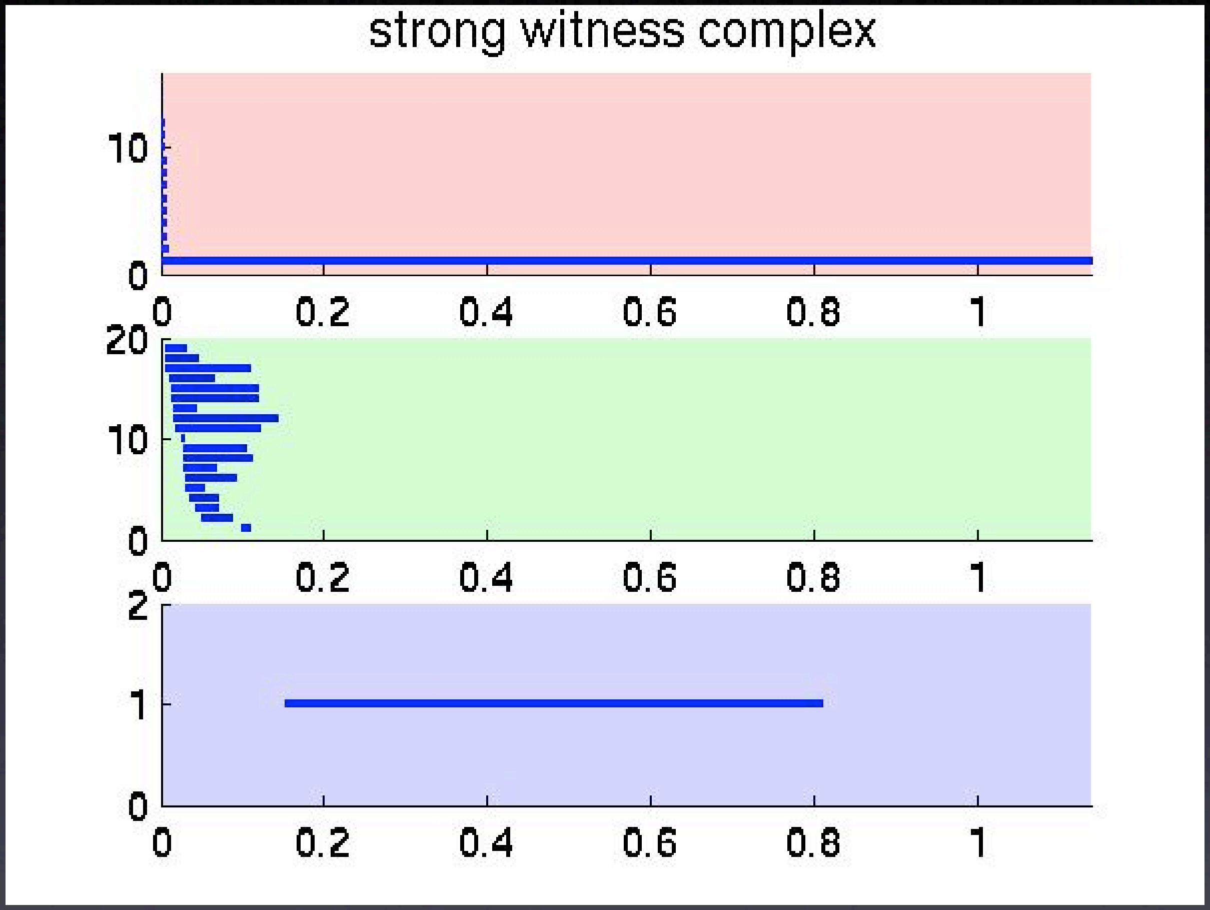
$\mathrm{Del}(A,X;\epsilon)$ | $\mathrm{Del}(A,B;\epsilon)$

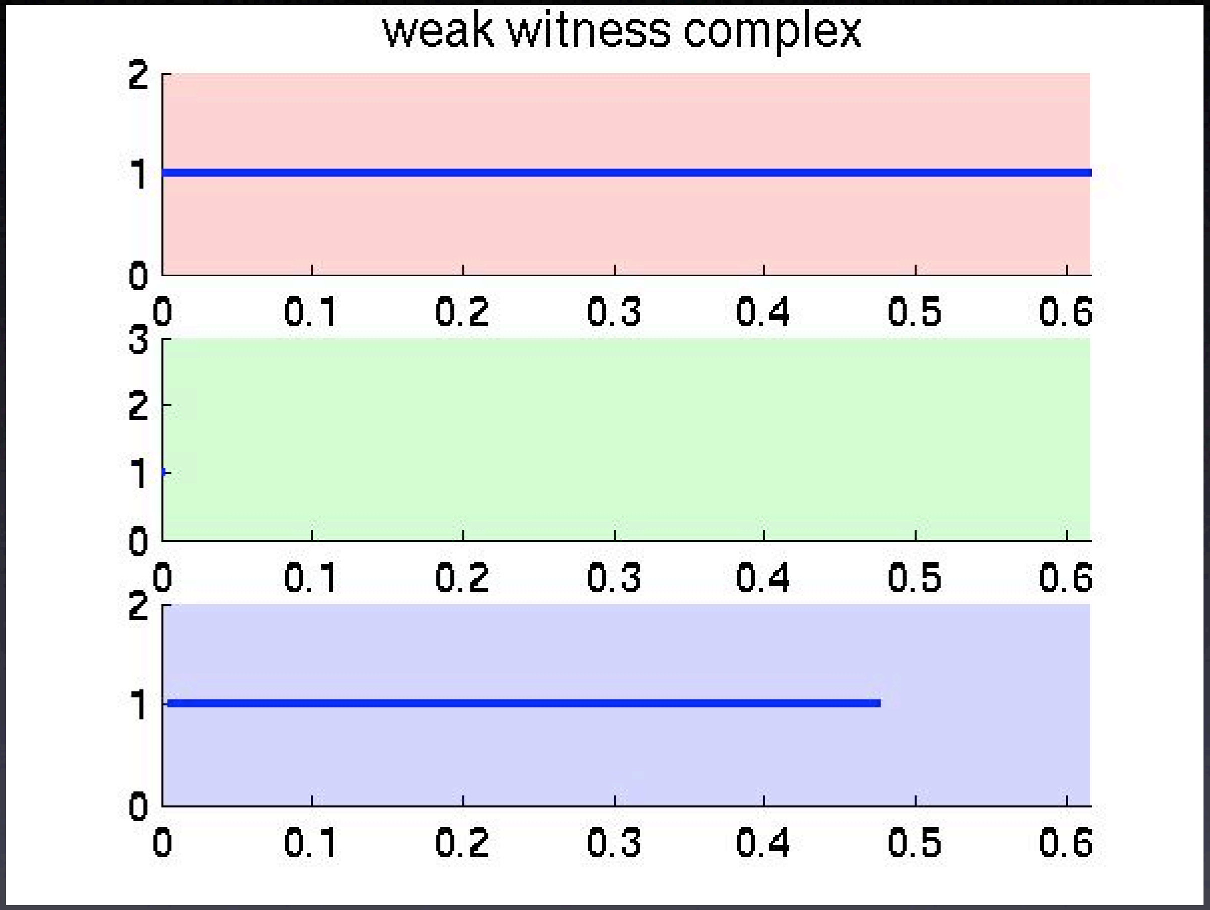
$\mathrm{Del}^w(A,X;\epsilon)$ | $\mathrm{Del}^w(A,B;\epsilon)$

Green means "plausibly equal"

Red means "clearly unequal"

Comparing strong and weak





Data points sampled from 2-sphere

Persistent homology

- Persistent homology algorithm (ELZ2000):
  - Given filtered simplicial complex $\{K_t, \to\}$
  - Input $(S, \partial S)$ in order of appearance of S

    Note: When S enters the filtration, the simplices of ∂S are already there.

  - Output persistent homology

$$\{H_*(K_t), \to\}$$

- What happens if you input the cells in reverse order?

Persistent relative cohomology

- Persistent homology algorithm (ELZ2000):
  - Given filtered simplicial complex $\{K_t, \rightarrow\}$
  - Input $(S, \delta S)$ in order of appearance of S

When S enters the reversed filtration, the simplices of δS are already there.

  - Output persistent relative cohomology

$$\{H^*(K,K_t), \rightarrow\}$$

(at time t, the missing cells are those of K_t).

- Next: exploit this for local homology calculations.

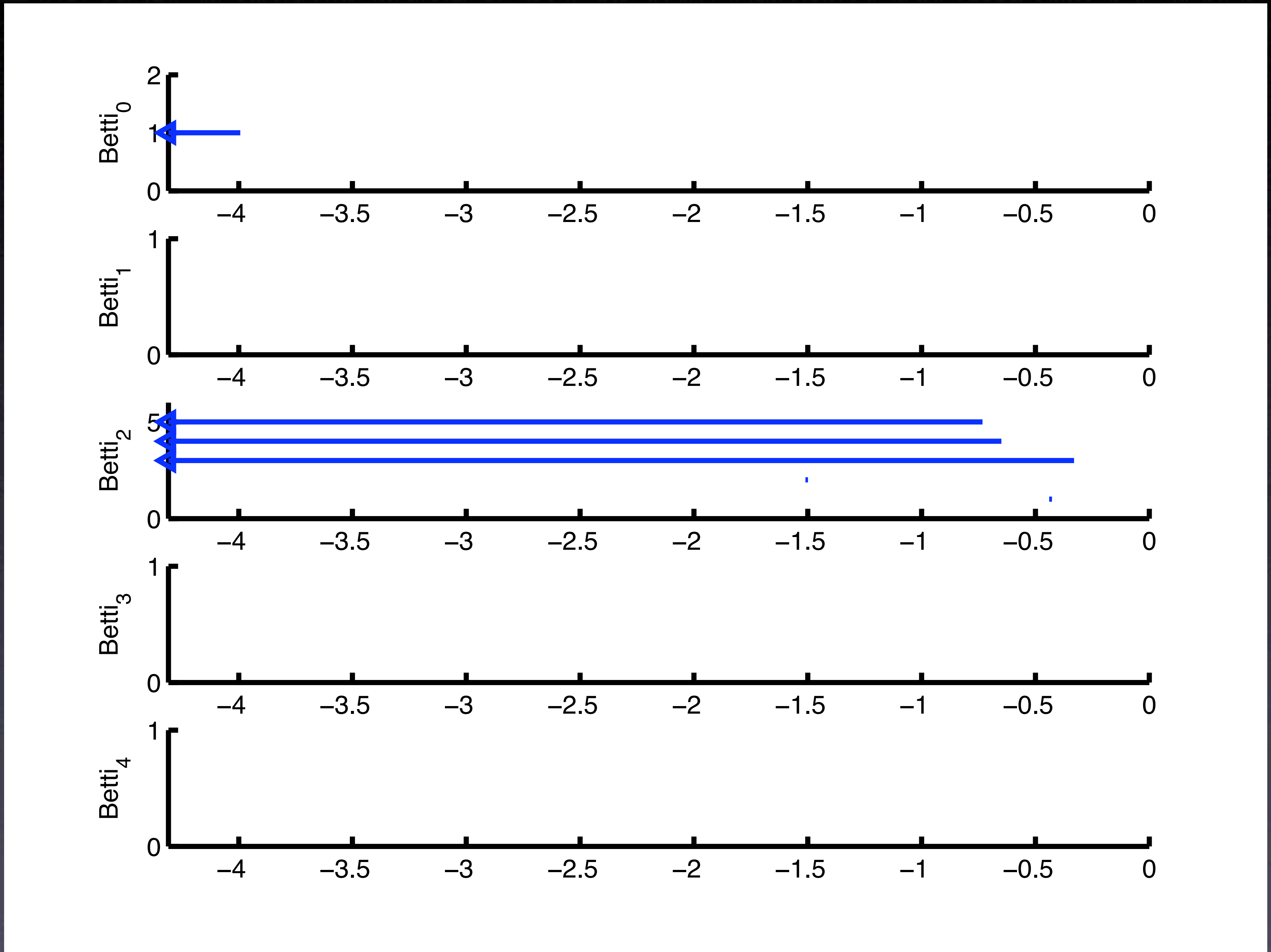
Local cohomology

- The local structure of X near a point x_0 is measured in homology by $H^*(X, X\text{-}x_0)$.
- Define a filtration X_t for $t<0$ which approaches $X\text{-}x_0$ as $t \to 0$ from below:

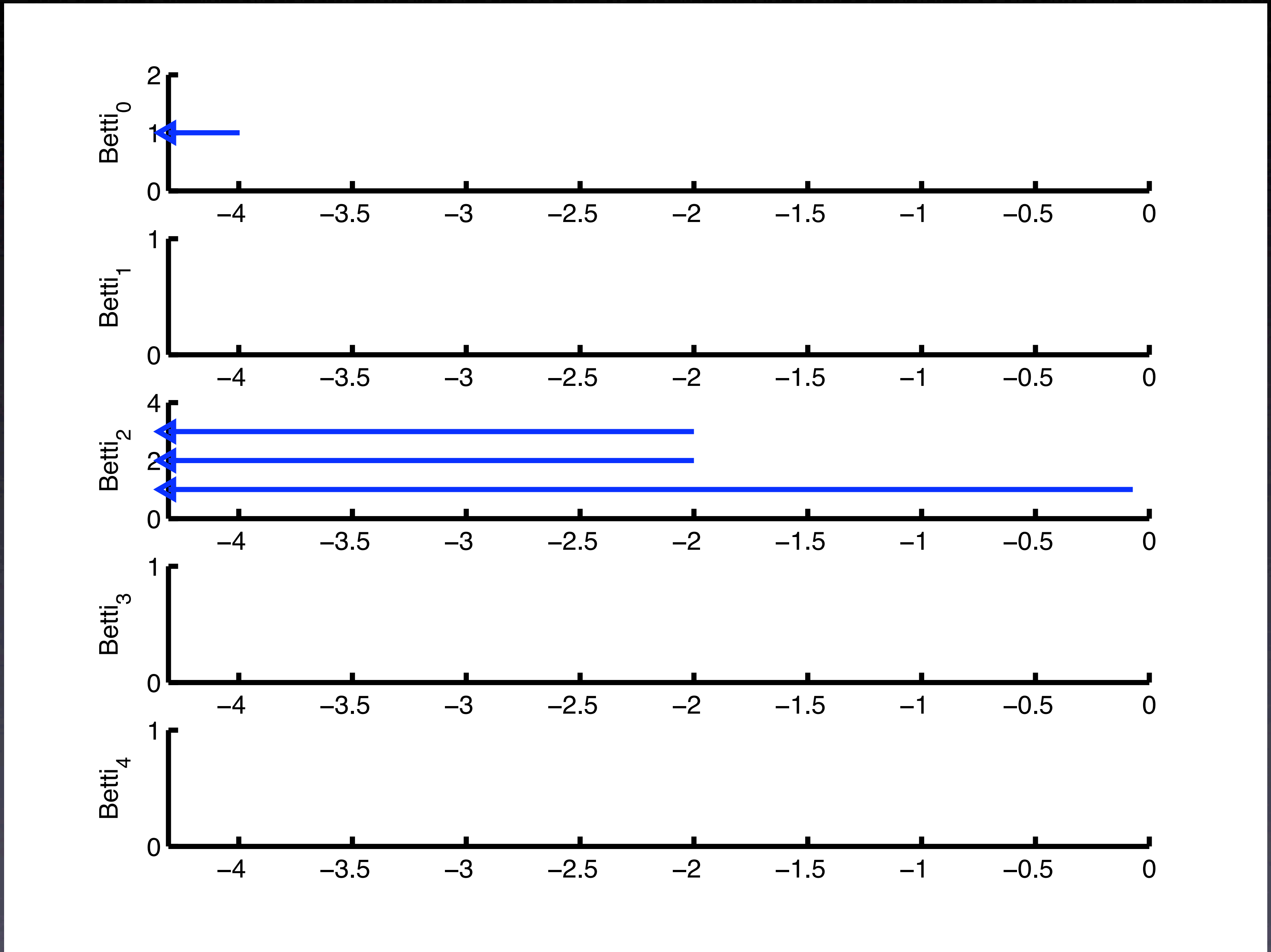
$$X_t = \{x \text{ in } X \mid d(x,x_0) > |t|\}$$

- For fixed ϵ and varying t, compute the persistent relative cohomology of $Del^w(A,B_t; \epsilon)$.
- The t-value of a given simplex S is the smallest t for which all the witnesses for $T \subseteq S$ exist.

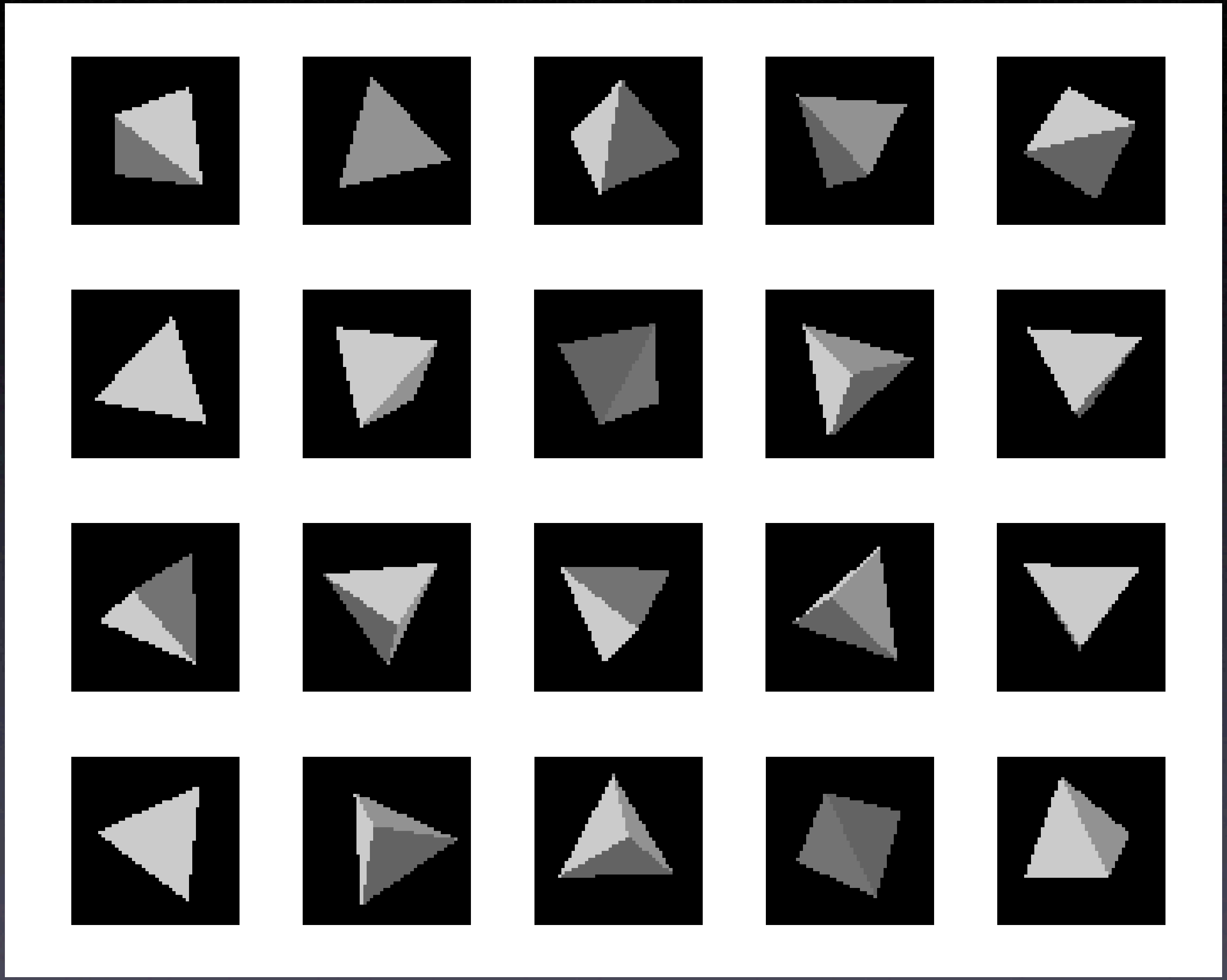
Two spheres meet on a circle



Two spheres meet on a circle

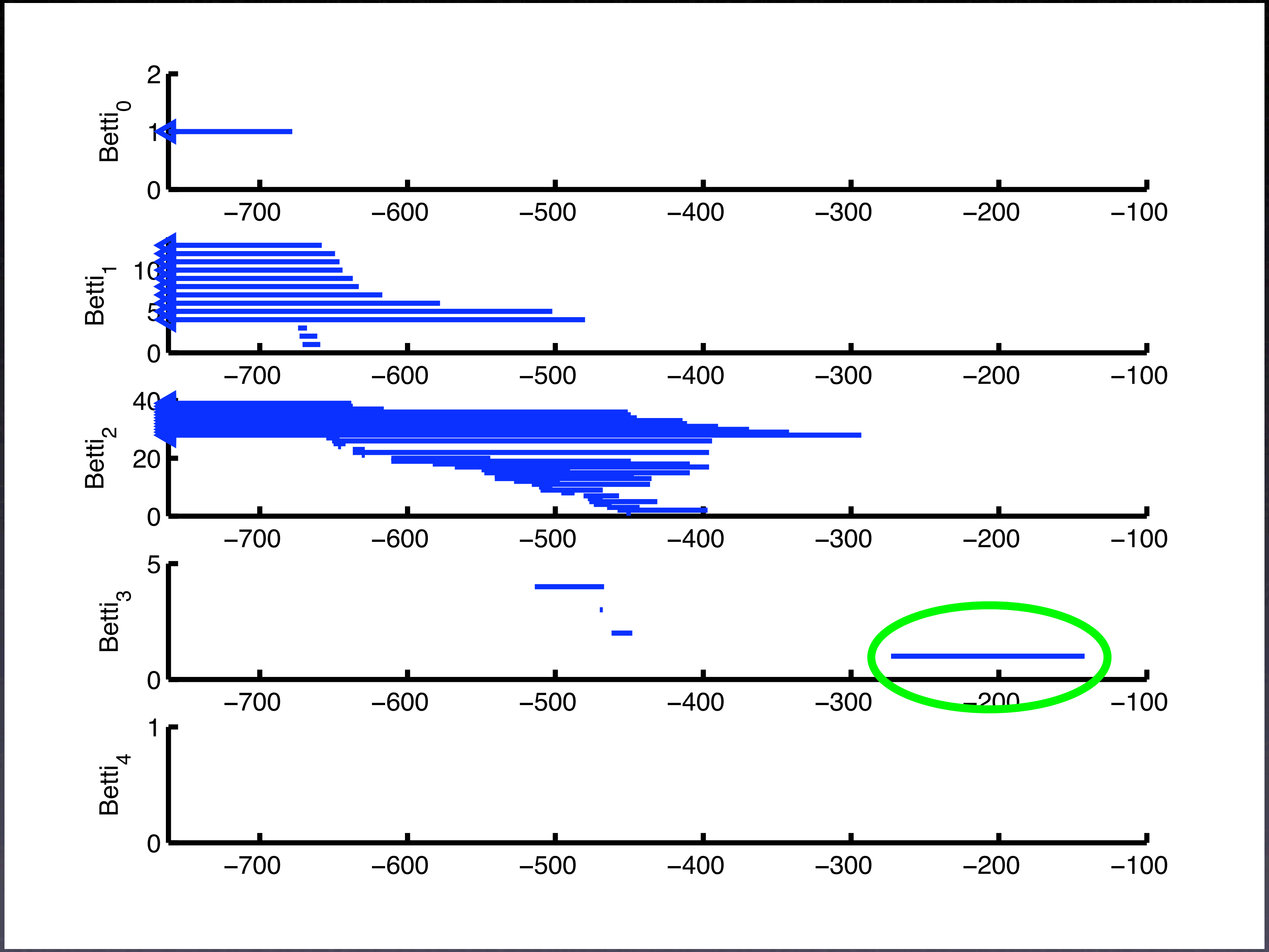


Space of tetrahedron images



Space of rotations = SO(3) is 3-dimensional

Space of tetrahedron images



Confession: I cheated a bit (by cherry-picking)

The End