Solutions Review Problems for Exam #1

1. Water leaks out a barrel at a rate proportional to the square root of the depth of the water at that time. If the water level starts at 36 inches and drops to 35 inches in 1 minute, how long will it take for the water to leak out of the barrel?

**Solution:** Let \( h = h(t) \) denote the water level in the barrel at time \( t \), where \( h \) is measured in inches and \( t \) in minutes. We then have that

\[
\frac{dh}{dt} = -k\sqrt{h}, \tag{1}
\]

where \( k \) is a constant of proportionality.

We can solve the equation in (1) by separating variables to obtain

\[
\int \frac{1}{\sqrt{h}} \, dh = - \int k \, dt,
\]

which integrates to

\[
2\sqrt{h} = -kt + c_1, \tag{2}
\]

where \( c_1 \) is an arbitrary constant. Dividing both sides of the equation in (2) by 2 and squaring, we obtain

\[
h(t) = \left( c - \frac{k}{2}t \right)^2, \tag{3}
\]

where we have set \( c = c_1/2 \).

In order to find what \( c \) in (3) is, we use the information \( h(0) = 36 \) to obtain

\[
c^2 = 36,
\]

from which we obtain that \( c = 6 \), so that (3) now becomes

\[
h(t) = \left( 6 - \frac{k}{2}t \right)^2. \tag{4}
\]

Next, use the information that \( h(1) = 35 \) to estimate the value of \( k \) in (4). We have that

\[
\left( 6 - \frac{k}{2} \right)^2 = 35,
\]
from which we obtain that
\[ k = 2(6 - \sqrt{35}) \approx 0.16784. \] (5)

To find the time, \( t \), at which all the water leaks out of the barrel, solve the equation
\[ h(t) = 0, \]
or
\[ (6 - \frac{k}{2}t)^2 = 0, \]
to obtain that
\[ t = \frac{12}{k}. \] (6)

Using the estimate for \( k \) in (5), we obtain from (6) that
\[ t \approx 71.5 \text{ minutes}. \]

Thus, it will take about 1 hour and 11.5 minutes for the water to leak out of the barrel. \( \Box \)

2. The rate at which a drug leaves the bloodstream and passes into the urine is proportional to the quantity of the drug in the blood at that time. If an initial dose of \( Q_0 \) is injected directly into the blood, 20\% is left in the blood after 3 hours.

(a) Write and solve a differential equation for the quantity, \( Q \), of the drug in the blood at time, \( t \), in hours.

**Solution:** Apply the conservation principle
\[ \frac{dQ}{dt} = \text{Rate of substance in} - \text{Rate of substance out}, \]
where
\[ \text{Rate of substance in} = 0 \]
and
\[ \text{Rate of substance out} = kQ, \]
where \( k \) is a constant of proportionality. Hence,
\[ \frac{dQ}{dt} = -kQ. \] (7)
(b) How much of the drug is left in the patient’s body after 6 hours if the patient is given 100 mg initially?

**Solution:** The solution to the differential equation (7) subject to the initial condition \( Q(0) = Q_o \) is given by

\[
Q(t) = Q_o e^{-kt}, \quad \text{for all } t \in \mathbb{R}.
\]  
(8)

To estimate the value of \( k \), we use the information that \( Q(3) = 0.2Q_o \) to obtain the equation

\[
Q_o e^{-3k} = 0.2Q_o,
\]

which can be solved for \( k \) to obtain

\[
k = -\frac{\ln(0.2)}{3} \approx 0.536479.
\]  
(9)

Next, use (8) to compute

\[
Q(6) = Q_o e^{-6k}.
\]  
(10)

Putting \( Q_o = 100 \) mg, and using the estimate for \( k \) in (9), we obtain from (10) that

\[
Q(6) \approx 100e^{-6(0.54)} \approx 4.0 \text{ mg}.
\]

\[\square\]

3. Use the Fundamental Theorem of Calculus to show that \( y(t) = y_o \exp(F(t)) \), where \( F \) is the antiderivative of \( f \) with \( F(0) = 0 \), is a solution to the initial value problem \( \frac{dy}{dt} = f(t)y, \quad y(0) = y_o \).

**Solution:** Apply the Chain Rule to obtain

\[
\frac{dy}{dt} = y_o \exp'(F(t))F'(t)
\]

\[
= y_o \exp(F(t))f(t)
\]

\[
= f(t)[y_o \exp(F(t))],
\]
which shows that 
\[ \frac{dy}{dt} = f(t)y. \]

Next, compute 
\[ y(0) = y_o \exp(F(0)) = y_o \exp(0) = y_o. \]

Hence, if \( F : I \to \mathbb{R} \) is differentiable over some open interval \( I \) which contains 0, with \( F' = f \) on \( I \), and \( F(0) = 0 \), then \( y(t) = y_o \exp(F(t)) \) for \( t \in I \) solves the initial value problem

\[
\begin{cases}
\frac{dy}{dt} = f(t)y \\
y(0) = y_o.
\end{cases}
\]

All

4. Find a solution to the initial value problem \( \frac{dy}{dt} = e^{t-y}, \quad y(0) = 1. \)

**Solution:** Write the differential equation as 
\[ \frac{dy}{dt} = e^{t}e^{-y}, \]

and separate variables to obtain

\[ \int e^y \, dy = \int e^t \, dt, \]

which integrates to

\[ e^y = e^t + c, \quad (11) \]

for arbitrary \( c \). Using the initial condition \( y(0) = 1 \) in (11) yields

\[ e = 1 + c, \]

from which we get that

\[ c = e - 1. \quad (12) \]

Substituting the value for \( c \) in (12) into the equation in (11) yields

\[ e^y = e^t + e - 1, \]

which can be solved for \( y \) to obtain

\[ y(t) = \ln[e^t + e - 1], \quad \text{for all } t \in \mathbb{R}. \]

\[ \square \]
5. Evaluate the following integrals

\begin{align*}
(a) \quad & \int_0^1 \frac{e^{-x}}{2 - e^{-x}} \, dx \\
(b) \quad & \int \frac{1}{x \ln x} \, dx \\
(c) \quad & \int_1^2 \frac{\ln x}{x} \, dx \\
(d) \quad & \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx
\end{align*}

**Solution:**

(a) Make the change of variables \( u = 2 - e^{-x} \), so that \( du = e^{-x} \, dx \). Then,

\[
\int_0^1 \frac{e^{-x}}{2 - e^{-x}} \, dx = \int_1^{2-1} \frac{1}{u} \, du = \ln(2 - e^{-1}).
\]

(b) Make the change of variables \( u = \ln x \), so that \( du = \frac{1}{x} \, dx \) and

\[
\int \frac{1}{x \ln x} \, dx = \int \frac{1}{u} \, du = \ln|u| + c = \ln|\ln x| + c.
\]

(c) Make the change of variables \( u = \ln x \), so that \( du = \frac{1}{x} \, dx \) and

\[
\int_1^2 \frac{\ln x}{x} \, dx = \int_0^{\ln 2} u \, du = \frac{1}{2} [\ln 2]^2.
\]

(d) Make the change of variables \( u = \sqrt{x} \) so that \( du = \frac{1}{2\sqrt{x}} \, dx \), and

\[
\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = 2 \int e^u \, du = 2e^u + c = 2e^{\sqrt{x}} + c.
\]

\( \square \)
6. The temperature in a hot iron decreases at a rate 0.11 times the difference between its present temperature and room temperature (20°C).

(a) Write a differential equation for the temperature of the iron.

**Solution:** Let \( u = u(t) \) denote the temperature of the hot iron at time \( t \). Then,
\[
\frac{du}{dt} = -0.11(u - 20),
\]
where \( u \) is measured in degrees Celsius and \( t \) in minutes. \( \square \)

(b) If the initial temperature of the rod is 100°C, and the time is measured in minutes, how long will it take for the rod to reach a temperature of 25°C?

**Solution:** The general solution of the differential equation in (13) is
\[
u(t) = 20 + ce^{-0.11 t}, \quad \text{for all } t \in \mathbb{R},
\]
for arbitrary constant \( c \).
To find the value of \( c \) in (14), we use the initial condition \( u(0) = 100 \) in (14) to obtain the equation
\[20 + c = 100,
\]
which yields
\[c = 80.
\]
Substituting the value of \( c \) in (15) into the expression for \( u \) in (14), we obtain that
\[u(t) = 20 + 80e^{-0.11 t}, \quad \text{for all } t \in \mathbb{R}.
\]
Next, we find the value of \( t \) for which \( u(t) = 25 \), or
\[20 + 80e^{-0.11 t} = 25,
\]
or
\[80e^{-0.11 t} = 5,
\]
which can be solved for \( t \) to yield
\[t = -\frac{\ln(1/16)}{0.11} = \frac{4 \ln 2}{0.11} = 25 \text{ minutes}.
\]
Thus, it will take about 25 minutes for the hot iron to reach the temperature or 25 degrees Celsius. \( \square \)