Math 131
Homework 2

Read chapters 2 and 3.1 of Rosenlicht.

1. Prove that any non-empty set of reals which is bounded from below has a greatest lower bound. That is, prove that the GLB Axiom follows from the LUB Axiom.

2. For each \( n \in \mathbb{N} \), let \( a_n \leq b_n \) and suppose that \([a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots\). Use the LUB Axiom to prove that there exists an \( x \) such that for every \( n \in \mathbb{N} \), \( x \in [a_n, b_n] \). In particular, don’t use results about sequences.


4. Show that
   \[
   d(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}
   \]
   is a metric for \( \mathbb{R}^n \).
   
   Hint: to prove the triangle inequality, fill in the details in the outline of steps given below.
   
   a) Suppose that \( a \geq 0 \), and \( b \) and \( c \) are real numbers such that for all \( \lambda \) we have \( a\lambda^2 + b\lambda + c \geq 0 \). Prove that \( b^2 \leq 4ac \).
   
   b) Show that \( \sum_{i=1}^{n} (u_i - \lambda v_i)^2 \) can be rewritten as \( a\lambda^2 + b\lambda + c \) where \( a \geq 0 \). Then use part a) to show that \( \sum_{i=1}^{n} u_i v_i \leq \left( \sum_{i=1}^{n} u_i^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} v_i^2 \right)^{\frac{1}{2}} \).
   
   c) Use part b) to show that
   \[
   \sum_{i=1}^{n} (x_i - y_i)^2 + \sum_{i=1}^{n} (y_i - z_i)^2 + 2 \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} (y_i - z_i)^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{n} (x_i - z_i)^2 \]
   \[
   \geq 2 \sum_{i=1}^{n} y_i^2 + 2 \sum_{i=1}^{n} x_i z_i - 2 \sum_{i=1}^{n} x_i y_i - 2 \sum_{i=1}^{n} y_i z_i + 2 \sum_{i=1}^{n} (x_i - y_i)(y_i - z_i).\]
   
   Use this inequality to prove the triangle inequality.