Homework 7: Hausdorff spaces and Metrizability

“Modern mathematics rests on the substructure of mathematical logic and the theory of sets... Upon this base rise the two pillars that support the whole edifice: general algebra and general topology.” - Lucienne Felix

1. Let $X$ be compact and Hausdorff and let $A$ be a closed subset of $X$. Define $\sim$ on $X$ by $x \sim y$ iff $x = y$ or $x, y \in A$. Prove that $X/\sim$ is both compact and Hausdorff.

2. Let $X$ be a Hausdorff space and let $f : X \to X$ be continuous. Define the fixed point set of $f$ to be the set $F = \{x \in X | f(x) = x\}$. Prove that $F$ is closed.

3. Let $X$ be a set with topologies $\tau$ and $\omega$. Suppose that $(X, \tau)$ is compact and $(X, \omega)$ is Hausdorff, and $\omega \subseteq \tau$. Prove that $\omega = \tau$.

4. Let $X$ denote the product of uncountably many copies of $\mathbb{R}$. Prove that $X$ is not metrizable.

5. We begin with some definitions. Let $X$ be a topological space. A collection of subsets $\tau$ of $X$ is said to be locally finite if for every $x \in X$ there is an open set $U$ containing $x$ such that $U$ intersects only finitely many sets in $\tau$. Let $\omega$ and $\tau$ both be collections of subsets of $X$, then $\tau$ is said to be a refinement of $\omega$ if for every $A \in \tau$ there is a $B \in \omega$ such that $A \subseteq B$. If all of the sets in $\tau$ are open then $\tau$ is said to be an open refinement of $\omega$. Now $X$ is said to be paracompact if it is Hausdorff and every open cover $\omega$ of $X$ has a locally finite open refinement which also covers $X$.

**Problem:** Suppose that $X$ is paracompact. Let $s \in X$ and let $T$ be a closed set which does not contain $s$. Prove that there exist disjoint open sets $U$ and $W$ such that $s \in U$ and $T \subseteq W$. 