(1) Find the interval of convergence of each of the following series. Be sure to check the endpoints.

(a) \[ \sum_{n=1}^{\infty} \frac{(x - 1)^n}{n^2} \] Interval of convergence \([0, 2]\).

(b) \[ \sum_{n=2}^{\infty} \frac{x^{n-1}}{2^n \ln(n)} \] Interval of convergence \([-2, 2)\).

(c) \[ \sum_{n=1}^{\infty} \frac{(x + 2)^{2n}}{n!} \] Interval of convergence \((-\infty, \infty)\).

(d) \[ \sum_{n=1}^{\infty} \frac{3^{n+1}n(x + 1)^n}{\ln(n + 1)} \] Interval of convergence \((-4/3, -2/3)\).

(e) \[ \sum_{n=0}^{\infty} n!(x - 2)^n \] Interval of convergence \([2, 2]\).

(f) \[ \sum_{n=0}^{\infty} \frac{n(x + 3)^{n+1}}{n^2 + 1} \] Interval of convergence \([-4, -2]\).

(2) Find the Taylor series of each of the following functions \(f(x)\) around the specified points \(a\). If you cannot find the general \(\sum\)-notation, write out the first 5 terms explicitly.

(a) \( f(x) = x^{3/2}; \ a = 1 \)

\[ P(x) = 1 + \frac{3}{2}(x - 1) + \frac{3}{8}(x - 1)^2 - \frac{1}{16}(x - 1)^3 + \frac{3}{128}(x - 1)^4 + \cdots \]

(b) \( f(x) = \sin(2x); \ a = \pi/8 \)

\[ P(x) = \frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2}(x - \frac{\pi}{8}) - 4\frac{\sqrt{2}}{2!}2(x - \frac{\pi}{8})^2 - 8\frac{\sqrt{2}}{3!}2(x - \frac{\pi}{8})^3 + 16\frac{\sqrt{2}}{4!}2(x - \frac{\pi}{8})^4 + \cdots \]

\[ = \sum_{n=0}^{\infty} \frac{2^n \sqrt{2}}{n!} (x - \frac{\pi}{8})^n \], where \(d_n\) stutter-oscillates: 1, 1, -1, -1, 1, \cdots. There is a nifty formula for \(d_n\) that requires new notation; ask if you’re interested.

(c) \( f(x) = \frac{1}{x^2}; \ a = 1 \)

\[ P(x) = \sum_{n=0}^{\infty} (-1)^n(n + 1)(x - 1)^n \]
(d) \( f(x) = e^{-3x}; \ a = 1/3 \)
\[
P(x) = \sum_{n=0}^{\infty} \frac{(-3)^n}{e(n!)} (x - \frac{1}{3})^n
\]

(e) \( f(x) = \frac{e^x - e^{-x}}{2}; \ a = 0 \)
\[
P(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{(This is the hyperbolic sine function; compare with the Taylor series of \( \sin(x) \))}
\]

(f) \( f(x) = x^3 - 3x^2 + 2x + 7; \ a = -1 \)
\[
P(x) = 1 + 11(x + 1) - 6(x + 1)^2 + (x + 1)^3 \quad \text{(expand this and you should get the same polynomial)}
\]

(3) Find the Taylor series of each of the following functions using your favorite method. This might include the original definition, or any of the shortcuts we have come across. Specify around which \( a \) you are finding the Taylor series.

(a) \( \frac{1}{x + 3} = \frac{1}{1 - [(-1)(x + 2)]} \)
\[
P(x) = \sum_{n=0}^{\infty} (-1)^n (x + 2)^n \quad \text{and} \quad a = -2 \quad \text{(by substitution in \( \frac{1}{1-x} \))}
\]

(b) \( \ln(1 - 3x) = \ln(1 + (-3x)) \)
\[
P(x) = \sum_{n=1}^{\infty} \frac{-(3^n)}{n} x^n \quad \text{and} \quad a = 0 \quad \text{(by substitution in \( \ln(1 + x) \))}
\]

(c) \( e^{-x+1} \)
\[
P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x - 1)^n \quad \text{and} \quad a = 1 \quad \text{(by substitution in \( e^x \))}
\]

(d) \( \frac{1}{(x + 1)^3} \)
\[
P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)(n+1)}{2} x^n \quad \text{and} \quad a = 0 \quad \text{(by differentiating \( \frac{1}{2x+1} \) twice)}
\]

(e) \( \cos^2(x) \)
\[
P(x) = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n} \quad \text{and} \quad a = 0 \quad \text{(by substitution in the identity \( \cos^2(x) = \frac{1}{2} + \frac{1}{2} \cos(2x) \))}
\]
\( f \int_0^x e^{t^2} \, dt \)

\[ P(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \] and \( a = 0 \) (by substituting in \( e^{t^2} \) and integrating term-by-term).

(4) Use Taylor series methods to compute each of the following expressions.

(a) \( \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} = -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1)^n}{n} = -1 + \ln(2) \)

by using \( \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} \).

(b) \( \frac{3}{2} - \frac{9}{3!} + \frac{27}{4!} - \frac{81}{5!} + \cdots = \sum_{n=2}^{\infty} \frac{(-1)^{n}3^{n-1}}{n!} = \frac{1}{3} \sum_{n=2}^{\infty} \frac{(-1)^{n}2^{n}}{n!} = \frac{1}{3} \left( -1 + 3 + \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} \right) \)

\( = \frac{1}{3} \left( -1 + 3 + e^{-3} \right) \) by using \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

(c) \( \lim_{x \to 0} \frac{\sin(x) - x + x^3/6}{x^5} = \lim_{x \to 0} \frac{x - x^3/3! + x^5/5! - x^7/7! + \cdots - x + x^3/6}{x^5} \)

\( = \lim_{x \to 0} \frac{x^5/5! - x^7/7! + \cdots}{x^5} = \lim_{x \to 0} (\frac{1}{5!} - x^2/7! + \cdots) = 1/5! \)

(d) \( \frac{\pi^2}{2^2} - \frac{\pi^4}{2^44!} + \frac{\pi^6}{2^66!} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(\pi/2)^{2n}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^{n}\frac{(\pi/2)^{2n}}{(2n)!} \)

\( = -\left( -1 + \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/2)^{2n}}{(2n)!} \right) = -(-1 + \cos(\pi/2)) = 1 \)

(5) The following questions all entail bounding the approximation error using Taylor’s remainder.

(a) How good is the approximation \( \sin(x) \approx x - x^3/6, \) if \( |x| < \pi/6? \)

Here we’re using the Taylor polynomial of degree 3 or 4; either is fine, since \( n \) is not specified (see solutions to hw 25 #21). Let’s say \( n = 4 \) here. So

\[ |R_4(x)| = \left| \frac{\cos(c)}{5!} x^5 \right| \leq \frac{(\pi/6)^5}{5!} \]

(b) Approximate \( \ln(0.9) \) using a Taylor polynomial of degree 4. What is the most that the corresponding error could be?

Using the approximation \( \ln(1 + x) \approx x - x^2/2 + x^3/3 - x^4/4, \) we see that \( \ln(0.9) \) corresponds to \( x = -0.1, \) and so

\( \ln(0.9) \approx (-0.1) - (-0.1)^2/2 + (-0.1)^3/3 - (-0.1)^4/4 \)
The error of the approximation is given by

\[ |R_4(-0.1)| = \left| \frac{24}{(1+c)^5}(-0.1)^5 \right| \leq \left| \frac{24}{(0.9)^5}(-0.1)^5 \right| \]

(c) Suppose we wish to approximate \( e^{-0.1} \) by a Taylor polynomial of degree \( n \). How big does \( n \) have to be if we wish that \( |e^{-0.1} - P_n(-0.1)| < 0.001? \)

The actual error of the approximation would be

\[ |R_n(-0.1)| = \left| \frac{e^c}{(n+1)!}(-0.1)^{n+1} \right| \leq \left| \frac{1}{(n+1)!}(-0.1)^{n+1} \right| \]

since \( c \) is between \( a = 0 \) and \( x = -0.1 \), and so \( e^c \) is no bigger than \( e^0 \). The first \( n \) for which

\[ \left| \frac{1}{(n+1)!}(-0.1)^{n+1} \right| < 0.001 \]

is \( n = 2 \), so that the approximation we want is

\[ e^{-0.1} \approx 1 + (-0.1) + (-0.1)^2/2! \]