Math 60 – Linear Algebra
Solutions to Homework 5

3.2 #7 We wish to determine whether any polynomial of degree 3 or less can be written as a linear combination of the given four polynomials. So let us consider the arbitrary polynomial $q(x) = ax^3 + bx^2 + cx + d$ – where $a, b, c$ and $d$ are real numbers – and ask whether that can be expressed as a linear combination of the four polynomials. In other words, can we find numbers $y_1, y_2, y_3, y_4$ such that $q(x) = y_1p_1(x) + y_2p_2(x) + y_3p_3(x) + y_4p_4(x)$? Expanding that expression, we get

$$ax^3 + bx^2 + cx + d = y_1(x^2 + 1) + y_2(x^2 + x + 1) + y_3(x^3 + x) + y_4(x^3 + x^2 + x + 1)$$

In other words, we wish to know whether the system of linear equations

$$\begin{align*}
y_3 + y_4 &= a \\
y_1 + y_2 + y_4 &= b \\
y_2 + y_3 + y_4 &= c \\
y_1 + y_2 + y_4 &= d
\end{align*}$$

always has a solution. The question can be answered either by Gaussian elimination, or by observing that the left-hand-sides of rows 2 and 4 are the same; so the system will not have solutions unless $b = d$. For instance, the polynomial $x^2$ is not in the span of the given four polynomials, and hence, these four polynomials do not span all of $\mathbb{P}^3$.

3.2 #14 Suppose $\vec{u}$ is an arbitrary vector in $V$. We wish to show that it can be written as a linear combination of the vectors $\vec{v} + \vec{w}$ and $\vec{v}$. We are given that $V = \text{Span}\{\vec{v}, \vec{w}\}$, and so we know that $\vec{u}$ can be written as a linear combination of $\vec{v}$ and $\vec{w}$. In other words, we know that there exists two numbers $a$ and $b$ such that $\vec{u} = a\vec{v} + b\vec{w}$. Now, since $b = a + (-a + b)$, we have

$$\vec{u} = a\vec{v} + (a + (-a + b))\vec{w}.$$ 

By distributivity of scalar multiplication over scalar addition, we get

$$\vec{u} = a\vec{v} + a\vec{w} + (-a + b)\vec{w}.$$ 

But now, we use the distributivity of scalar multiplication over vector addition to get

$$\vec{u} = a(\vec{v} + \vec{w}) + (-a + b)\vec{w}$$

and so we have written $\vec{u}$ as a linear combination of the two vectors $\vec{v} + \vec{w}$ and $\vec{w}$.

3.3 #4 To show linear dependence, we must find a nontrivial linear combination of these vectors which equals the zero vector:

$$x(2, 2, 6, 0) + y(0, -1, 0, 1) + z(1, 2, 3, 3) + w(1, -1, 3, 2) = (0, 0, 0, 0)$$

Changing that relationship into a system of linear equations, we get:
\[
\begin{align*}
2x + z + w &= 0 \\
2x - y + 2z - w &= 0 \\
6x + 3z + 3w &= 0 \\
y + 3z - 2w &= 0
\end{align*}
\]
\[
\begin{pmatrix}
2 & 0 & 1 & 1 & 0 \\
2 & -1 & 2 & 1 & 0 \\
6 & 0 & 3 & 3 & 0 \\
0 & 1 & 3 & -2 & 0
\end{pmatrix}
\] 
\[
\begin{pmatrix}
2 & 0 & 1 & 1 & 0 \\
0 & -1 & 1 & -2 & 0 \\
0 & 0 & 4 & -4 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

From the echelon form of the matrix, we can now tell that, since \( w \) is a free variable, there will be infinitely many solutions to the homogeneous system of equations, and so, in particular, one of the corresponding linear combinations will be trivial, and the four vectors are indeed linearly dependent. Now, to write one as a linear combination of the others, we need to find one of these solutions.

\[
\begin{pmatrix}
1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 1 & -1 & 2 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\] 
\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and so \( \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -r \\ -r \\ r \\ r \end{pmatrix} \) is the set of all solutions to this system of equations. To get a nontrivial solution, set \( r = 1 \), so that \( x = y = -1 \) and \( z = w = 1 \). We can now write

\[
(1, -1, 3, -2) = (2, 2, 6, 0) + (0, -1, 0, 1) - (1, 2, 3, 3)
\]

Now, to show that the remaining vectors are linearly independent, we start with the linear combination

\[
x(2, 2, 6, 0) + y(0, -1, 0, 1) + z(1, 2, 3, 3) = (0, 0, 0, 0)
\]

and try to show that the only solution is the trivial one. But the system of linear equations that arises is the same as the previous one, except for the last column on the left-hand side:

\[
\begin{align*}
2x + z &= 0 \\
2x - y + 2z &= 0 \\
6x + 3z &= 0 \\
y + 3z &= 0
\end{align*}
\]
\[
\begin{pmatrix}
2 & 0 & 1 & 0 \\
2 & -1 & 2 & 0 \\
6 & 0 & 3 & 0 \\
0 & 1 & 3 & 0
\end{pmatrix}
\] 
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

so the only solution is the trivial one \( x = y = z = 0 \), and so the three remaining vectors are linearly independent.

3.3 #12 We wish to show that a set of vectors that looks like \( \{\vec{0}, \vec{v}_1, \cdots, \vec{v}_n\} \) is necessarily linearly dependent. It would be enough to show that one of these vectors can be written as a linear combination of the others. Observe that

\[
0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_n = \vec{0} + \cdots + \vec{0}
\]
since by theorem 1.4, 0 times any vector equals the zero vector. Now, using the additive
identity axiom, we get
\[ 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_n = \vec{0} \]
And so we’ve written one of the vectors (namely, \( \vec{0} \)) as a linear combination of the other
vectors (\( \vec{v}_1 \cdots , \vec{v}_n \)), proving that the original set of vectors must be linearly dependent.

3.3 #14 We are given that \( \{ \vec{v}, \vec{w} \} \) is a linearly independent set. We want to prove that, if
\[ x(\vec{v} + \vec{w}) + y\vec{w} = \vec{0}, \]
then we must have \( x = y = 0 \). But
\[ x(\vec{v} + \vec{w}) + y\vec{w} = (x\vec{v} + x\vec{w}) + y\vec{w} \]
by axiom 5. And, using axiom 2, we get
\[ x(\vec{v} + \vec{w}) + y\vec{w} = x\vec{v} + (x\vec{w} + y\vec{w}) \]
and finally, using axiom 6, we get
\[ x(\vec{v} + \vec{w}) + y\vec{w} = x\vec{v} + (x + y)\vec{w}. \]
So, if \( x(\vec{v} + \vec{w}) + y\vec{w} = \vec{0} \), then \( x\vec{v} + (x + y)\vec{w} = \vec{0} \). And, since \( \vec{v} \) and \( \vec{w} \) are linearly independent,
\( x = 0 \) and \( x + y = 0 \), so that \( x = y = 0 \), proving that \( \{ \vec{v} + \vec{w}, \vec{w} \} \) must be linearly independent.

3.3 #17 Theorem 3.5 states that if we can write a vector in a set as a linear combination
of the others, then that set is linearly independent.

a As we proved in class, two vectors are linearly dependent if one of them is a scalar multiple
of the other; that’s a consequence of the fact that a linear combination of one vector is just
a scalar multiple of that vector. Here, it’s easy to see that \((-3, 1, 0)\) is not a multiple of
\((6, 2, 0)\) since, for instance, 6 and 2 are both positive, whereas \(-3\) and \(1\) are of opposite signs.
So the vectors are linearly independent.

b Eyeballing the three vectors, we can see that \((23, 43, 63) = 3(1, 1, 1) + 2(10, 20, 30)\), so the
vectors are indeed linearly dependent.

c Let’s illustrate this with an example. If \((1, 2, 3, 0)\) is written as a linear combination
of the other vectors in the set, then that linear combination cannot include the fourth vector
\((1, 2, 3, 4)\) since the 4 would cause the fourth coordinate of \((1, 2, 3, 0)\) to be nonzero. But that
just leaves the first two vectors, which have 0 for their third coordinate, and so we cannot
obtain the 3 in \((1, 2, 3, 0)\) through such a combination. A similar argument holds for the
other vectors, showing that the set is linearly independent.

d Using exercise 12 above, we see that
\[
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
and so the three vectors (matrices) are linearly dependent.

e Since \( p_1(x) = (x + 1)^3 = x^3 + 3x^2 + 3x + 1 \), we can write it as
\[ p_1(x) = 3p_2(x) + p_3(x) \]
so these vectors (polynomials) are linearly dependent.

3.4 #3 We need to show that this set of vectors is linearly independent, and that it spans $\mathbb{R}^3$. To show independence, we must prove that whenever
\[
x(1, 2, 3) + y(0, 1, 2) + z(0, 0, 1) = (0, 0, 0)
\]
then necessarily $x = y = z = 0$. And to show that the three vectors span $\mathbb{R}^3$, we must show that the system of equations
\[
x(1, 2, 3) + y(0, 1, 2) + z(0, 0, 1) = (a, b, c)
\]
always has a solution for an arbitrary vector $(a, b, c) \in \mathbb{R}^3$. Those two systems of equations have the same coefficients on the left-hand side, so we can combine the Gaussian eliminations:
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & a \\
2 & 1 & 0 & 0 & b \\
3 & 2 & 1 & 0 & c
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & a \\
0 & 1 & 0 & b-2a & 0 \\
0 & 2 & 1 & c-3a & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & a \\
0 & 1 & 0 & b-2a & 0 \\
0 & 0 & 1 & c-3a-2(b-2a) & 0
\end{bmatrix}
\]
The only solution to the homogeneous system of linear equations is $x = y = z = 0$, so the vectors are linearly independent. And, since there are no all-zero rows on the left-hand side, the (second) system of equations will always have a solution for any vector $(a, b, c) \in \mathbb{R}^3$, so the three vectors span $\mathbb{R}^3$. Therefore, the three vectors form a basis for $\mathbb{R}^3$.

3.4 #12 Since the set of vectors $\{\vec{v}, \vec{w}\}$ forms a basis for the vector space $V$, it must be linearly independent, and it must span $V$ (by the definition of basis). Now, by section 3.2, exercise 14, we know that $\{\vec{v} + \vec{w}, \vec{w}\}$ must span $V$ as well. And, by section 3.3, exercise 14, we know that $\{\vec{v} + \vec{w}, \vec{w}\}$ must be linearly independent. Therefore, $\{\vec{v} + \vec{w}, \vec{w}\}$ forms a basis for $V$.

3.5 #15 The Contraction Theorem allows us to obtain a basis from a set of vectors that spans the given vector space. The method entails throwing away vectors that can be written as linear combinations of the other vectors. That in turn entails solving the homogeneous system of linear equations, and as we saw in class, writing the vectors that correspond to free variables in terms of vectors that correspond to leading variables. Or, in other words, just throwing away the vectors that correspond to free variables. So we start out with the homogeneous system $x_1(3, 1, 1) + x_2(1, 5, 3) + x_3(-2, 2, 1) + x_4(5, 4, 3) + x_5(1, 1, 1) = (0, 0, 0)$, and so we row-reduce using Gaussian elimination:
\[
\begin{bmatrix}
3 & 1 & -2 & 5 & 1 & 0 \\
1 & 5 & 2 & 4 & 1 & 0 \\
1 & 3 & 1 & 3 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 5 & 2 & 4 & 1 & 0 \\
3 & 1 & -2 & 5 & 1 & 0 \\
1 & 3 & 1 & 3 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 5 & 2 & 4 & 1 & 0 \\
0 & -14 & -8 & -7 & -2 & 0 \\
0 & 2 & 7 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 5 & 2 & 4 & 1 & 0 \\
0 & -14 & -8 & -7 & -2 & 0 \\
0 & 0 & 1 & 2 & 0 & 0
\end{bmatrix}
\]
And so, since $x_4$ and $x_5$ are free variables, we can eliminate the 4th and 5th vectors from the set to obtain the $\mathbb{R}^3$ basis $\{(3, 1, 1), (1, 5, 3), (-2, 2, 1)\}$.