1.9

1. We have three types of elements that need to be assigned to 21 houses so that exactly seven of each type are assigned. The number of ways to do this is the multinomial coefficient

\[
\binom{21}{7,7,7} = 399,072,960.
\]

4. There are \( \binom{10}{3,3,2,1,1} \) arrangements of the 10 letters of four distinct types. All of them are equally likely, and only one spells statistics. So, the probability is \( \frac{1}{\binom{10}{3,3,2,1,1}} = \frac{1}{50400} \).

6. There are \( 6^7 \) possible outcomes for the seven dice. If each of the six numbers is to appear at least once among the seven dice, then one number must appear twice and each of the other five numbers must appear once. Suppose first that the number 1 appears twice and each of the other numbers appears once. The number of outcomes of this type in the sample space is equal to the number of different arrangements of the symbols 1, 1, 2, 3, 4, 5, 6, which is \( \frac{7!}{2!(1)!} = \frac{7!}{2} \). There is an equal number of outcomes for each of the other five numbers which might appear twice among the seven dice. Therefore, the total number of outcomes in which each number appears at least once is \( \frac{6(7)!}{(2)6^7} \), and the probability of this event is

\[
\frac{6(7)!}{(2)6^7} = \frac{7!}{2(6^6)}.
\]

7. There are \( \binom{25}{10,8,7} \) ways of distributing the 25 cards to the three players. There are \( \binom{12}{6,2,4} \) ways of distributing the 12 red cards to the players so that each receives the designated number of red cards. There are then \( \binom{13}{4,6,3} \) ways of distributing the other 13 cards to the players, so that each receives the designated total number of cards. The product of these last two numbers of ways is, therefore, the number of ways of distributing the 25 cards to the players so that each receives the designated number of red cards and the designated total number of cards. So, the final probability is

\[
\binom{12}{6,2,4} \binom{13}{4,6,3} \binom{25}{10,8,7}.
\]

9. There are \( \binom{52}{13,13,13,13} \) ways of distributing the cards to the four players. Call these four players A, B, C, and D. There is only one way of distributing the cards so that player A receives all red cards, player B receives all yellow cards, player C receives all blue cards, and player D receives all green cards. However, there are \( 4! \) ways of assigning the four colors to the four players and therefore there are \( 4! \) ways of distributing the cards so that each player receives 13 cards of the same color. So, the probability we need is

\[
\frac{4!}{\binom{52}{13,13,13,13}} = \frac{4!(13!)^4}{52!} \approx 4.474 \times 10^{-28}.
\]
4. This is a case of the matching problem with \( n = 3 \). We are asked to find \( p_3 \). By Eq. (1.10.10) in the text, this equals
\[
P_3 = 1 - \frac{1}{2} + \frac{1}{6} = \frac{2}{3}.
\]

5. Determine first the probability that at least one guest will receive the proper hat. This probability is the value \( p_n \) specified in the matching problem, with \( n = 4 \), namely
\[
P_4 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{5}{8}.
\]
So, the probability that no guest receives the proper hat is \( 1 - \frac{5}{8} = 3/8 \).

7. Let \( A_1 \) denote the event that no student from the freshman class is selected, and let \( A_2, A_3, \) and \( A_4 \) denote the corresponding events for the sophomore, junior, and senior classes, respectively. The probability that at least one student will be selected from each of the four classes is equal to \( 1 - \Pr(A_1 \cup A_2 \cup A_3 \cup A_4) \). We shall evaluate \( \Pr(A_1 \cup A_2 \cup A_3 \cup A_4) \) by applying Theorem 1.10.2. The event \( A_1 \) will occur if and only if the 15 selected students are sophomores, juniors, or seniors. Since there are 90 such students out of a total of 100 students, we have
\[
\Pr(A_1) = \frac{90}{15} / \frac{100}{15}.
\]
The values of \( \Pr(A_i) \) for \( i = 2, 3, 4 \) can be obtained in a similar fashion. Next, the event \( A_1A_2 \) will occur if and only if the 15 selected students are juniors or seniors. Since there are a total of 70 juniors and seniors, we have
\[
\Pr(A_1A_2) = \frac{70}{15} / \frac{100}{15}.
\]
The probability of each of the six events of the form \( A_iA_j \) for \( i < j \) can be obtained in this way. Next the event \( A_1A_2A_3 \) will occur if and only if all 15 selected students are seniors. Therefore, \( \Pr(A_1A_2A_3) = \frac{40}{15} / \frac{100}{15} \). The probabilities of the events \( A_1A_2A_4 \) and \( A_1A_3A_4 \) can also be obtained in this way. It should be noted, however, that \( \Pr(A_2A_3A_4) = 0 \) since it is impossible that all 15 selected students will be freshmen. Finally, the event \( A_1A_2A_3A_4 \) is also obviously impossible, so
\[
\Pr(A_1A_2A_3A_4) = 0.
\]

So, the probability we want is
\[
1 - \left[ \frac{90}{15} + \frac{80}{15} + \frac{70}{15} + \frac{60}{15} \right. \\
\left. + \frac{100}{15} + \frac{100}{15} + \frac{100}{15} \right. \\
\left. + \frac{70}{15} + \frac{60}{15} + \frac{50}{15} \right. \\
\left. + \frac{50}{15} + \frac{40}{15} + \frac{30}{15} \right. \\
\left. + \frac{40}{15} + \frac{30}{15} \right. \\
\left. + \frac{30}{15} + \frac{20}{15} \right].
\]

9. Let \( p_n = 1 - q_n \). As discussed in the text, \( p_{10} < p_{300} < 0.63212 < p_{53} < p_{21} \). Since \( p_n \) is smallest for \( n = 10 \), then \( q_n \) is largest for \( n = 10 \).