The motivation

We can simulate real numbers on the interval [0,1]. We’d like to be able to simulate variables from other distributions. In fact, we’d like to be able to simulate observations from the following distribution:

\[
\text{pdf: } g(*) = \lambda e^{-\lambda *} \quad * \geq 0 \\
\text{cdf: } G(*) = 1 - e^{-\lambda *} \quad * \geq 0
\]

The set up

Let \( X \) be a uniform [0,1] random variable. That is, \( f_X(x) = 1 \quad 0 \leq x \leq 1; F_X(x) = x \quad 0 \leq x \leq 1. \)

Let \( Y = G^{-1}(X) \). What is the distribution of \( Y \)?

Note:

\[
X = 1 - e^{-Y\lambda} \\
Y = -\ln(1 - X)/\lambda
\]

The solution

\[
F_Y(y) = P(Y \leq y) = P(G^{-1}(X) \leq y) \\
= P(X \leq G(y)) \\
= F_X(G(y)) \\
= G(y)
\]

That is, if we let \( Y = G^{-1}(X) \), then the random variable \( Y \) will have exactly the distribution for which we were hoping.

The implications

The relationship above holds in both directions. That is, if \( Y \) has any distribution \( G \), then \( X = G(Y) \) will have a uniform distribution on [0,1].

\[
F_X(x) = P(X \leq x) \\
= P(G(Y) \leq x) = P(Y \leq G^{-1}(x)) \\
= G(G^{-1}(x)) = x \quad 0 \leq x \leq 1
\]

Which proves that \( X \) has a uniform distribution on [0,1].

How does it work?

1. (a) Find a random uniform observation, \( x^* \)  
   (b) \( G^{-1}(x^*) \) will be the random exponential observation we simulate.

2. (a) Find a random observation from any distribution, \( y^* \)  
   (b) \( G(y^*) \) will be the random uniform [0,1] observation we simulate.