Assignment #3

Due on Friday, September 27, 2019

Read Chapter 3, on *Indirect Methods,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let $\mathcal{A} = \{y \in C^1([0,1],\mathbb{R}) \mid y(0) = 0 \text{ and } y(1) = 1\}$. Define $J\colon \mathcal{A} \to \mathbb{R}$ by

$$J(y) = \int_0^1 [2e^x y(x) + (y'(x))^2] \, \mathrm{d}x \quad \text{for all } y \in \mathcal{A}.$$

Give the Euler–Lagrange equation associated with J and, if possible, solve it subject to the boundary conditions in $\mathcal{A}$.

2. Let $J\colon \mathcal{A} \to \mathbb{R}$ be defined by $J(y) = \displaystyle\int_1^2 [2(y(x))^2 + x^2(y'(x))^2] \, \mathrm{d}x$ for all $y \in \mathcal{A}$, where $\mathcal{A} = \{y \in C^1([1,2],\mathbb{R}) \mid y(1) = 1 \text{ and } y(2) = 5\}$. Give the Euler–Lagrange equation associated with J and, if possible, solve it subject to the boundary conditions in $\mathcal{A}$.

   Suggestion: Look for solutions of the differential equation of the form $y(x) = x^m$, for $x \in [1,2]$, where m needs to be determined so that y solves the Euler–Lagrange equation.

3. Let $J\colon \mathcal{A} \to \mathbb{R}$ be defined by $J(y) = \displaystyle\int_5^{10} \sqrt{x}\sqrt{1 + (y'(x))^2} \, \mathrm{d}x$ for all $y \in \mathcal{A}$, where $\mathcal{A} = \{y \in C^1([5,10],\mathbb{R}) \mid y(5) = 4 \text{ and } y(10) = 6\}$. Give the Euler–Lagrange equation associated with J and, if possible, solve it subject to the boundary conditions in $\mathcal{A}$.

4. **The Brachistochrone Problem**. The Euler–Lagrange equation associated with the functional defined in the discussion of the Brachistochrone problem in class, and in the lecture notes, can be written in the form

$$\frac{d}{dx}\left[\frac{u'}{\sqrt{1 + (u')^2}\sqrt{u}}\right] = -\frac{\sqrt{1 + (u')^2}}{2u^{3/2}}, \qquad \text{for } 0 < x < x_1, \tag{1}$$

Evaluate the derivative on the left–hand side of the equation in (1) and simplify to obtain from (1) that

$$(u')^2 + 2uu'' + 1 = 0 \quad \text{for } 0 < x < x_1, \tag{2}$$

where u'' denotes the second derivative of u.

5. **The Brachistochrone Problem, Continued**. Multiply on both sides of (2) by u' to get

$$(u')^3 + 2uu'u'' + u' = 0 \quad \text{for } 0 < x < x_1. \tag{3}$$

(a) Show that the differential equation in (3) can be written as

$$\frac{d}{dx}[u + u(u')^2] = 0. \tag{4}$$

(b) Integrate the equation in (4) to obtain a first–order differential equation for u.