Notes on the Calculus of Variations and Optimization

Preliminary Lecture Notes

Adolfo J. Rumbos

Contents

Chapter 1

Preface

This course is an introduction to the Calculus of Variations and its applications to the theory of differential equations, in particular, boundary value problems. The calculus of variations is a subject as old as the Calculus of Newton and Leibniz. It arose out of the necessity of looking at physical problems in which an optimal solution is sought; e.g., which configurations of molecules, or paths of particles, will minimize a physical quantity like the energy or the action? Problems like these are known as **variational problems**.

From its beginnings, the Calculus of Variations has been intimately connected with the theory of differential equations; in particular, the theory of boundary value problems. Sometimes a variational problem leads to a differential equation that can be solved, and this gives the desired optimal solution. On the other hand, variational methods can be successfully used to find solutions of otherwise intractable problems in nonlinear partial differential equations. This interplay between the theory of boundary value problems for differential equations and the calculus of variations will be one of the major themes in the course.

We begin the course with an example involving surfaces that span a wire loop in space. Out of all such surfaces, we would like to find, if possible, the one that has the smallest possible surface area. If such a surface exists, we call it a **mimimal surface**. This example will serve to motivate a large portion of what we will be doing in this course. The minimal surface problem is an example of a variational problem.

In a variational problem, out of a class of functions (e.g., functions whose graphs in three–dimensional space yield surface spanning a given loop) we seek to find one that optimizes (minimizes or maximizes) a certainty quantity (e.g., the surface area of the surface). There are two approaches to solving this kind of problems: the direct approach and the indirect approach. In the direct approach, we try to find a minimizer or a maximizer of the quantity, in some cases, by considering sequences of functions for which the quantity under study approaches a maximum or a minimum, and then extracting a subsequence of the functions that converge in some sense to the sought after optimal solution.

In the indirect method of the Calculus of Variations, which was developed first historically, we first find necessary conditions for a given function to be an optimizer for the quantity. In cases in which we assume that functions in the class under study are differentiable, these conditions, sometimes, come in the form of a differential equations, or system of differential equations, that the functions must satisfy, in conjunction with some boundary conditions. This process leads to a boundary value problem. If the boundary value problem can be solved, we can obtain a candidate for an optimizer of the quantity (a critical "point"). The next step in the process is to show that the given candidate is an optimizer. This can be done, in some cases, by establishing some sufficient conditions for a function to be an optimizer. The indirect method in the Calculus of Variations is reminiscent of the optimization procedure that we first learn in a first single–variable Calculus course.

Conversely, some classes of boundary value problems have a particular structure in which solutions are optimizers (minimizers, maximizers, or, in general, critical "points") of a certain quantity over a class of functions. Thus, these differential equations problems can, in theory, be solved by finding optimizers of a certain quantity. In some cases, the existence of optimizers can be achieved by a direct method in the Calculus of Variations. This provides an approach, known as the **variational approach** in the theory of differential equations.

Chapter 2

Examples of a Variational Problems

2.1 Minimal Surfaces

Imagine you take a twisted wire loop, as that pictured in Figure 2.1.1, and dip it into a soap solution. When you pull it out of the solution, a soap film spanning the wire loop develops. We are interested in understanding the mathematical properties of the film, which can be modeled by a smooth surface in three

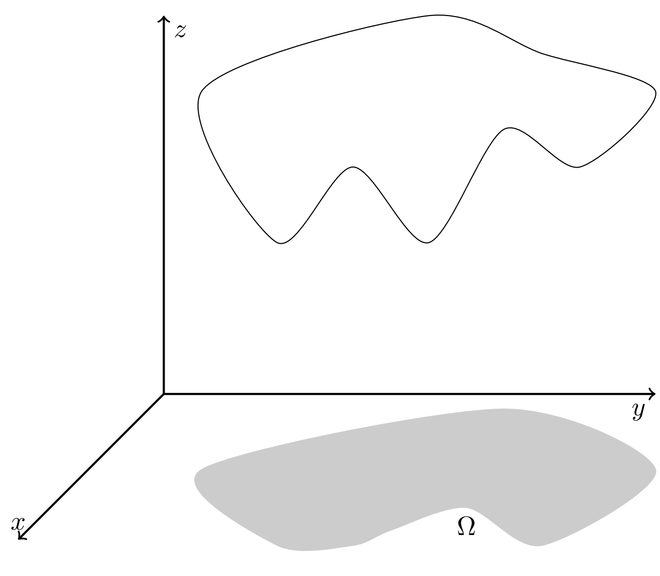

Figure 2.1.1: Wire Loop

dimensional space. Specifically, the shape of the soap film spanning the wire loop, can be modeled by the graph of a smooth function, $u\colon \overline{\Omega} \to \mathbb{R}$, defined on the closure of a bounded region, Ω, in the xy–plane with smooth boundary $\partial\Omega$. The physical explanation for the shape of the soap film relies on the variational principle that states that, at equilibrium, the configuration of the film must be such that the energy associated with the surface tension in the film must be the lowest possible. Since the energy associated with surface tension in the film is proportional to the area of the surface, it follows from the least–energy principle that a soap film must minimize the area; in other words, the soap film spanning the wire loop must have the shape of a smooth surface in space containing the wire loop with the property that it has the smallest possible area among all smooth surfaces that span the wire loop. In this section we will develop a mathematical formulation of this variational problem.

The wire loop can be modeled by the curve determined by the set of points:

$$(x, y, g(x, y)), \qquad \text{for } (x, y) \in \partial\Omega,$$

where $\partial\Omega$ is the smooth boundary of a bounded open region Ω in the xy–plane (see Figure 2.1.1), and g is a given function defined in a neighborhood of $\partial\Omega$, which is assumed to be continuous. A surface, S, spanning the wire loop can be modeled by the image of a C^1 map

$$\Phi\colon \Omega \to \mathbb{R}^3$$

given by

$$\Phi(x, y) = (x, y, u(x, u)), \qquad \text{for all } x \in \overline{\Omega}, \tag{2.1}$$

where $\overline{\Omega} = \Omega \cup \partial R$ is the closure of Ω, and

$$u\colon \overline{\Omega} \to \mathbb{R}$$

is a function that is assumed to be C^2 in Ω and continuous on $\overline{\Omega}$; we write

$$u \in C^2(\Omega) \cap C(\overline{\Omega}).$$

Let $\mathcal{A}_g$ denote the collection of functions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying

$$u(x, y) = g(x, y), \quad \text{for all } (x, y) \in \partial\Omega;$$

that is,

$$\mathcal{A}_g = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) \mid u = g \text{ on } \partial\Omega\}. \tag{2.2}$$

Next, we see how to compute the area of the surface $S_u = \Phi(\Omega)$, where Φ is the map given in (2.1) for $u \in \mathcal{A}_g$, where $\mathcal{A}_g$ is the class of functions defined in (2.2).

The grid lines $x = c$ and $y = d$, for arbitrary constants c and d, are mapped by the parametrization Φ into curves in the surface S_u given by

$$y \mapsto \Phi(c, y)$$

and

$$x \mapsto \Phi(x, d),$$

respectively. The tangent vectors to these paths are given by

$$\Phi_y = \left(0, 1, \frac{\partial u}{\partial y}\right) \tag{2.3}$$

and

$$\Phi_x = \left(1, 0, \frac{\partial u}{\partial x}\right), \tag{2.4}$$

respectively. The quantity

$$\|\Phi_x \times \Phi_y\| \ \Delta x \Delta y \tag{2.5}$$

gives an approximation to the area of the portion of the surface S_u that results from mapping the rectangle $[x, x + \Delta x] \times [y, y + \Delta y]$ in the region Ω to the surface S_u by means of the parametrization Φ given in (2.1). Adding up all the contributions in (2.5), while refining the grid, yields the following formula for the area S_u:

$$\text{area}(S_u) = \iint_\Omega \|\Phi_x \times \Phi_y\| \ dxdy. \tag{2.6}$$

Using the definitions of the tangent vectors Φ_x and Φ_y in (2.3) and (2.4), respectively, we obtain that

$$\Phi_x \times \Phi_y = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right),$$

so that

$$\|\Phi_x \times \Phi_y\| = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2},$$

or

$$\|\Phi_x \times \Phi_y\| = \sqrt{1 + |\nabla u|^2},$$

where $|\nabla u|$ denotes the Euclidean norm of ∇u. We can therefore write (2.6) as

$$\text{area}(S_u) = \iint_\Omega \sqrt{1 + |\nabla u|^2} \ dxdy. \tag{2.7}$$

The formula in (2.7) allows us to define a map

$$A \colon \mathcal{A}_g \to \mathbb{R}$$

by

$$A(u) = \iint_\Omega \sqrt{1 + |\nabla u|^2} \ dxdy, \qquad \text{for all } u \in \mathcal{A}_g, \tag{2.8}$$

which gives the area of the surface parametrized by the map $\Phi \colon \overline{\Omega} \to \mathbb{R}^3$ given in (2.1) for $u \in \mathcal{A}_g$. We will refer to the map $A \colon \mathcal{A}_g \to \mathbb{R}$ defined in (2.8) as the area **functional**. With the new notation we can restate the variational problem of this section as follows:

Problem 2.1.1 (Variational Problem 1)**.** *Out of all functions in $\mathcal{A}_g$, find one such that*

$$A(u) \leqslant A(v), \quad \text{for all } v \in \mathcal{A}_g. \tag{2.9}$$

That is, find a function in $\mathcal{A}_g$ that minimizes the area functional in the class $\mathcal{A}_g$.

Problem 2.1.1 is an instance of what has been known as Plateau's problem in the Calculus of Variations. The mathematical question surrounding Pateau's problem was first formulated by Euler and Lagrange around 1760. In the middle of the 19^{th} century, the Belgian physicist Joseph Plateu conducted experiments with soap films that led him to the conjecture that soap films that form around wire loops are of minimal surface area. It was not until 1931 that the American mathematician Jesse Douglas and the Hungarian mathematician Tibor Radó, independently, came up with the first mathematical proofs for the existence of minimal surfaces. In this section we will derive a necessary condition for the existence of a solution to Problem 2.1.1, which is expressed in terms of a partial differential equation (PDE) that $u \in \mathcal{A}_g$ must satisfy, the minimal surface equation.

Suppose we have found a solution, $u \in \mathcal{A}_g$, of Problem 2.1.1 in $u \in \mathcal{A}_g$. Let $\varphi\colon \overline{\Omega} \to \mathbb{R}$ by a C^∞ function with compact support in Ω; we write $\varphi \in C_c^\infty(\Omega)$ (we show a construction of such function in the Appendix). It then follows that

$$u + t\varphi \in \mathcal{A}_g, \quad \text{for all } t \in \mathbb{R}, \tag{2.10}$$

since φ vanishes in a neighborhood of $\partial\Omega$ and therefore $u + t\varphi = g$ on $\partial\Omega$. It follows from (2.10) and (2.9) that

$$A(u) \leqslant A(u + t\varphi), \quad \text{for all } t \in \mathbb{R}. \tag{2.11}$$

Consequently, the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = A(u + t\varphi), \quad \text{for all } t \in \mathbb{R}, \tag{2.12}$$

has a local minimum at 0, by virtue of (2.11) and (2.12). It follows from this observation that, if f is differentiable at 0, then

$$f'(0) = 0. \tag{2.13}$$

We will see next that, since we are assuming that $u \in C^2(R) \cap C(\overline{\Omega})$ and $\varphi \in C_c^\infty(\Omega)$, f is indeed differentiable. To see why this is the case, use (2.12) and (2.8) to compute

$$f(t) = \iint_\Omega \sqrt{1 + |\nabla(u + t\varphi)|^2}\; dxdy, \quad \text{for all } t \in \mathbb{R}, \tag{2.14}$$

where

$$\nabla(u + t\varphi) = \nabla u + t\nabla\varphi, \quad \text{for all } t \in \mathbb{R},$$

by the linearity of the differential operator ∇. It then follows that

$$\begin{aligned}|\nabla(u+t\varphi)|^2 &= (\nabla u+t\nabla\varphi)\cdot(\nabla u+t\nabla\varphi)\\ &= \nabla u\cdot\nabla u+t\nabla u\cdot\nabla\varphi+t\nabla\varphi\cdot\nabla u+t^2\nabla\varphi\cdot\nabla\varphi\\ &= |\nabla u|^2+2t\nabla u\cdot\nabla\varphi+t^2|\nabla\varphi|^2,\end{aligned}$$

so that, substituting into (2.14),

$$f(t)=\iint_\Omega\sqrt{1+|\nabla u|^2+2t\nabla u\cdot\nabla\varphi+t^2|\nabla\varphi|^2}\;dxdy,\qquad\text{for all } t\in\mathbb{R}. \tag{2.15}$$

Since the integrand in (2.15) is C^1, we can differentiate under the integral sign (see Proposition B.1.1 in Section B.1 of Appendix B on page 147 of these notes) to get

$$f'(t)=\iint_\Omega\frac{\nabla u\cdot\nabla\varphi+t|\nabla\varphi|^2}{\sqrt{1+|\nabla u|^2+2t\nabla u\cdot\nabla\varphi+t^2|\nabla\varphi|^2}}\;dxdy, \tag{2.16}$$

for all $t\in\mathbb{R}$. Thus, f is differentiable and, substituting 0 for t in (2.16),

$$f'(0)=\iint_\Omega\frac{\nabla u\cdot\nabla\varphi}{\sqrt{1+|\nabla u|^2}}\;dxdy. \tag{2.17}$$

Hence, if u is a minimizer of the area functional in $\mathcal{A}_g$, it follows from (2.12) and (2.17) that

$$\iint_\Omega\frac{\nabla u\cdot\nabla\varphi}{\sqrt{1+|\nabla u|^2}}\;dxdy=0,\qquad\text{for all } \varphi\in C_c^\infty(\Omega). \tag{2.18}$$

The statement in (2.18) provides a necessary condition for the existence of a minimizer of the area functional in $\mathcal{A}_g$. We will next see how (2.18) gives rise to a PDE that $u\in C^2(\Omega)\cap C(\overline{\Omega})$ must satisfy in order for it to be minimizer of the area functional in $\mathcal{A}_g$.

First, we rewrite the integral on the left–hand side of (2.18) as

$$\iint_\Omega\frac{\nabla u\cdot\nabla\varphi}{\sqrt{1+|\nabla u|^2}}\;dxdy=\iint_\Omega\frac{u_x\varphi_x+u_y\varphi_y}{\sqrt{1+|\nabla u|^2}}\;dxdy,$$

or

$$\begin{aligned}\iint_\Omega\frac{\nabla u\cdot\nabla\varphi}{\sqrt{1+|\nabla u|^2}}\;dxdy &= \iint_\Omega\frac{u_x}{\sqrt{1+|\nabla u|^2}}\varphi_x\;dxdy\\ &\quad+\iint_\Omega\frac{u_y}{\sqrt{1+|\nabla u|^2}}\varphi_y\;dxdy.\end{aligned} \tag{2.19}$$

Next, we integrate by parts the first integral on the right–hand side of (2.19) using the integration by parts formula in (B.37) (see Section B.3 in Appendix

B) to get

$$\begin{aligned}\iint_\Omega \frac{u_x}{\sqrt{1+|\nabla u|^2}}\varphi_x \; dxdy &= \int_{\partial\Omega} \frac{\varphi u_x n_1}{\sqrt{1+|\nabla u|^2}} \; ds \\ &\quad - \iint_\Omega \varphi \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+|\nabla u|^2}}\right) \; dxdy,\end{aligned} \tag{2.20}$$

where n_1 is the first component of the outward unit normal, $\widehat{n}$, to the boundary, $\partial\Omega$, of Ω, and ds is the element of arclength along $\partial\Omega$.

Similar calculations to those leading to (2.20) can be used to obtain

$$\begin{aligned}\iint_\Omega \frac{u_y}{\sqrt{1+|\nabla u|^2}}\varphi_y \; dxdy &= \int_{\partial\Omega} \frac{\varphi u_y n_2}{\sqrt{1+|\nabla u|^2}} \; ds \\ &\quad - \iint_\Omega \varphi \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right) \; dxdy,\end{aligned} \tag{2.21}$$

where n_2 is the second component of the outward unit normal, $\widehat{n}$, to $\partial\Omega$.

Adding the expressions in (2.20) and (2.21), we see that the expression in (2.19) can now be written as

$$\begin{aligned}\iint_\Omega \frac{\nabla u\cdot\nabla\varphi}{\sqrt{1+|\nabla u|^2}} \; dxdy &= \int_{\partial\Omega} \frac{\varphi\nabla u\cdot\widehat{n}}{\sqrt{1+|\nabla u|^2}} \; ds \\ &\quad - \iint_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\varphi \; dxdy.\end{aligned} \tag{2.22}$$

In view of (2.22), we see that we can now rewrite (2.18) as

$$\int_{\partial\Omega} \varphi\frac{\nabla u\cdot\widehat{n}}{\sqrt{1+|\nabla u|^2}} \; ds - \iint_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\varphi \; dxdy + = 0, \tag{2.23}$$

for all $\varphi \in C_c^\infty(\Omega)$, where the first integral in (2.23) is a line integral around the boundary of Ω.

Now, since $\varphi \in C_c^\infty(\Omega)$ vanishes in a neighborhood of the boundary of Ω, it follows from (2.23) that

$$\iint_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\varphi \; dxdy = 0, \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{2.24}$$

By virtue of the assumption that u is a C^2 functions, it follows that the divergence term of the integrand (2.24) is continuous on Ω, it follows from the statement in (2.24) that

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \quad \text{in } \Omega, \tag{2.25}$$

by virtue of the Fundamental Lemmas in the Calculus of Variations (see Section 3.2 in these notes).

The equation in (2.25) is a second order nonlinear PDE known as the **minimal surface equation**. It provides a necessary condition for a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to be a minimizer of the area functional in $\mathcal{A}_g$. Since, we are also assuming that $u \in \mathcal{A}_g$, we get that u must solve the boundary value problem (BVP):

$$\left\{\begin{array}{rcll} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) & = & 0 & \text{in } \Omega; \\ \\ u & = & g & \text{on } \partial\Omega. \end{array}\right. \tag{2.26}$$

The BVP in (2.26) is called the **Dirichlet problem** for the minimal surface equation.

The PDE in (2.25) can also be written as

$$(1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} = 0, \quad \text{in } \Omega, \tag{2.27}$$

where the subscripted symbols read as follows:

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y},$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2},$$

and

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = u_{yx}. \tag{2.28}$$

The fact that the "mixed" second partial derivatives in (2.28) are equal follows from the assumption that u is a C^2 function.

The equation in (2.27) is a nonlinear, second order, elliptic PDE.

2.2 The Linearized Minimal Surface Equation

For the case in which the wire loop in the previous section is very close to a horizontal plane (see Figure 2.2.2), it is reasonable to assume that, if $u \in \mathcal{A}_g$, $|\nabla u|$ is very small throughout Ω. We can therefore use the linear approximation

$$\sqrt{1+t} \approx 1 + \frac{1}{2}t, \quad \text{for small } |t|, \tag{2.29}$$

to approximate the area function in (2.8) by

$$A(u) \approx \iint_\Omega \left[1 + \frac{1}{2}|\nabla u|^2\right] dxdy, \quad \text{for all } u \in \mathcal{A}_g,$$

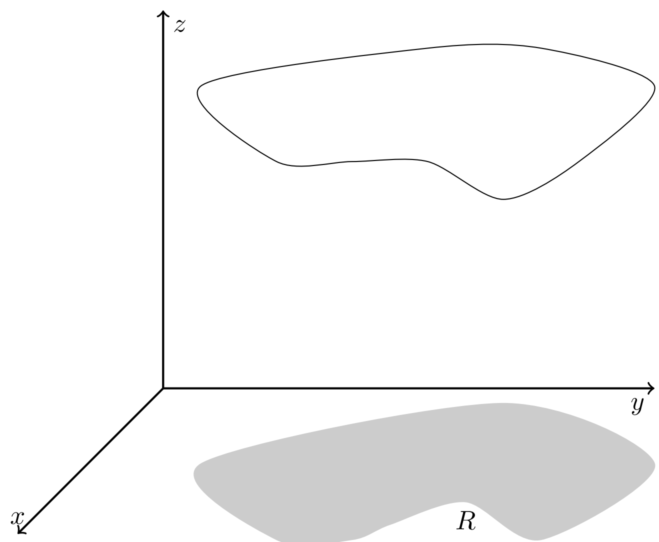

Figure 2.2.2: Almost Planar Wire Loop

so that

$$A(u) \approx \text{area}(\Omega) + \frac{1}{2}\iint_\Omega |\nabla u|^2\, dxdy, \quad \text{for all } u \in \mathcal{A}_g. \tag{2.30}$$

The integral on the right–hand side of the expression in (2.30) is known as the **Dirichlet Integral**. We will use it in these notes to define the Dirichlet functional, $\mathcal{D}\colon \mathcal{A}_g \to \mathbb{R}$,

$$\mathcal{D}(u) = \frac{1}{2}\iint_\Omega |\nabla u|^2\, dxdy, \quad \text{for all } u \in \mathcal{A}_g. \tag{2.31}$$

Thus, in view of (2.30) and (2.31),

$$A(u) \approx \text{area}(\Omega) + \mathcal{D}(u), \quad \text{for all } u \in \mathcal{A}_g. \tag{2.32}$$

Thus, according to (2.32), for wire loops close to a horizontal plane, minimal surfaces spanning the wire loop can be approximated by solutions to the following variational problem,

Problem 2.2.1 (Variational Problem 2)**.** *Out of all functions in $\mathcal{A}_g$, find one such that*

$$\mathcal{D}(u) \leqslant \mathcal{D}(v), \quad \textit{for all } v \in \mathcal{A}_g. \tag{2.33}$$

It can be shown that a necessary condition for $u \in \mathcal{A}_g$ to be a solution to the Variational Problem 2.2.1 is that u solves the boundary value problem

$$\left\{\begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega; \\ & & & \\ u & = & g & \text{on } \partial\Omega, \end{array}\right. \tag{2.34}$$

where

$$\Delta u = u_{xx} + u_{yy},$$

the two–dimensional Laplacian. The BVP in (2.34) is called the Dirichlet Problem for Laplace's equation.

2.3 Vibrating String

Consider a string of length L (imagine a guitar string or a violin string) whose ends are located at $x = 0$ and $x = L$ along the x–axis (see Figure 2.3.3). We assume that the string is made of some material of (linear) density $\rho(x)$ (in units of mass per unit length). Assume that the string is fixed at the end–points and

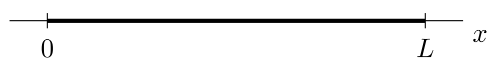

Figure 2.3.3: String of Length L at Equilibrium

is tightly stretched so that there is a constant tension, τ, acting tangentially along the string at all times. We would like to model what happens to the string after it is plucked to a configuration like that pictured in Figure 2.3.4 and then released. We assume that the shape of the plucked string is described

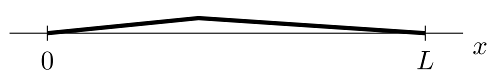

Figure 2.3.4: Plucked String of Length L

by a continuous function, f, of x, for $x \in [0, L]$. At any time $t \geqslant 0$, the shape of the string is described by a function, u, of x and t; so that $u(x, t)$ gives the vertical displacement of a point in the string located at x when the string is in the equilibrium position pictured in Figure 2.3.3, and at time $t \geqslant 0$. We then have that

$$u(x, 0) = f(x), \quad \text{for all } x \in [0, L]. \tag{2.35}$$

In addition to the initial condition in (2.35), we will also prescribe the initial speed of the string,

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \quad \text{for all } x \in [0, L], \tag{2.36}$$

where g is a continuous function of x; for instance, if the plucked string is released from rest, then $g(x) = 0$ for all $x \in [0, L]$. We also have the boundary conditions,

$$u(0, t) = u(L, t) = 0, \text{ for all } t, \tag{2.37}$$

which model the assumption that the ends of the string do not move.

The question we would like to answer is: Given the initial conditions in (2.35) and (2.36), and the boundary conditions in (2.37), can we determine the shape of the string, $u(x,t)$, for all $x \in [0, L]$ and all times $t > 0$? We will answer this questions in a subsequent chapter in these notes. In this section, though, we will derive a necessary condition in the form of a PDE that u must satisfy in order for it to describe the motion of the vibrating string.

To find the PDE governing the motion of the string, we will formulate the problem as a variational problem. We will use Hamilton's Principle in Mechanics, or the **Principle of Least Action** (see [GPJS01, pp. 34–35]). This principle states that the path that configurations of a mechanical system take from time $t = 0$ to $t = T$ is such that a quantity called the **action** is minimized (or optimized) along the path. The action is defined by

$$A = \int_0^T [K(t) - V(t)]\, dt, \tag{2.38}$$

where $K(t)$ denotes the kinetic energy of the system at time t, and $V(t)$ its potential energy at time t.

For the case of a string whose motion is described by small vertical displacements $u(x,t)$, for all $x \in [0, L]$ and all times t, the kinetic energy is given by

$$K(t) = \frac{1}{2}\int_0^L \rho(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2\, dx. \tag{2.39}$$

To see how (2.39) comes about, note that the kinetic energy of a particle of mass m is

$$K = \frac{1}{2}mv^2,$$

where v is the speed of the particle. Thus, for a small element of the string whose projection on the x–axis is the interval $[x, x+\Delta x]$, so that its approximate length is Δx, the kinetic energy is, approximately,

$$\Delta K \approx \frac{1}{2}\rho(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2\, \Delta x. \tag{2.40}$$

Thus, adding up the kinetic energies in (2.40) over all elements of the string adding in length to L, and letting $\Delta x \to 0$, yields the expression in (2.39), which we rewrite as

$$K(t) = \frac{1}{2}\int_0^L \rho u_t^2\, dx, \qquad \text{for all } t, \tag{2.41}$$

where u_t denotes the partial derivative of u with respect to t.

To compute the potential energy of the string, we compute the work done by the tension, τ, along the string in stretching the string from its equilibrium length of L, to the length at time t given by

$$\int_0^L \sqrt{1 + u_x^2}\, dx; \tag{2.42}$$

so that

$$V(t) = \tau \left[\int_0^L \sqrt{1+u_x^2}\ dx - L \right], \quad \text{for all } t. \tag{2.43}$$

Since we are considering small vertical displacements of the string, we can linearize the expression in (2.42) by means of the linear approximation in (2.29) to get

$$\int_0^L \sqrt{1+u_x^2}\ dx \approx \int_0^L [1 + \frac{1}{2}u_x^2]\ dx = L + \frac{1}{2}\int_0^L \frac{1}{2}u_x^2\ dx,$$

so that, substituting into (2.43),

$$V(t) \approx \frac{1}{2}\int_0^L \tau u_x^2\ dx, \quad \text{for all } t. \tag{2.44}$$

Thus, in view of (2.38), (2.41) and (2.44), we consider the problem of optimizing the quantity

$$A(u) = \int_0^T \int_0^L \left[\frac{1}{2}\rho u_t^2 - \frac{1}{2}\tau u_x^2 \right]\ dxdt, \tag{2.45}$$

where we have substituted the expressions for $K(t)$ and $V(t)$ in (2.41) and (2.44), respectively, into the expression for the action in (2.38).

We will use the expression for the action in (2.45) to define a functional in the class of functions $\mathcal{A}$ defined as follows: Let $R = (0, L) \times (0, T)$, the Cartesian product of the open intervals $(0, L)$ and $(0, T)$. Then, R is an open rectangle in the xt–plane. We say that $u \in \mathcal{A}$ if $u \in C^2(R) \cap C(\overline{R})$, and u satisfies the initial conditions in (2.35) and (2.36), and the boundary conditions in (2.37). Then, the action functional,

$$A \colon \mathcal{A} \to \mathbb{R},$$

is defined by the expression in (2.45), so that

$$A(u) = \frac{1}{2}\iint_R \left[\rho u_t^2 - \tau u_x^2\right]\ dxdt, \quad \text{for } u \in \mathcal{A}. \tag{2.46}$$

Next, for $\varphi \in C_c^\infty(R)$, note that $u + s\varphi \in \mathcal{A}$, since φ has compact support in R, and therefore φ and all its derivatives are 0 on ∂R. We can then define a real valued function $h \colon \mathbb{R} \to \mathbb{R}$ by

$$h(s) = A(u + s\varphi), \quad \text{for } s \in \mathbb{R}, \tag{2.47}$$

Using the definition of the functional A in (2.46), we can rewrite $h(s)$ in (2.47) as

$$\begin{aligned} h(s) &= \frac{1}{2}\iint_R \left[\rho[(u+s\varphi)_t]^2 - \tau[(u+s\varphi)_x]^2\right]\ dxdt \\ &= \frac{1}{2}\iint_R \left[\rho[u_t + s\varphi_t]^2 - \tau[u_x + s\varphi_x]^2\right]\ dxdt, \end{aligned}$$

so that

$$h(s) \quad = \quad A(u) + s \iint_R [\rho u_t \varphi_t - \tau u_x \varphi_x] \; dxdt + s^2 A(\varphi), \tag{2.48}$$

for $s \in \mathbb{R}$, where we have used the definition of the action functional in (2.46).

It follows from (2.48) that h is differentiable and

$$h'(s) \quad = \quad \iint_R [\rho u_t \varphi_t - \tau u_x \varphi_x] \; dxdt + 2sA(\varphi), \quad \text{for } s \in \mathbb{R}. \tag{2.49}$$

The principle of least action implies that, if u describes the shape of the string, then $s = 0$ must be a critical point of h. Hence, $h'(0) = 0$ and (2.49) implies that

$$\iint_R [\rho u_t \varphi_t - \tau u_x \varphi_x] \; dxdt = 0, \quad \text{for } \varphi \in C_c^\infty(R), \tag{2.50}$$

is a necessary condition for $u(x,t)$ to describe the shape of a vibrating string for all times t.

Next, we use the integration by parts formulas

$$\iint_R \psi \frac{\partial \varphi}{\partial x} \; dxdt = \int_{\partial R} \psi\varphi n_1 \; ds - \iint_R \frac{\partial \psi}{\partial x} \varphi \; dxdt,$$

for C^1 functions ψ and φ, where n_1 is the first component of the outward unit normal, $\vec{n}$, on ∂R (wherever this vector is defined), and

$$\iint_R \psi \frac{\partial \varphi}{\partial t} \; dxdt = \int_{\partial R} \psi\varphi n_2 \; ds - \iint_R \frac{\partial \psi}{\partial t} \varphi \; dxdt,$$

where n_2 is the second component of the outward unit normal, $\vec{n}$, to obtain

$$\iint_R \rho u_t \varphi_t \; dxdt = \int_{\partial R} \rho u_t \varphi n_2 \; ds - \iint_R \frac{\partial}{\partial t}[\rho u_t] \varphi \; dxdt,$$

so that

$$\iint_R \rho u_t \varphi_t \; dxdt = - \iint_R \frac{\partial}{\partial t}[\rho u_t] \varphi \; dxdt, \tag{2.51}$$

since φ has compact support in R.

Similarly,

$$\iint_R \tau u_x \varphi_x \; dxdt = - \iint_R \frac{\partial}{\partial x}[\tau u_x] \varphi \; dxdt. \tag{2.52}$$

Next, substitute the results in (2.51) and (2.52) into (2.50) to get

$$\iint_R \left[\frac{\partial}{\partial t}[\rho u_t] - \frac{\partial}{\partial x}[\tau u_x] \right] \varphi \; dxdt = 0, \quad \text{for } \varphi \in C_c^\infty(R). \tag{2.53}$$

Thus, applying the Fundamental Lemma of the Calculus of Variations (see the next chapter in these notes), we obtain from (2.53) that

$$\rho \frac{\partial^2 u}{\partial t^2} - \tau \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{in } R, \tag{2.54}$$

since we area assuming that u is C^2, ρ is a continuous function of x, and τ is constant.

The PDE in (2.54) is called the one–dimensional **wave equation**. It is sometimes written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho}\frac{\partial^2 u}{\partial x^2},$$

or

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{2.55}$$

where

$$c^2 = \frac{\tau}{\rho},$$

for the case in which ρ is assumed to be constant.

The wave equation in (2.54), or (2.55), is a second order, linear, hyperbolic PDE.

Chapter 3

Indirect Methods in the Calculus of Variations

We begin this chapter by discussing a very simple problem in the Calculus of Variations: Given two points in the plane, find a smooth path connecting those two points that has the shortest length. A solution of this problem is called a **geodesic curve** between the two points. This example is simple because we know the answer to this question from Euclidean geometry. Nevertheless, the solutions that we present here serve to illustrate both the direct and indirect methods in the calculus of variation. It will certainly be a good introduction to the indirect methods.

3.1 Geodesics in the plane

Let P and Q denote two points in the xy–plane with coordinates (x_o, y_o) and (x_1, y_1), respectively. We consider the class, $\mathcal{A}$, of smooth paths that connect P to Q. One of theses paths is shown in Figure 3.1.1. We assume that paths are given parametrically by the pair of functions

$$(x(s), y(s)), \qquad \text{for } s \in [0,1],$$

where $x\colon [0,1] \to \mathbb{R}$ and $y\colon [0,1] \to \mathbb{R}$ are differentiable functions with continuous derivatives in some oven interval that contains $[0,1]$, such that

$$(x(0), y(0)) = (x_o, y_o) \quad \text{and} \quad (x(1), y(1)) = (x_1, y_1).$$

We may write this more succinctly as

$$\mathcal{A} = \{(x,y) \in C^1([0,1], \mathbb{R}^2) \mid (x(0), y(0)) = P \ \text{ and } \ (x(1), y(1)) = Q\}. \tag{3.1}$$

We define a **functional**, $J\colon \mathcal{A} \to \mathbb{R}$, by

$$J(x,y) = \int_0^1 \sqrt{(x'(s))^2 + (y'(s))^2}\, ds, \qquad \text{for all } (x,y) \in \mathcal{A}. \tag{3.2}$$

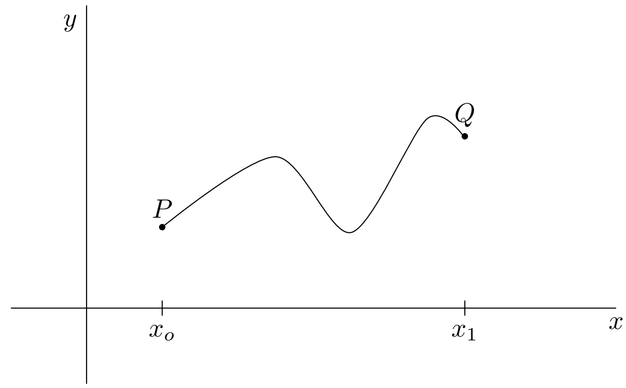

Figure 3.1.1: Path connecting P and Q

Thus, $J(x, y)$ is the arc–length along the path from P to Q parametrized by

$$(x(s), y(s)), \qquad \text{for } s \in [0, 1].$$

We would like to solve the following variational problem:

Problem 3.1.1 (Geodesic Problem 1)**.** *Out of all paths in $\mathcal{A}$, find one, (x, y), such that*

$$J(x, y) \leqslant J(u, v), \quad \textit{for all } (u, v) \in \mathcal{A}. \tag{3.3}$$

Observe that the expression for J in (3.2) can be written as

$$J(x, y) = \int_0^1 |(x'(s), y'(s))| \, ds, \quad \text{for all } (x, y) \in \mathcal{A}, \tag{3.4}$$

where $|(\cdot, \cdot)|$ in the integrand in (3.4) denotes the Euclidean norm in $\mathbb{R}^2$.

We will first use this example to illustrate the direct method in the Calculus of Variations. We begin by showing that the functional J defined in (3.4) is bounded below in $\mathcal{A}$ by $\|Q - P\|$, or

$$|(x_1, y_1) - (x_o, y_o)| = \sqrt{(x_1 - x_o)^2 + (y_1 - y_o)^2},$$

the Euclidean distance from P to Q; that is

$$|Q - P| \leqslant J(u, v), \quad \text{for all } (u, v) \in \mathcal{A}. \tag{3.5}$$

Indeed, it follows from the Fundamental Theorem of Calculus that

$$(u(1), v(1)) - (u(0), v(0)) = \int_0^1 (u'(s), v'(s)) \, ds,$$

for any $(u, v) \in \mathcal{A}$, or

$$Q - P = \int_0^1 (u'(s), v'(s)) \, ds, \quad \text{for any } (u, v) \in \mathcal{A}. \tag{3.6}$$

Now, using the fact that $|Q-P|^2 = (Q-P)\cdot(Q-P)$, the dot product (or Euclidean inner product) of $Q-P$ with itself, we obtain from (3.6) that

$$|Q-P|^2 = (Q-P)\cdot\int_0^1 (u'(s),v'(s))\;ds, \quad \text{for any } (u,v)\in\mathcal{A},$$

or

$$|Q-P|^2 = \int_0^1 (Q-P)\cdot(u'(s),v'(s))\;ds, \quad \text{for any } (u,v)\in\mathcal{A}. \tag{3.7}$$

Thus, applying the Cauchy–Schwarz inequality to the integrand of the integral on the right–hand side of (3.7), we get that

$$|Q-P|^2 \leqslant \int_0^1 |Q-P|\;|(u'(s),v'(s))|\;ds, \quad \text{for any } (u,v)\in\mathcal{A},$$

or

$$|Q-P|^2 \leqslant |Q-P|\int_0^1 |(u'(s),v'(s))|\;ds, \quad \text{for any } (u,v)\in\mathcal{A},$$

or

$$|Q-P|^2 \leqslant |Q-P|\;J(u,v), \quad \text{for any } (u,v)\in\mathcal{A}. \tag{3.8}$$

For the case in which $P\neq Q$, we see that the estimate in (3.5) follows from the inequality in (3.8).

Now, it follows from (3.5) that the functional J is bounded from below in $\mathcal{A}$ by $\|Q-P\|$. Hence, the infimum of J over $\mathcal{A}$ exists and

$$|Q-P| \leqslant \inf_{(u,v)\in\mathcal{A}} J(u,v). \tag{3.9}$$

Next, we see that the infimum in (3.9) is attained and is $|Q-P|$. Indeed, let

$$(x(s),y(s)) = P + s(Q-P), \quad \text{for } s\in[0,1], \tag{3.10}$$

the straight line segment connecting P to Q.

Note that

$$(x(0),y(0)) = P \quad \text{and} \quad (x(1),y(1)) = Q,$$

and (x,y) is a differentiable path with

$$(x'(s),y'(s)) = Q-P, \quad \text{for all } s\in[0,1]. \tag{3.11}$$

Thus, (x,y) belongs to the class $\mathcal{A}$ defined in (3.1).

Using the definition of the functional J in (3.4) and the fact in (3.11), compute

$$J(x,y) = \int_0^1 |Q-P|\;ds = |Q-P|.$$

Consequently, we get from (3.9) that

$$\inf_{(u,v)\in\mathcal{A}} J(u,v) = |Q-P|. \tag{3.12}$$

Furthermore, the infimum in (3.12) is attained on the path given in (3.10). This is in accord with the notion from elementary Euclidean geometry that the shortest distance between two points is attained along a straight line segment connecting the points.

Next, we illustrate the indirect method in the Calculus of Variations, which is the main topic of this chapter.

We consider the special parametrization

$$(x(s), y(s)) = (s, y(s)), \qquad \text{for } x_o \leqslant s \leqslant x_1,$$

where $y\colon [x_o, x_1] \to \mathbb{R}$ is a continuous function that is differentiable, with continuous derivative, in an open interval that contains $[x_o, x_1]$. Here we are assuming that

$$x_o < x_1,$$

$$y(x_o) = y_o \quad \text{and} \quad y(x_1) = y_1.$$

Thus, the path connecting P and Q is the graph of a differentiable function over the interval $[x_1, x_2]$. This is illustrated in Figure 3.1.1.

We'll have to define the class $\mathcal{A}$ and the functional $J\colon \mathcal{A} \to \mathbb{R}$ in a different way. Put

$$\mathcal{A} = \{y \in C^1([x_o, x_1], \mathbb{R}) \mid y(x_o) = y_o \text{ and } y(x_1) = y_1\}, \tag{3.13}$$

and

$$J(y) = \int_{x_o}^{x_1} \sqrt{1 + (y'(s))^2}\, ds, \qquad \text{for all } y \in \mathcal{A}. \tag{3.14}$$

We would like to solve the variational problem:

Problem 3.1.2 (Geodesic Problem 2)**.** *Out of all functions in $\mathcal{A}$, find one, y, such that*

$$J(y) \leqslant J(v), \qquad \textit{for all } v \in \mathcal{A} \tag{3.15}$$

In the indirect method, we assume that we have a solution of the optimization problem, and then deduce conditions that this function must satisfy; in other words, we find a necessary condition for a function in a competing class to be a solution. Sometimes, the necessary conditions can be used to find a candidate for a solution of the optimization problem (a critical "point"). The next step in the indirect method is to verify that the candidate indeed solves the optimization problem.

Thus, assume that there is a function $y \in \mathcal{A}$ that solves Geodesic Problem 2; that is, y satisfy the estimates in (3.15). Next, let $\eta\colon [x_o, x_1] \to \mathbb{R}$ denote a C^1 function such that $\eta(x_o) = 0$ and $\eta(x_1) = 0$ (we will see later in these notes how this function η can be constructed). It then follows from the definition of $\mathcal{A}$ in (3.13) that

$$y + t\eta \in \mathcal{A}, \qquad \text{for all } t \in \mathbb{R}.$$

It then follows from (3.15) that

$$J(y) \leqslant J(y + t\eta), \quad \text{for all } t \in \mathbb{R}. \tag{3.16}$$

Next, define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = J(y + t\eta), \quad \text{for all } t \in \mathbb{R}. \tag{3.17}$$

It follows from (3.16) and the definition of f in (3.17) that f has a minimum at 0; that is

$$f(t) \geqslant f(0), \quad \text{for all } t \in \mathbb{R}.$$

Thus, if it can be shown that the function f defined in (3.17) is differentiable, we get the necessary condition

$$f'(0) = 0, \tag{3.18}$$

which can be written in terms of J as

$$\frac{d}{dt}[J(y + t\eta)]\bigg|_{t=0} = 0. \tag{3.19}$$

Next, we proceed to show that the function

$$f(t) = J(y + t\eta) = \int_{x_o}^{x_1} \sqrt{1 + (y'(s) + t\eta'(s))^2}\; ds, \quad \text{for } t \in \mathbb{R},$$

is a differentiable function of t.

Observe that we can write f as

$$f(t) = \int_{x_o}^{x_1} \sqrt{1 + (y'(s))^2 + 2ty'(s)\eta'(s) + t^2(\eta'(s))^2}\; ds, \quad \text{for } t \in \mathbb{R}. \tag{3.20}$$

To see show that f is differentiable, we need to see that we can differentiate the expression on the right–hand side of (3.20) under the integral sign. This follows from the fact that the partial derivative of the integrand on the right–hand side of 3.20,

$$\frac{\partial}{\partial t}\sqrt{1 + (y'(s))^2 + 2ty'(s)\eta'(s) + t^2(\eta'(s))^2},$$

or

$$\frac{y'(s)\eta'(s) + t(\eta'(s))^2}{\sqrt{1 + (y'(s))^2 + 2ty'(s)\eta'(s) + t^2(\eta'(s))^2}},$$

for t, y and s, for s in some open interval containing $[x_o, x_1]$. It then follows from the results in Appendix B.1 that f given in (3.20) is differentiable and

$$f'(t) = \int_{x_o}^{x_1} \frac{y'(s)\eta'(s) + t(\eta'(s))^2}{\sqrt{1 + (y'(s))^2 + 2ty'(s)\eta'(s) + t^2(\eta'(s))^2}}\; ds, \quad \text{for } t \in \mathbb{R}, \tag{3.21}$$

(see Proposition B.1.1). Evaluating the expression for $f'(t)$ in (3.21) at $t = 0$, we obtain that

$$f'(0) = \int_{x_o}^{x_1} \frac{y'(s)\eta'(s)}{\sqrt{1 + (y'(s))^2}}\; ds. \tag{3.22}$$

Thus, in view of (3.18) and (3.22), we see that a necessary condition for $y \in \mathcal{A}$, where $\mathcal{A}$ is given in (3.13), to be a minimizer of the functional $J\colon \mathcal{A} \to \mathbb{R}$ given in (3.14), is that

$$\int_{x_o}^{x_1} \frac{y'(s)}{\sqrt{1+(y'(s))^2}}\ \eta'(s)\ ds = 0, \qquad \text{for all } \eta \in C^1_o([x_o,x_1],\mathbb{R}), \tag{3.23}$$

where

$$C^1_o([x_o,x_1],\mathbb{R}) = \{\eta \in C^1([x_o,x_1],\mathbb{R}) \mid \eta(x_o) = 0 \text{ and } \eta(x_1) = 0\}, \tag{3.24}$$

the class of C^1, real–valued functions in $[x_o, x_1]$ that vanish at the end–points of the interval $[x_o, x_1]$.

We will see in the next section that if the condition in (3.23) holds true for every $\eta \in C^1_o([x_o,x_1],\mathbb{R})$, then

$$\frac{y'(s)}{\sqrt{1+(y'(s))^2}} = c_1, \qquad \text{for all } s \in [x_o, x_1], \tag{3.25}$$

where c_1 is a constant (see the second fundamental lemma in the Calculus of Variations, Lemma 3.2.8 on page 31 in these notes).

Now, squaring on both sides of (3.25),

$$\frac{(y'(s))^2}{1+(y'(s))^2} = c_1^2, \qquad \text{for all } s \in [x_o, x_1]. \tag{3.26}$$

It follows from (3.26) that $c_1^2 \neq 1$ (otherwise we would conclude that $1 = 0$, which is impossible). Hence, we can solve (3.26) for $(y'(s))^2$ to obtain

$$(y'(s))^2 = \frac{c^2}{1-c_1^2}, \qquad \text{for all } s \in [x_o, x_1], \tag{3.27}$$

from which we conclude that

$$y'(s) = c_2, \qquad \text{for all } s \in [x_o, x_1], \tag{3.28}$$

where c_2 is a constant.

We can solve the differential equation in (3.28) to obtain the general solution

$$y(s) = c_2 s + c_3, \qquad \text{for all } s \in [x_o, x_1], \tag{3.29}$$

where c_1 and c_2 is a constants.

Since we are also assuming that $y \in \mathcal{A}$, where $\mathcal{A}$ is given in (3.13), it follows that y must satisfy the boundary conditions

$$y(x_o) = y_o \quad \text{and} \quad y(x_1) = y_1.$$

We therefore get the following system of equations that c_2 and c_3 must solve

$$\begin{cases} c_2 x_o + c_3 = y_o; \\ c_2 x_1 + c_3 = y_1, \end{cases}$$

or

$$\begin{cases} x_o c_2 + c_3 = y_o; \\ x_1 c_2 + c_3 = y_1. \end{cases} \tag{3.30}$$

Solving the system in (3.30) for c_2 and c_3 yields

$$c_2 = \frac{y_1 - y_o}{x_1 - x_o} \quad \text{and} \quad c_3 = \frac{x_1 y_o - x_o y_1}{x_1 - x_o}.$$

Thus, using the expression for y in (3.29),

$$y(s) = \frac{y_1 - y_o}{x_1 - x_o}\; s + \frac{x_1 y_o - x_o y_1}{x_1 - x_o}. \tag{3.31}$$

Note that the expression in (3.31) is the equation of a straight line that goes through the points (x_o, y_o) and (x_1, y_1). Thus, we have shown that a candidate for a minimizer of the arc–length functional J defined in (3.14) over the class given in (3.13) is a straight line segment connecting the point P to the point Q. It remains to show that the function y in (3.31) is in indeed a minimizer of J in $\mathcal{A}$, and that it is the only minimizer of J in $\mathcal{A}$. This will be done in a subsequent section in these notes.

3.2 Fundamental Lemmas in the Calculus of Variations

In the previous section we found a necessary condition for a function $y \in \mathcal{A}$, where

$$\mathcal{A} = \{y \in C^1([x_o, x_1], \mathbb{R}) \mid y(x_o) = y_o \text{ and } y(x_1) = y_1\}, \tag{3.32}$$

to be a minimizer of the arc–length functional $J\colon \mathcal{A} \to \mathbb{R}$,

$$J(y) = \int_{x_o}^{x_1} \sqrt{1 + (y'(x))^2}\; dx, \quad \text{for } y \in C^1([x_o, x_1], \mathbb{R}), \tag{3.33}$$

over the class $\mathcal{A}$.

We found that, if $y \in \mathcal{A}$ is a minimizer of J over the class $\mathcal{A}$, then y must satisfy the condition

$$\int_{x_o}^{x_1} \frac{y'(x)}{\sqrt{1 + (y'(x))^2}}\; \eta'(s)\; dx = 0, \quad \text{for all } \eta \in C_o^1([x_o, x_1], \mathbb{R}). \tag{3.34}$$

This is a necessary condition for $y \in \mathcal{A}$ to be a minimizer of the functional $J\colon \mathcal{A} \to \mathbb{R}$ defined in (3.33).

We then invoked a fundamental lemma in the Calculus of Variations to deduce that the condition in (3.34) implies that

$$\frac{y'(x)}{\sqrt{1 + (y'(x))^2}} = c_1, \quad \text{for all } x \in [x_o, x_1], \tag{3.35}$$

and for some constant c_1. This is also a necessary condition for $y \in \mathcal{A}$ being a minimizer of the functional $J\mathcal{A} \to \mathbb{R}$ defined in (3.33), where $\mathcal{A}$ given in given in (3.32).

We will see in this section that the differential equation in (3.35) follows from the condition in (3.34) provided that the function

$$\frac{y'}{\sqrt{1+(y')^2}}$$

is known to be continuous. This is the case, for instance, if y is assumed to come from certain classes of differentiable functions defined on a closed and bounded interval. We'll start by defining these classes of functions.

Let $a, b \in \mathbb{R}$, and assume that $a < b$.

Definition 3.2.1. The class $C([a,b],\mathbb{R})$ consists of real–valued functions,

$$f\colon [a,b] \to \mathbb{R},$$

defined on the closed and bounded interval $[a,b]$, and which are assumed to be continuous on $[a,b]$. It can be shown that $C([a,b],\mathbb{R})$ is a linear space (or vector space) in which the operations are point–wise addition

$$(f+g)(x) = f(x) + g(x), \quad \text{for all } x \in [a,b], \text{ and all } f, g \in C([a,b],\mathbb{R}),$$

and scalar multiplication

$$(cf)(x) = cf(x), \quad \text{for all } x \in [a,b], \text{ all } c \in \mathbb{R} \text{ and all } f \in C([a,b],\mathbb{R}).$$

Definition 3.2.2. The class $C_o([a,b],\mathbb{R})$ consists of all functions in $C([a,b],\mathbb{R})$ that vanish at a and b; in symbols,

$$C_o([a,b],\mathbb{R}) = \{f \in C([a,b],\mathbb{R}) \mid f(a) = 0 \text{ and } f(b) = 0\}.$$

We note that $C_o([a,b],\mathbb{R})$ is a linear subspace of $C([a,b],\mathbb{R})$.

Definition 3.2.3. The class $C^1([a,b],\mathbb{R})$ consists of all functions $f\colon U \to \mathbb{R}$ that are differentiable in an open interval U that contains $[a,b]$ and such that f' is continuous on $[a,b]$. We note that $C^1([a,b],\mathbb{R})$ is a linear subspace of $C([a,b],\mathbb{R})$.

Definition 3.2.4. The class $C^1_o([a,b],\mathbb{R})$ consists of all functions $f \in C^1([a,b],\mathbb{R})$ that vanish at the end–points of the interval $[a,b]$; thus,

$$C^1_o([a,b],\mathbb{R}) = \{f \in C^1([a,b],\mathbb{R}) \mid f(a) = 0 \text{ and } f(b) = 0\}.$$

We begin by stating and proving the following basic lemma for the class $C([a,b],\mathbb{R})$.

Lemma 3.2.5 (Basic Lemma 1)**.** Let $f \in C([a,b],\mathbb{R})$ and assume that $f(x) \geqslant 0$ for all $x \in [a,b]$. Suppose that

$$\int_a^b f(x)\ dx = 0.$$

Then, $f(x) = 0$ for all $x \in [a,b]$.

Proof: Suppose that $f \in C([a,b],\mathbb{R})$, $f(x) \geqslant 0$ for all $x \in [a,b]$, and

$$\int_a^b f(x)\ dx = 0. \tag{3.36}$$

Assume, by way of contradiction, that there exists $x_o \in (a,b)$ with $f(x_o) > 0$. Then, since f is continuous at x_o, there exists $\delta > 0$ such that $(x_o - \delta, x_o + \delta) \subset (a,b)$ and

$$x \in (x_o - \delta, x_o + \delta) \Rightarrow |f(x) - f(x_o)| < \frac{f(x_o)}{2}. \tag{3.37}$$

Now, using the triangle inequality, we obtain the estimate

$$f(x_o) \leqslant |f(x) - f(x_o)| + |f(x)|;$$

so that, in view of (3.37) and the assumption that f is nonnegative on $[a,b]$,

$$f(x_o) < \frac{f(x_o)}{2} + f(x), \qquad \text{for } x_o - \delta < x < x_o + \delta,$$

from which we get that

$$f(x) > \frac{f(x_o)}{2}, \qquad \text{for } x_o - \delta < x < x_o + \delta. \tag{3.38}$$

It follows from (3.38) that

$$\int_{x_o-\delta}^{x_o+\delta} f(x)\ dx > \int_{x_o-\delta}^{x_o+\delta} \frac{f(x_o)}{2}\ dx = \delta f(x_o).$$

Thus, since we are assuming that $f \geqslant 0$ on $[a,b]$,

$$\int_a^b f(x)\ dx \geqslant \int_{x_o-\delta}^{x_o+\delta} f(x)\ dx > \delta f(x_o) > 0,$$

which is in direct contradiction with the assumption in (3.36). Consequently, $f(x) = 0$ for all $x \in (a,b)$. By the continuity of f on $[a,b]$, we also get that $f(x) = 0$ for all $x \in [a,b]$, and the proof of the lemma is now complete. ■

Lemma 3.2.6 (Basic Lemma 2)**.** Let $f \in C([a,b],\mathbb{R})$ and assume that

$$\int_c^d f(x)\ dx = 0, \qquad \text{for every } (c,d) \subset (a,b).$$

Then, $f(x) = 0$ for all $x \in [a,b]$.

Proof: Suppose that $f \in C([a,b],\mathbb{R})$ and

$$\int_c^d f(x)\, dx = 0, \qquad \text{for every } (c,d) \subset (a,b). \tag{3.39}$$

Arguing by contradiction, assume that $f(x_o) \neq 0$ for some $x_o \in (a,b)$. Without loss of generality, we may assume that $f(x_o) > 0$. Then, by the continuity of f, there exists $\delta > 0$ such that $(x_o - \delta, x_o + \delta) \subset (a,b)$ and

$$x \in (x_o - \delta, x_o + \delta) \Rightarrow |f(x) - f(x_o)| < \frac{f(x_o)}{2},$$

or

$$x \in (x_o - \delta, x_o + \delta) \Rightarrow f(x_o) - \frac{f(x_o)}{2} < f(x) < f(x_o) + \frac{f(x_o)}{2},$$

from which we get that

$$f(x) > \frac{f(x_o)}{2}, \qquad \text{for } x_o - \delta < x < x_o + \delta. \tag{3.40}$$

It follows from (3.40) that

$$\int_{x_o-\delta}^{x_o+\delta} f(x)\, dx > \int_{x_o-\delta}^{x_o+\delta} \frac{f(x_o)}{2}\, dx = \delta f(x_o) > 0,$$

which is in direct contradiction with (3.39). Hence, it must be the case that $f(x) = 0$ for $a < x < b$. It then follows from the continuity of f that $f(x) = 0$ for all $x \in [a,b]$. ∎

Next, we state and prove the first fundamental lemma of the Calculus of Variations. A version of this result is presented as a Basic Lemma in Section 3–1 in [Wei74].

Lemma 3.2.7 (Fundamental Lemma 1)**.** Let $G \in C([a,b],\mathbb{R})$ and assume that

$$\int_a^b G(x)\eta(x)\, dx = 0, \qquad \text{for every } \eta \in C_o([a,b],\mathbb{R}).$$

Then, $G(x) = 0$ for all $x \in [a,b]$.

Proof: Let $G \in C([a,b],\mathbb{R})$ and suppose that

$$\int_a^b G(x)\eta(x)\, dx = 0, \qquad \text{for every } \eta \in C_o([a,b],\mathbb{R}). \tag{3.41}$$

Arguing by contradiction, assume that there is $x_o \in (a,b)$ with $G(x_o) \neq 0$. Without loss of generality, we may also assume that $G(x_o) > 0$. Then, since G is continuous on $[a,b]$, there exists $\delta > 0$ such that $(x_o - \delta, x_o + \delta) \subset (a,b)$ such that

$$G(x) > \frac{G(x_o)}{2}, \qquad \text{for } x_o - \delta < x < x_o + \delta. \tag{3.42}$$

Put $x_1 = x_o - \delta$ and $x_2 = x_o + \delta$ and define $\eta\colon [a,b] \to \mathbb{R}$ by

$$\eta(x) = \begin{cases} 0, & \text{if } a \leqslant x \leqslant x_1; \\ (x - x_1)(x_2 - x), & \text{if } x_1 < x \leqslant x_2; \\ 0, & \text{if } x_2 < x \leqslant b. \end{cases} \tag{3.43}$$

Note that $\eta \in C([a,b])$ and that $\eta(a) = \eta(b) = 0$; so that, $\eta \in C_o([a,b],\mathbb{R})$. Observe also, from the definition of η in (3.43) that

$$\eta(x) > 0, \qquad \text{for } x_1 < x < x_2. \tag{3.44}$$

It also follows from the definition of η in 3.43 that

$$\int_a^b G(x)\eta(x)\ dx = \int_{x_1}^{x_2} G(x)\eta(x)\ dx;$$

so that, in view of (3.44) and (3.42),

$$\int_a^b G(x)\eta(x)\ dx > \frac{G(x_o)}{2} \int_{x_1}^{x_2} \eta(x)\ dx > 0,$$

which is in direct contradiction with the assumption in (3.41). Consequently, $G(x) = 0$ for all $x \in (a,b)$. The continuity of G on $[a,b]$ then implies that $G(x) = 0$ for all $x \in [a,b]$. ■

The following result is the one that we used in the example presented in the previous section. We shall refer to it as the second fundamental lemma in the Calculus of Variations.

Lemma 3.2.8 (Fundamental Lemma 2)**.** Let $G \in C([a,b],\mathbb{R})$ and assume that

$$\int_a^b G(x)\eta'(x)\ dx = 0, \qquad \text{for every } \eta \in C_o^1([a,b],\mathbb{R}).$$

Then, $G(x) = c$ for all $x \in [a,b]$, where c is a constant.

Proof: Let $G \in C([a,b],\mathbb{R})$ and assume that

$$\int_a^b G(x)\eta'(x)\ dx = 0, \qquad \text{for every } \eta \in C_o^1([a,b],\mathbb{R}). \tag{3.45}$$

Put

$$c = \frac{1}{b-a}\int_a^b G(x)\ dx, \tag{3.46}$$

the average value of G over $[a,b]$.

Define $\eta\colon [a,b] \to \mathbb{R}$ by

$$\eta(x) = \int_a^x (G(t) - c)\ dt, \qquad \text{for } x \in [a,b]. \tag{3.47}$$

Then, η is a differentiable function, by virtue of the Fundamental Theorem of Calculus, with

$$\eta'(x) = G(x) - c, \qquad \text{for } x \in [a, b], \tag{3.48}$$

which defines a continuous function on $[a, b]$. Consequently, $\eta \in C^1([a, b], \mathbb{R})$.

Observe also that, in view of (3.47),

$$\eta(a) = \int_a^a (G(t) - c)\ dt = 0,$$

and

$$\eta(b) = \int_a^b (G(t) - c)\ dt = \int_a^b G(t)\ dt - \int_a^b c\ dt = 0,$$

where we have used the definition of c in (3.46). It then follows that $\eta \in C_o^1([a, b], \mathbb{R})$.

Next, compute

$$\begin{aligned}\int_a^b (G(x) - c)^2\ dx &= \int_a^b (G(x) - c)(G(x) - c)\ dx \\ &= \int_a^b (G(x) - c)\eta'(x)\ dx,\end{aligned}$$

where we have used (3.48); so that,

$$\int_a^b (G(x) - c)^2\ dx = \int_a^b G(x)\eta'(x)\ dx - c\int_a^b \eta'(x).$$

Thus, using the assumption in (3.45) and the Fundamental Theorem of Calculus,

$$\int_a^b (G(x) - c)^2\ dx = 0, \tag{3.49}$$

since $\eta \in C_o^1([a, b], \mathbb{R})$.

It follows from (3.49) and the Basic Lemma 1 (Lemma 3.2.5) that $G(x) = c$ for all $x \in [a, b]$, since $(G - c)^2 \geqslant 0$ is $[a, b]$. ■

The following lemma combines the results of the first and the second fundamental lemmas in the Calculus of Variations.

Lemma 3.2.9 (Fundamental Lemma 3)**.** Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be continuous real values functions defined on $[a, b]$. Assume that

$$\int_a^b [f(x)\eta(x) + g(x)\eta'(x)]\ dx = 0, \qquad \text{for every } \eta \in C_o^1([a, b], \mathbb{R}).$$

Then, g is differentiable in (a, b) and

$$\frac{d}{dx}[g(x)] = f(x), \qquad \text{for all } x \in (a, b).$$

Proof: Let $f \in C([a,b],\mathbb{R})$, $g \in C([a,b],\mathbb{R})$, and assume that

$$\int_a^b [f(x)\eta(x) + g(x)\eta'(x)]\ dx = 0, \qquad \text{for every } \eta \in C_o^1([a,b],\mathbb{R}). \tag{3.50}$$

Put

$$F(x) = \int_a^x f(t)\ dt, \qquad \text{for } x \in [a,b]. \tag{3.51}$$

Then, by the Fundamental Theorem of Calculus, F is differentiable in (a,b) and

$$F'(x) = f(x), \qquad \text{for } x \in (a,b). \tag{3.52}$$

Next, let $\eta \in C_o^1([a,b],\mathbb{R})$ and use integration by parts to compute

$$\begin{aligned}\int_a^b f(x)\eta(x)\ dx &= f(x)\eta(x)\Big|_a^b - \int_a^b F(x)\eta'(x)\ dx \\ &= -\int_a^b F(x)\eta'(x)\ dx,\end{aligned}$$

since $\eta(a) = \eta(b) = 0$; consequently, we can rewrite the condition in (3.50) as

$$\int_a^b [g(x) - F(x)]\eta'(x)\ dx = 0, \qquad \text{for every } \eta \in C_o^1([a,b],\mathbb{R}). \tag{3.53}$$

We can then apply the second fundamental lemma (Lemma 3.2.8) to obtain from (3.53) that

$$g(x) - F(x) = C, \qquad \text{for all } x \in [a,b]$$

and some constant C, from which we get that

$$g(x) = F(x) + C, \qquad \text{for all } x \in [a,b]. \tag{3.54}$$

It follows from (3.54), (3.51) and the Fundamental Theorem of Calculus that g is differentiable with derivative

$$g'(x) = F'(x), \qquad \text{for all } x \in (a,b);$$

so that $g'(x) = f(x)$ for all $x \in (a,b)$, in view of (3.52). ■

3.3 The Euler–Lagrange Equations

In the previous two sections we saw how the second fundamental lemma in the Calculus of Variations (Lemma 3.2.8) can be used to obtain the differential equation

$$\frac{y'}{\sqrt{1+(y')^2}} = c_1, \tag{3.55}$$

where c_1 is a constant, as a necessary condition for $y \in C^1([x_o, x_1], \mathbb{R})$ to be a minimizer of the arc–length functional, $J\colon C^1([x_o, x_1], \mathbb{R}) \to \mathbb{R}$:

$$J(y) = \int_{x_o}^{x_1} \sqrt{1 + (y'(x))^2}\, dx, \quad \text{for } C^1([x_o, x_1], \mathbb{R}), \tag{3.56}$$

over the class of functions

$$\mathcal{A} = \{y\colon C^1([x_o, x_1], \mathbb{R}) \mid y(x_o) = y_o \text{ and } y(x_1) = y_1\}. \tag{3.57}$$

The differential equation in (3.55) and the boundary conditions

$$y(x_o) = y_o \text{ and } y(x_1) = y_1,$$

defining the class $\mathcal{A}$ in (3.57), constitute a boundary value problem. In Section 3.1 we were able to solve this boundary value problem to obtain the solution in (3.31), a straight line segment from the point (x_o, y_o) to the point (x_1, y_1). This is a candidate for a minimizer of the arc–length functional J given in (3.56) over the class $\mathcal{A}$ in (3.57).

In this section we illustrate the procedure employed in the Sections 3.1 and 3.2 in the case of a general functional $J\colon C^1([a, b], \mathbb{R}) \to \mathbb{R}$ of the form

$$J(y) = \int_a^b F(x, y(x), y'(x))\, dx, \quad \text{for } y \in C^1([a, b], \mathbb{R}), \tag{3.58}$$

where $F\colon [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function of three variables. An example of a function F in the integrand in (3.58) with value $F(x, y, z)$ is

$$F(x, y, z) = \sqrt{1 + z^2}, \quad \text{for all } (x, y, z) \in [x_o, x_1] \times \mathbb{R} \times \mathbb{R},$$

which is used in the definition of the arc–length functional J given in (3.56). In the following example we provide another example of $F(x, y, z)$ that comes up in the celebrated **brachistochrone problem**, or the curve of shortest descent time. A version of this problem is also discussed on page 19 of [Wei74] (note that in that version of the problem, the positive y–axis points downwards, while in the version discussed here, it points upwards as shown in Figure 3.3.2).

Example 3.3.1 (Brachistochrone Problem)**.** Given points P and Q in a vertical plane, with P higher that Q and to the left of Q (see Figure 3.3.2), find the curve connecting P to Q along which a particle of mass m descends from P to Q in the shortest possible time, assuming that only the force of gravity is acting on the particle.

The sketch in Figure 3.3.2 shows a possible curve of descent from P to Q. Observe also from the figure that we have assigned coordinates $(0, y_o)$ to P and (x_1, y_1) to Q, where $x_1 > 0$ and $y_o > y_1$.

We assume that the path from P to Q is the graph of a C^1 function $y\colon [0, x_1] \to \mathbb{R}$ such that

$$y(0) = y_o \quad \text{and} \quad y(x_1) = y_1.$$

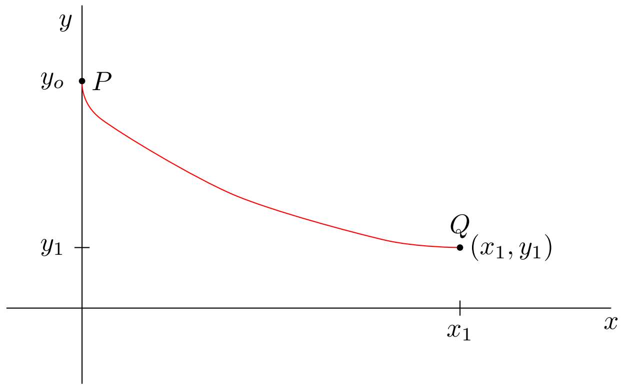

Figure 3.3.2: Path descending from P to Q

The arc-length along the path from the point P to any point on the path, as a function of x, is then given by

$$s(x) = \int_0^x \sqrt{1 + (y'(t))^2}\, dt, \qquad \text{for } 0 \leqslant x \leqslant x_1;$$

so that,

$$s'(x) = \sqrt{1 + (y'(x))^2}, \qquad \text{for } 0 < x < x_1, \tag{3.59}$$

by the Fundamental Theorem of Calculus.

The speed, v, of the particle along the path at any point on the curve is given by

$$v = \frac{ds}{dt}. \tag{3.60}$$

We can use (3.59) and (3.60) to obtain a formula for the descent time, T, of the particle:

$$T = \int_0^{x_1} \frac{s'(x)}{v}\, dx,$$

or

$$T = \int_0^{x_1} \frac{\sqrt{1 + (y'(x))^2}}{v}\, dx. \tag{3.61}$$

It remains to compute v in the denominator of the integrand in (3.61).

The speed of the particle will depend on the location of the particle along the path. If we assume that the particle is released from rest, the $v = 0$ at P. To find the speed at other points on the path, we will use the law of conservation of energy, which says that the total mechanical energy of the system is conserved; that is, the total energy remains constant throughout the motion. The total energy of this particular system is the sum of the kinetic energy and the potential energy of the particle of mass m.

$$\text{Total Energy} = \text{Kinetic Energy} + \text{Potential Energy}.$$

At P the particle is at rest, so

$$\text{Kinetic Energy at } P = 0,$$

while its potential energy is

$$\text{Potential Energy at } P = mgy_o,$$

where g is the gravitational acceleration. Thus,

$$\text{Total Energy at } P = mgy_o.$$

At any point $(x, y(x))$ on the path the total energy is

$$\text{Total Energy at } (x,y) = \frac{1}{2}mv^2 + mgy.$$

Thus, the law of conservation of energy implies that

$$\frac{1}{2}mv^2 + mgy = mgy_o,$$

or

$$\frac{1}{2}v^2 + gy = gy_o,$$

after cancelling m; so that,

$$v^2 = 2g(y_o - y),$$

from which we get that

$$v = \sqrt{2g(y_o - y)}. \tag{3.62}$$

This gives us an expression for v as a function of y. Note that we need to assume that all the paths connecting P to Q under consideration must satisfy $y(x) < y_o$ for all $0 < x < x_1$.

Substituting the expression for v in (3.62) into the denominator of the integrand in (3.61), we obtain the expression for the time of descent

$$T(y) = \int_0^{x_1} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{2g(y_o - y(x))}}\, dx, \tag{3.63}$$

for $y \in C^1([0, x_1], \mathbb{R})$ in the class

$$\mathcal{A} = \{y \in C^1([0, x_1], \mathbb{R}) \mid y(0) = y_o,\ y(x_1) = y_1, \text{ and } y(x) < y_o \text{ for } 0 < x < x_1\}. \tag{3.64}$$

We would like to minimize the time of descent functional in (3.63) over the class of functions $\mathcal{A}$ defined in (3.64). Note that, if $y \in \mathcal{A}$ is a minimizer of the functional T given in (3.63), then y is also a minimizer of

$$\sqrt{2g}T(y) = \int_0^{x_1} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{y_o - y(x)}}\, dx, \quad \text{for } y \in \mathcal{A}.$$

Thus, we will seek a minimizer of the functional $J\colon \mathcal{A} \to \mathbb{R}$ given by

$$J(y) = \int_0^{x_1} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{y_o - y(x)}}\, dx, \quad \text{for } y \in \mathcal{A}. \tag{3.65}$$

The functional J derived in Example 3.3.1 (the Brachistochrone Problem), corresponds to a function $F\colon [0,x_1]\times(-\infty,y_o)\times\mathbb{R}\to\mathbb{R}$ defined by

$$F(x,y,z)=\frac{\sqrt{1+z^2}}{\sqrt{y_o-y}},\qquad \text{for } (x,y,z)\in[0,x_1]\times(-\infty,y_o)\times\mathbb{R},$$

in the general functional J given in (3.58). We will see many more examples of this general class of functionals in these notes and in the homework assignments.

The general variational problem we would like to consider in this section is the following:

Problem 3.3.2 (General Variational Problem 1)**.** Given real numbers a and b such that $a<b$, let $F\colon [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ denote a continuous function of three variables, (x,y,z), with $x\in[a,b]$, and y and z in the set of real numbers (in some cases, as in the Brachistochrone problem, we might need to restrict the values of y and z as well). Define the functional $J\colon C^1([a,b],\mathbb{R})\to\mathbb{R}$ by

$$J(y)=\int_a^b F(x,y(x),y'(x))\;dx,\qquad \text{for } y\in C^1([a,b],\mathbb{R}),\tag{3.66}$$

For real numbers y_o and y_1, consider the class of functions

$$\mathcal{A}=\{y\in C^1([a,b],\mathbb{R}) \mid y(a)=y_o,\ y(b)=y_1\}.\tag{3.67}$$

If possible, find $y\in\mathcal{A}$ such that

$$J(y)\leqslant J(v),\qquad \text{for all } v\in\mathcal{A},\tag{3.68}$$

or

$$J(y)\geqslant J(v),\qquad \text{for all } v\in\mathcal{A}.\tag{3.69}$$

Thus, in Problem 3.3.2 we seek a minimum, in the case of (3.68), of the functional J in (3.66) over the class $\mathcal{A}$ in (3.67) (a minimization problem), or we seek a maximum, in the case of (3.69), of J over $\mathcal{A}$ (a maximization problem). In general, we call either problem (3.68) or (3.69) an optimization problem.

We will see in these notes that, in order to answer the questions posed in the General Variational Problem 1, we need to impose additional conditions on the function F. We will see what those conditions are as we attempt to solve the problem.

Suppose that we know *a priori* that the functional J defined in (3.66) is bounded from below in $\mathcal{A}$ given in (3.67). Thus, it makes sense to ask whether there exists $y\in\mathcal{A}$ at which J is minimized; that is, is there a $y\in\mathcal{A}$ for which (3.68) holds true? We begin by assuming that this is the case; that is, there exists $y\in\mathcal{A}$ for which

$$J(y)\leqslant J(v),\qquad \text{for all } v\in\mathcal{A}.\tag{3.70}$$

As in one of our solutions of the geodesic problem presented in Section 3.1, we next seek to find necessary conditions for $y \in \mathcal{A}$ to be a minimizer of J defined in (3.66) over the class $\mathcal{A}$ given in (3.68).

Let $\eta\colon [a,b] \to \mathbb{R}$ denote a C^1 function such that $\eta(a) = 0$ and $\eta(b) = 0$; so that $\eta \in C^1_o([a,b],\mathbb{R})$. It then follows from the definition of $\mathcal{A}$ in (3.67) that

$$y + t\eta \in \mathcal{A}, \qquad \text{for all } t \in \mathbb{R}.$$

Thus, we get from (3.70) that

$$J(y) \leqslant J(y + t\eta), \qquad \text{for all } t \in \mathbb{R}. \tag{3.71}$$

Define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(t) = J(y + t\eta), \qquad \text{for all } t \in \mathbb{R}. \tag{3.72}$$

It follows from (3.70) and the definition of g in (3.72) that g has a minimum at 0; that is

$$g(t) \geqslant g(0), \qquad \text{for all } t \in \mathbb{R}.$$

Thus, if it can be shown that the function g defined in (3.72) is differentiable, we get the necessary condition

$$g'(0) = 0,$$

which can be written in terms of J as

$$\frac{d}{dt}[J(y + t\eta)]\Big|_{t=0} = 0. \tag{3.73}$$

Hence, to obtain the necessary condition in (3.73), we need to make sure that the map

$$t \mapsto J(y + t\eta), \qquad \text{for } t \in \mathbb{R}, \tag{3.74}$$

is a differentiable function of t at $t = 0$, where, according to (3.66),

$$J(y + t\eta) = \int_a^b F(x, y(x) + t\eta(x), y'(x) + t\eta'(x))\; dx, \qquad \text{for } t \in \mathbb{R}. \tag{3.75}$$

According to Proposition B.1.1 in Appendix B.1 in these notes, the question of differentiability of the map in (3.74) reduces to whether or not the partial derivatives

$$\frac{\partial}{\partial y}[F(x,y,z)] = F_y(x,y,z) \quad \text{and} \quad \frac{\partial}{\partial z}[F(x,y,z)] = F_z(x,y,z),$$

are continuous on $[a,b] \times \mathbb{R} \times \mathbb{R}$. Thus, in addition to F being continuous, we will assume that F has continuous partial derivatives with respect to y and with respect to z, F_y and F_z, respectively. Making these additional assumptions, we can apply Proposition B.1.1 to the expression in (3.75) to obtain, using the Chain Rule as well,

$$\frac{d}{dt}[J(y+t\eta)] = \int_a^b [F_y(x, y+t\eta, y'+t\eta')\eta + F_z(x, y+t\eta, y'+t\eta')\eta']\ dx, \quad (3.76)$$

for all $t \in \mathbb{R}$, where we have written y for $y(x)$, y' for $y'(x)$, η for $\eta(x)$, and η' for $\eta'(x)$ in the integrand of the integral on the right–hand side of (3.76).

Substituting 0 for t in (3.76) we then obtain that

$$\frac{d}{dt}[J(y+t\eta)]\Big|_{t=0} = \int_a^b [F_y(x, y, y')\eta + F_z(x, y, y')\eta']\ dx.$$

Thus, the necessary condition in (3.73) for $y \in \mathcal{A}$ to be a minimizer of J in $\mathcal{A}$, for the case in which F is continuous with continuous partial derivatives F_y and F_z, is that

$$\int_a^b [F_y(x, y, y')\eta + F_z(x, y, y')\eta']\ dx = 0, \qquad \text{for all } \eta \in C_o^1([a,b],\mathbb{R}). \quad (3.77)$$

Since we are assuming that F_y and F_z are continuous, we can apply the third fundamental lemma in the Calculus of Variations (Lemma 3.2.9) to obtain from (3.77) that the map

$$x \mapsto F_z(x, y(x), y'(x)), \qquad \text{for } x \in [a,b],$$

is differentiable for all $x \in (a,b)$ and

$$\frac{d}{dx}[F_z(x, y(x), y'(x))] = F_y(x, y(x), y'(x)), \qquad \text{for all } x \in (a,b). \quad (3.78)$$

The differential equation in (3.78) is called the **Euler–Lagrange** equation associated with the functional J defined in (3.66). It gives a necessary condition for a function $y \in \mathcal{A}$ to be an optimizer of J over the class $\mathcal{A}$ given in (3.67). We restate this fact, along with the assumptions on F, in the following proposition.

Proposition 3.3.3 (Euler–Lagrange Equation)**.** Let $a, b \in \mathbb{R}$ be such that $a < b$ and let $F\colon [a,b] \times \mathbb{R} \times \mathbb{R}$ be a continuous function of three variables $(x, y, x) \in [a,b] \times \mathbb{R} \times \mathbb{R}$ with continuous partial derivatives with respect to y and with respect to z, F_y and F_z, respectively.

Define $J\colon C^1([a,b],\mathbb{R}) \to \mathbb{R}$ by

$$J(y) = \int_a^b F(x, y(x), y'(x))\ dx, \qquad \text{for } y \in C^1([a,b],\mathbb{R}). \quad (3.79)$$

For real numbers y_o and y_1, define

$$\mathcal{A} = \{y \in C^1([a,b],\mathbb{R}) \mid y(a) = y_o,\ y(b) = y_1\}. \quad (3.80)$$

A necessary condition for $y \in \mathcal{A}$ to be a minimizer, or a maximizer, of J over the class $\mathcal{A}$ is that y solves the Euler–Lagrange equation

$$\frac{d}{dx}[F_z(x, y(x), y'(x))] = F_y(x, y(x), y'(x)), \qquad \text{for all } x \in (a,b). \quad (3.81)$$

In the indirect method of the Calculus of Variations, as it applies to the General Variational Problem 1 (Problem 3.3.2), we first seek for a function $y \in \mathcal{A}$, where $\mathcal{A}$ is given in (3.80), that solves the Euler–Lagrange equation in (3.81). This leads to the **two–point boundary value problem**

$$\begin{cases} \dfrac{d}{dx}[F_z(x, y(x), y'(x))] = F_y(x, y(x), y'(x)), & \text{for all } x \in (a, b); \\ \\ y(a) = y_o, \ y(b) = y_1. \end{cases} \tag{3.82}$$

A solution of the boundary value problem in (3.82) will be a candidate for a minimizer, or a maximizer, of the functional J in (3.79) over the class $\mathcal{A}$ given in (3.80).

The second step in the indirect method is to verify that the candidate is a minimizer or a maximizer. In the remainder of this section we give examples of the boundary value problems in (3.82) involving the Euler–Lagrange equation. In subsequent sections we will see how to verify that a solution of the boundary value problem in (3.82) yields a minimizer for a large class of problems.

Example 3.3.4 (Geodesic Problem 2, Revisited)**.** In Problem 3.1.2 (The Geodesic Problem 2), we looked at the problem of minimizing the functional,

$$J\colon C^1([x_o, x_1], \mathbb{R}) \to \mathbb{R},$$

given by

$$J(y) = \int_{x_o}^{x_1} \sqrt{1 + (y'(x))^2}\, dx, \qquad \text{for all } y \in C^1([x_o, x_1], \mathbb{R}), \tag{3.83}$$

over the class

$$\mathcal{A} = \{y \in C^1([x_o, x_1], \mathbb{R}) \mid y(x_o) = y_o, \ y(x_1) = y_1\}. \tag{3.84}$$

In this case,

$$F(x, y, z) = \sqrt{1 + z^2}, \qquad \text{for } (x, y, z) \in [x_o, x_1] \times \mathbb{R} \times \mathbb{R}.$$

So that,

$$F_y(x, y, z) = 0 \ \text{ and } \ F_z(x, y, z) = \frac{z}{\sqrt{1 + z^2}}, \ \text{ for } (x, y, z) \in [x_o, x_1] \times \mathbb{R} \times \mathbb{R}.$$

The Euler–Lagrange equation associated with the functional in (3.83) is then

$$\frac{d}{dx}\left[\frac{y'(x)}{\sqrt{1 + (y'(x))^2}}\right] = 0, \qquad \text{for } x \in (x_o, x_1).$$

Integrating this equation yields

$$\frac{y'(x)}{\sqrt{1 + (y'(x))^2}} = c_1, \qquad \text{for } x \in (x_o, x_1),$$

for some constant of integration c_1. This is the same equation in (3.25) that we obtained in the solution of Geodesic Problem 2. The solution of this equation subject to the boundary conditions

$$y(x_o) = y_o \quad \text{and} \quad y(x_1) = y_1$$

was given in (3.31); namely,

$$y(x) = \frac{y_1 - y_o}{x_1 - x_o}\, x + \frac{x_1 y_o - x_o y_1}{x_1 - x_o}, \qquad \text{for } x_o \leqslant x \leqslant x_1. \tag{3.85}$$

The graph of the function y given in (3.85) is a straight line segment form the point (x_o, y_o) to the point (x_1, y_1). We will see in a subsequent section that the function in (3.85) is the unique minimizer of the arc–length functional defined in (3.83) over the class $\mathcal{A}$ given in (3.84).

Example 3.3.5 (The Brachistochrone Problem, Revisited)**.** In Example 3.3.1 we saw that the time of descent from point $P(0, y_o)$ to point $Q(x_1, y_1)$, where $x_1 > 0$ and $y_o > y_1$, along a path that is the graph of a function $y \in C^1([0, x_1], \mathbb{R})$, with $y(0) = y_o$ and $y(x_1) = y_1$, is proportional to

$$J(y) = \int_0^{x_1} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{y_o - y(x)}}\, dx, \qquad \text{for } y \in \mathcal{A}, \tag{3.86}$$

where $\mathcal{A}$ is the class

$$\mathcal{A} = \{y \in C^1([0, x_1], \mathbb{R}) \mid y(0) = y_o,\ y(x_1) = y_1,\ \text{and } y(x) < y_o \text{ for } 0 < x < x_1\}. \tag{3.87}$$

In this case, the function F corresponding to the functional J in (3.86) is given by

$$F(x, y, z) = \frac{\sqrt{1 + z^2}}{\sqrt{y_o - y}}, \qquad \text{for } 0 \leqslant x \leqslant x_1,\ y < y_o,\ \text{and } z \in \mathbb{R}.$$

We then have that

$$F_y(x, y, z) = \frac{\sqrt{1 + z^2}}{2(y_o - y)^{3/2}}, \qquad \text{for } 0 \leqslant x \leqslant x_1,\ y < y_o,\ \text{and } z \in \mathbb{R},$$

and

$$F_z(x, y, z) = \frac{z}{\sqrt{1 + z^2}\sqrt{y_o - y}}, \qquad \text{for } 0 \leqslant x \leqslant x_1,\ y < y_o,\ \text{and } z \in \mathbb{R}.$$

Thus, the Euler–Lagrange equations associated with the functional J given in (3.86) is

$$\frac{d}{dx}\left[\frac{y'(x)}{\sqrt{1 + (y'(x))^2}\sqrt{y_o - y(x)}}\right] = \frac{\sqrt{1 + (y'(x))^2}}{2(y_o - y(x))^{3/2}}, \qquad \text{for } 0 < x < x_1. \tag{3.88}$$

To simplify the differential equation in (3.88), we will assume that y is twice differentiable. We will also introduce a new variable, u, which is a function of x defined by

$$u(x) = y_o - y(x), \quad \text{where } y(x) < y_o \text{ for } 0 < x < x_1; \tag{3.89}$$

so that, $u(x) > 0$ for $0 < x < x_1$ and

$$y(x) = y_o - u(x), \quad \text{where } u(x) > 0 \text{ for } 0 < x < x_1, \tag{3.90}$$

and

$$y' = -u'. \tag{3.91}$$

We can then rewrite the equation in (3.88) as

$$\frac{d}{dx}\left[\frac{-u'(x)}{\sqrt{1+(u'(x))^2}\sqrt{u(x)}}\right] = \frac{\sqrt{1+(u'(x))^2}}{2(u(x)^{3/2}}, \quad \text{for } 0 < x < x_1,$$

or

$$\frac{d}{dx}\left[\frac{u'}{\sqrt{1+(u')^2}\sqrt{u}}\right] = -\frac{\sqrt{1+(u')^2}}{2u^{3/2}}, \quad \text{for } 0 < x < x_1, \tag{3.92}$$

where we have written u for $u(x)$ and u' for $u'(x)$. Next, we proceed to evaluate the derivative on the left–hand side of the equation in (3.92) and simplify to obtain from (3.92) that

$$(u')^2 + 2uu'' + 1 = 0 \quad \text{for } 0 < x < x_1, \tag{3.93}$$

where u'' denotes the second derivative of u. Multiply on both sides of (3.93) by u' to get

$$(u')^3 + 2uu'u'' + u' = 0, \quad \text{for } 0 < x < x_1,$$

which can in turn be written as

$$\frac{d}{dx}[u + u(u')^2] = 0. \tag{3.94}$$

Integrating the differential in (3.94) yields

$$u(1 + (u')^2) = C, \quad \text{for } 0 < x < x_1,$$

and some constant C, which we can solve for $(u')^2$ to get

$$(u')^2 = \frac{C-u}{u}, \quad \text{for } 0 < x < x_1. \tag{3.95}$$

Next, we solve (3.95) for u' to get

$$u' = \sqrt{\frac{C-u}{u}}, \quad \text{for } 0 < x < x_1, \tag{3.96}$$

where we have taken the positive square root in (3.96) in view of (3.91), since y decreases with increasing x.

Our goal now is to find a solution of the differential equation in (3.96) subject to the conditions

$$u = 0 \text{ when } x = 0, \tag{3.97}$$

according to (3.89), amd

$$u = y_o - y_1 \text{ when } x = x_1. \tag{3.98}$$

Using the Chain Rule, we can rewrite (3.96) as

$$\frac{dx}{du} = \sqrt{\frac{u}{C-u}}, \qquad \text{for } 0 < u < y_o - y_1. \tag{3.99}$$

The graph of a solution of (3.99) will be a smooth path connecting the point $(0, 0)$ to the pint $(x_1, y_o - y_1)$ in the xu–plane as pictured in Figure 3.3.3. We

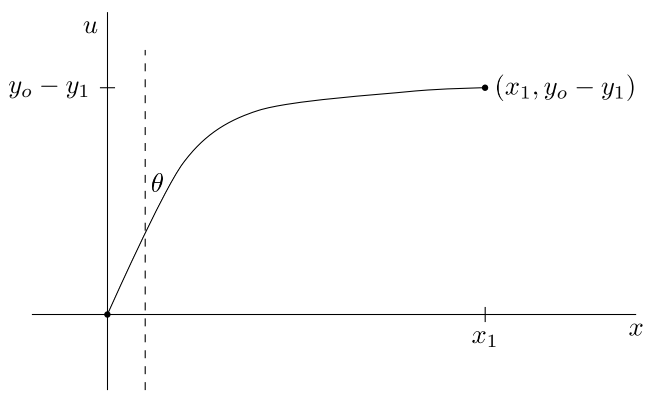

Figure 3.3.3: Shortest "descent" time path in the xu–plane

can also obtain the path as a parametrized curve

$$x = x(\theta),\ u = u(\theta), \qquad \text{for } \theta_o < \theta < \theta_1, \tag{3.100}$$

where θ is the angle the tangent line to the curve makes with a vertical line (see the sketch in Figure 3.3.3). We then have that

$$\frac{dx}{du} = \tan\theta; \tag{3.101}$$

so that, using (3.99),

$$\frac{u}{C-u} = \frac{\sin^2\theta}{\cos^2\theta}, \qquad \text{for } \theta_o < \theta < \theta_1. \tag{3.102}$$

Solving the equation in (3.102) for u yields

$$u = C\sin^2\theta, \tag{3.103}$$

where we have used the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$; thus, using the trigonometric identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta),$$

$$u(\theta) = \frac{C}{2}(1 - \cos 2\theta), \qquad \text{for } \theta_o < \theta < \theta_1. \tag{3.104}$$

In view of the condition in (3.97) we see from (3.104) that $\theta_o = 0$; so that,

$$u(\theta) = \frac{C}{2}(1 - \cos 2\theta), \qquad \text{for } 0 < \theta < \theta_1. \tag{3.105}$$

To find the parametric expression for x in terms of θ, use the Chain Rule to obtain

$$\frac{dx}{d\theta} = \frac{dx}{du}\frac{du}{d\theta};$$

so that, in view of (3.101) and (3.103)

$$\frac{dx}{d\theta} = \frac{\sin\theta}{\cos\theta} \cdot 2C\sin\theta\cos\theta,$$

which which we get

$$\frac{dx}{d\theta} = 2C\sin^2\theta,$$

or

$$\frac{dx}{d\theta} = C(1 - \cos 2\theta), \qquad \text{for } 0 < \theta < \theta_1. \tag{3.106}$$

Integrating the differential equation in (3.106) and using the boundary condition in (3.97), we obtain that

$$x(\theta) = C(\theta - \frac{1}{2}\sin 2\theta), \qquad \text{for } 0 < \theta < \theta_1,$$

which we can rewrite as

$$x(\theta) = \frac{C}{2}(2\theta - \sin 2\theta) \qquad \text{for } 0 < \theta < \theta_1. \tag{3.107}$$

Putting together the expressions in (3.105) and (3.107), denoting $\frac{C}{2}$ by a, and introducing a new parameter $t = 2\theta$, we obtain the parametric equations

$$\begin{cases} x(t) = at - a\sin t; \\ u(t) = a - a\cos t, \end{cases} \tag{3.108}$$

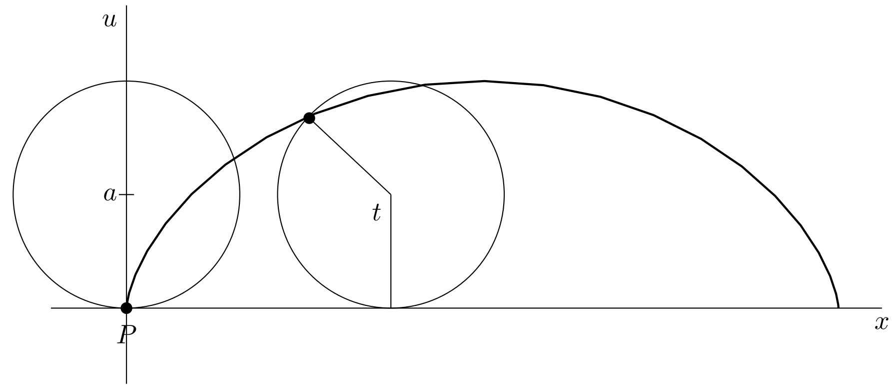

Figure 3.3.4: Cycloid

for $0 \leqslant t \leqslant t_1$, which are the parametric equations of a **cycloid**. This is the curve traced by a point, P, on a circle of radius a and center $(0, a)$, which starts at the origin when $t = 0$, as the circle rolls on the x–axis in the positive direction (see Figure 3.3.4). The parameter t gives the angle the vector from the center of the circle to P makes with a vertical vector emanating from the center and pointing downwards; this is shown in Figure 3.3.4.

To find a curve parametrized by (3.108) that goes through the point

$$(x_1, y_o - y_1),$$

so that the boundary condition in (3.98) is satisfied, we need to find a and t_1 such that

$$\left\{\begin{array}{rcl} at_1 - a\sin t_1 & = & x_1 \\ a - a\cos t_1 & = & y_o - y_1. \end{array}\right. \tag{3.109}$$

We will show presently that the system in (3.109) can always be solved for positive values of x_1 and $y_o - y_1$ by an appropriate choice of a.

The sketch in Figure 3.3.5 shows a cycloid generated by rolling a circle of radius 1 along the x–axis in the positive direction. Assume, for the sake of illustration, that the point $(x_1, y_o - y_1)$ lies above the cycloid and draw the line segment joining the origin in the xu–plane to the point $(x_1, y_o - y_1)$. The line will meet the cycloid at exactly one point; we labeled that point P_1 in the sketch in Figure 3.3.5. Observe that, in this case, the distance from the origin to P_1 is shorter than the distance from the origin to $(x_1, y_o - y_1)$. However, by increasing the value of $a > 1$ in (3.108) we can get another cycloid that meets the line segment from $(0, 0)$ to $(x_1, y_o - y_1)$ at a point whose distance from the origin is bigger than that from P_1 to $(0, 0)$; see the sketch in Figure 3.3.5. According to the parametric equations in (3.108), the distance from any point on a cycloid generated by a circle of radius a is given by

$$\|(x(t), u(t))\| = a\sqrt{t^2 + 2 - 2t\sin t - 2\cos t}, \qquad \text{for } 0 \leqslant t \leqslant 2\pi. \tag{3.110}$$

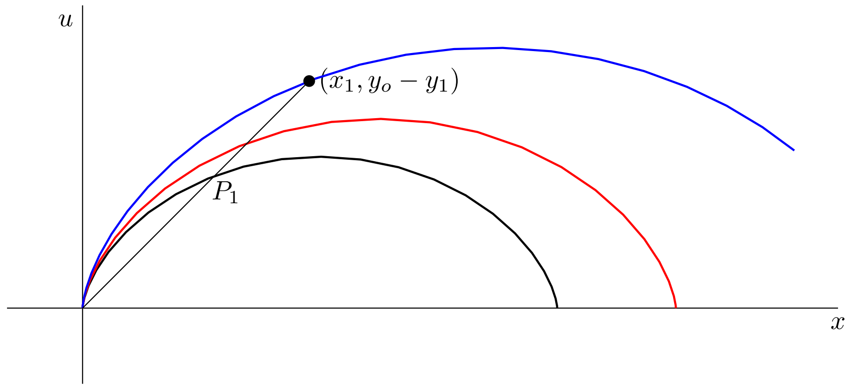

Figure 3.3.5: Solving the system in (3.109)

Observe that, for $0 < t < 2\pi$,

$$\|(x(t), u(t))\| \to \infty \quad \text{as} \quad a \to \infty.$$

Thus, since the distance defined in (3.110) is an increasing and continuous function of a, it follows from the intermediate value theorem that there exists a value of a such that the cycloid generated by a circle of radius a goes through the point $(x_1, y_o - y_1)$; this is also shown in Figure 3.3.5. On the other hand, if the point $(x_1, y_o - y_1)$ is below the original cycloid, we can decrease the radius $a < 1$ of the circle generating the cycloid until we reach the point $(x_1, y_o - y_1)$.

Once the value of $a > 0$ is determined, we can find the value of t_1 by solving the second equation in (3.109) to obtain

$$t_1 = \cos^{-1}\left(\frac{a - (y_o - y_1)}{a}\right).$$

A sketch of the curve obtained in this fashion is shown in Figure 3.3.6.

To get the solution of the Euler–Lagrange equation in (3.88) in the xy–plane subject to the boundary conditions

$$y(0) = y_o \quad \text{and} \quad y(x_1) = y_1,$$

we use the transformation equation in (3.89) to get from (3.108) the parametric equations

$$\begin{cases} x(t) = at - a\sin t; \\ y(t) = y_o - a + a\cos t, \end{cases} \tag{3.111}$$

for $0 \leqslant t \leqslant t_1$, where a and t_1 have already been determined.

A sketch of this curve is shown in Figure 3.3.7. This curve is the graph of a twice–differentiable function, $y\colon [0, x_1] \to \mathbb{R}$, that solves the two–point boundary

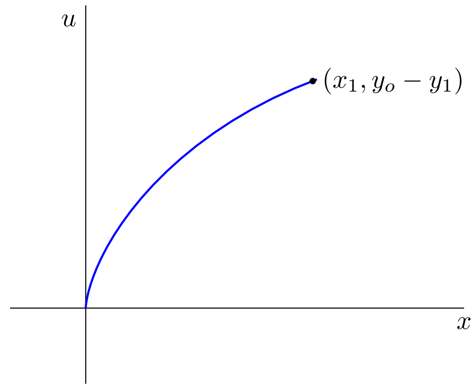

Figure 3.3.6: Sketch of solution of (3.93) subject to $u(0) = 0$ and $u(x_1) = y_o - y_1$

value problem

$$\left\{\begin{array}{rcl} \dfrac{d}{dx}\left[\dfrac{y'}{\sqrt{1+(y')^2}\sqrt{y_o - y}}\right] & = & \dfrac{\sqrt{1+(y')^2}}{2(y_o - y)^{3/2}}; \\ \\ y(0) & = & y_o; \\ y(x_1) & = & y_1. \end{array}\right. \tag{3.112}$$

The solution of the two–point boundary value problem in (3.112) described in Example 3.3.5 is a candidate for a minimizer of the descent time functional in (3.63). We have not shown that this function is indeed a minimizer. In the next chapter we shall see how to show that the solution of the boundary value problem in (3.112) provides the curve of fastest descent from the point $P(0, y_o)$ to the point $Q(x_1, y_1)$ in Figure 3.3.2.

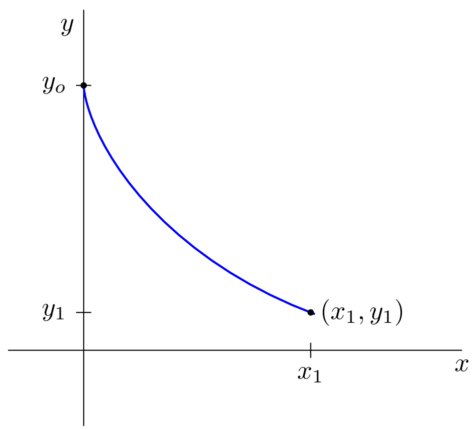

Figure 3.3.7: Sketch of solution of (3.88) subject to $y(0) = y_o$ and $y(x_1) = y_1$

Chapter 4

Convex Minimization

The functionals we encountered in the Geodesic Example 2 (see Example 3.1.2) and in the Brachistochrone Problem (Example 3.3.1),

$$J(y) = \int_{x_o}^{x_1} \sqrt{1 + (y'(x))^2}\, dx, \quad \text{for all } y \in C^1([x_o, x_1], \mathbb{R}), \tag{4.1}$$

and

$$J(y) = \int_{0}^{x_1} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{y_o - y(x)}}\, dx, \quad \text{for all } y \in \mathcal{A}, \tag{4.2}$$

where

$$\mathcal{A} = \{y \in C^1([0, x_1], \mathbb{R}) \mid y(0) = y_o, y(x_1) = x_1, y(x) < y_o \text{ for } 0 < x < x_1\},$$

respectively, are strictly convex functionals. We will see in this chapter that for this class of functionals we can prove the existence of a unique minimizer. The functionals in (4.1) and (4.2) are also Gâteaux differentiable. We begin this chapter with a discussion of Gâteaux differentiability.

4.1 Gâteaux Differentiability

We consider the general situation of a functional $J\colon V \to \mathbb{R}$ defined on a linear space V. Let V_o denote a nontrivial subspace of V; that is, V_o is not the trivial subspace $\{\mathbf{0}\}$. For every vectors u in V and v in V_o, define the real valued function, $g\colon \mathbb{R} \to \mathbb{R}$, of a single variable as follows

$$g(t) = J(u + tv), \quad \text{for all } t \in \mathbb{R}; \tag{4.3}$$

that is, the function g gives the values of J along a line through u in the direction of $v \neq 0$. We will focus on the special case in which the function g is differentiable at $t = 0$. If this is the case, we say that the functional J is **Gâteaux differentiable** at u in the direction of v and we denote $g'(0)$ by

$dJ(u;v)$, and call it the Gâteaux derivative of J at u in the direction of v; so that, according to the definition of g in (4.3),

$$dJ(u;v) = \frac{d}{dt}[J(u+tv)]\Big|_{t=0}. \tag{4.4}$$

The existence of the expression on the right–hand side of (4.4) translate into the existence of the limit defining $g'(0)$, or

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t}.$$

Here is the formal definition of Gâteaux differentiability.

Definition 4.1.1 (Gâteaux Differentiability)**.** Let V be a normed linear space, V_o be a nontrivial subspace of V, and $J\colon V \to \mathbb{R}$ be a functional defined on V. We say that J is Gâteaux differentiable at $u \in V$ in the direction of $v \in V_o$ if the limit

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} \tag{4.5}$$

exists. If the limit in (4.5) exists, we denote it by the symbol $dJ(u;v)$ and call it the Gâteaux derivative of J at u in the direction of v, or the first variation of J at u in the direction of v. Thus, if J is Gâteaux differentiable at u in the direction of v, its Gâteaux derivative at u in the direction of v is given by

$$dJ(u;v) = \frac{d}{dt}[J(u+tv)]\Big|_{t=0}, \tag{4.6}$$

or, in view of (4.5),

$$dJ(u;v) = \lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} \tag{4.7}$$

We now present a few examples of Gâteaux differentiable functionals in various linear spaces, and their Gâteaux derivatives. In practices, we usually compute (if possible) the derivative of $J(u+tv)$ with respect to t, and then evaluate it at $t=0$ (see the right–hand side of the equation in (4.6)).

Example 4.1.2 (The Dirichlet Integral)**.** Let Ω denote an open, bounded subset of $\mathbb{R}^n$. Let $C^1(\overline{\Omega}, \mathbb{R})$ denote the space or real–valued functions $u\colon \overline{\Omega} \to \mathbb{R}$ whose partial derivatives exist, and are continuous in an open subset, U, that contains $\overline{\Omega}$. Define $J\colon C^1(\overline{\Omega}, \mathbb{R}) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx, \qquad \text{for all } u \in C^1(\overline{\Omega}, \mathbb{R}). \tag{4.8}$$

The expression $|\nabla u|$ in the integrand on the right–hand–side of (4.8) is the Euclidean norm of the gradient of u,

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = (u_x, u_y), \qquad \text{if } n = 2,$$

or

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = (u_x, u_y, u_z), \quad \text{if } n = 3;$$

so that

$$|\nabla u|^2 = (u_x)^2 + (u_y)^2, \quad \text{if } n = 2,$$

or

$$|\nabla u|^2 = (u_x)^2 + (u_y)^2 + (u_y)^2, \quad \text{if } n = 3.$$

The differential dx in the integral on the right–hand side of (4.8) represents the element of area, $dxdy$, in the case in which $\Omega \subset \mathbb{R}^2$, or the element of volume, $dxdydz$, in the case in which $\Omega \subset \mathbb{R}^3$. Thus, if $n = 2$ the integral on the right–hand side of (4.8) is a double integral over the plane region $\Omega \subset \mathbb{R}^2$; while, if $n = 3$ the integral in (4.8) is a triple integral over the region in three–dimensional Euclidean space.

We denote by $C^1_c(\overline{\Omega}, \mathbb{R})$ the space of functions $v \in C^1(\overline{\Omega}, \mathbb{R})$ that that have compact support contained in Ω. We shall show in this example that the functional $J\colon C^1(\overline{\Omega}, \mathbb{R}) \to \mathbb{R}$ defined in (4.8) is Gâteaux differentiable at every $u \in C^1(\overline{\Omega}, \mathbb{R})$ in the direction of every $v \in C^1_c(\overline{\Omega}, \mathbb{R})$ and

$$dJ(u; v) = \int_\Omega \nabla u \cdot \nabla v \; dx, \quad \text{for all } u \in C^1(\overline{\Omega}, \mathbb{R}) \text{ and all } v \in C^1_c(\overline{\Omega}, \mathbb{R}), \tag{4.9}$$

where $\nabla u \cdot \nabla v$ denotes the dot product of the gradients of u and v; thus,

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y, \quad \text{if } n = 2,$$

or

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y + u_z v_z, \quad \text{if } n = 3.$$

For $u \in C^1(\overline{\Omega}, \mathbb{R})$ and $v \in C^1_c(\overline{\Omega}, \mathbb{R})$, use the definition of J in (4.8) to compute

$$\begin{aligned} J(u + tv) &= \frac{1}{2}\int_\Omega |\nabla(u + tv)|^2 \; dx \\ &= \frac{1}{2}\int_\Omega |\nabla u + t\nabla v)|^2 \; dx, \end{aligned}$$

where we have used the linearity of the differential operator ∇. Thus, using the fact that the square of the Euclidean norm of a vector is the dot product of the vector with itself,

$$\begin{aligned} J(u + tv) &= \frac{1}{2}\int_\Omega (\nabla u + t\nabla v) \cdot \nabla u + t\nabla v) \; dx \\ &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + 2t\nabla u \cdot \nabla v + t^2|\nabla v|^2) \; dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 + t\int_\Omega \nabla u \cdot \nabla v \; dx + t^2\int_\Omega |\nabla v|^2 \; dx. \end{aligned}$$

Thus, using the definition of J in (4.8),

$$J(u+tv) = J(u) + t\int_\Omega \nabla u\cdot\nabla v\ dx + t^2 J(v), \quad \text{for all } t\in\mathbb{R}. \tag{4.10}$$

Observe that the right–hand side of the expression in (4.10) is a quadratic polynomial in t. Hence, it is differentiable in t and

$$\frac{d}{dt}[J(u+tv)] = \int_\Omega \nabla u\cdot\nabla v\ dx + 2tJ(v), \quad \text{for all } t\in\mathbb{R}.$$

from which we get that

$$\frac{d}{dt}[J(u+tv)]\Big|_{t=0} = \int_\Omega \nabla u\cdot\nabla v\ dx. \tag{4.11}$$

It follows from (4.11) that J is Gâteaux differentiable at every $u\in C^1(\overline{\Omega},\mathbb{R})$ in the direction of every $v\in C^1_c(\overline{\Omega},\mathbb{R})$, and its Gâteaux derivative is as claimed in (4.9).

Example 4.1.3. Let $a,b\in\mathbb{R}$ be such that $a<b$ and let

$$F\colon [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$$

be a continuous function of three variables $(x,y,x)\in[a,b]\times\mathbb{R}\times\mathbb{R}$ with continuous partial derivatives with respect to y and with respect to z, F_y and F_z, respectively. Put $V=C^1([a,b],\mathbb{R})\to\mathbb{R}$ and $V_o = C^1_o([a,b],\mathbb{R})\to\mathbb{R}$.

Define $J\colon V\to\mathbb{R}$ by

$$J(y) = \int_a^b F(x,y(x),y'(x))\ dx, \quad \text{for } y\in V. \tag{4.12}$$

Then, J is Gâteaux differentiable at every $y\in V$ in the direction of every $\eta\in V_o$, and its Gâteaux derivative is given by

$$dJ(y,\eta) = \int_a^b [F_y(x,y,y')\eta + F_z(x,y,y')\eta']\ dx, \quad \text{for } y\in V \text{ and } \eta\in V_o. \tag{4.13}$$

For $y\in V$ and $\eta\in V_o$, use (4.12) to compute

$$J(y+t\eta) = \int_a^b F(x,y(x)+t\eta(x),y'(x)+t\eta'(x))\ dx, \quad \text{for } t\in\mathbb{R}. \tag{4.14}$$

Thus, according to Proposition B.1.1 in Appendix B.1 in these notes, the question of differentiability of the map

$$t\mapsto J(y+t\eta), \quad \text{for } t\in\mathbb{R}, \tag{4.15}$$

reduces to whether the partial derivatives

$$\frac{\partial}{\partial y}[F(x,y,z)] = F_y(x,y,z) \quad \text{and} \quad \frac{\partial}{\partial z}[F(x,y,z)] = F_z(x,y,z)$$

are continuous in $[a,b]\times\mathbb{R}\times\mathbb{R}$. This is one of the assumptions we have made in this example.

Now, it follows from the Chain Rule that

$$\frac{\partial}{\partial t}[F(\cdot, y+t\eta, y'+t\eta')] = F_y(\cdot, y+t\eta, y'+t\eta')\eta + F_z(\cdot, y+t\eta, y'+t\eta')\eta',$$

for all $t\in\mathbb{R}$, where where the dot in the first argument of the functions F_y and F_z indicates dependence on x, and where we have written y for $y(x)$, y' for $y'(x)$, η for $\eta(x)$, and η' for $\eta'(x)$. Thus, since we are assuming that y and η are C^1 functions, we see that the assumptions for Proposition B.1.1 in Appendix B.1 hold true. We therefore conclude that the map in (4.15) is differentiable and

$$\frac{d}{dt}[J(y+t\eta)] = \int_a^b [F_y(x,y+t\eta,y'+t\eta')\eta + F_z(x,y+t\eta,y'+t\eta')\eta']\ dx, \tag{4.16}$$

for all $t\in\mathbb{R}$, where we have written y for $y(x)$, y' for $y'(x)$, η for $\eta(x)$, and η' for $\eta'(x)$ in the integrand of the integral on the right–hand side of (4.15).

Setting $t=0$ in (4.16) we then obtain that

$$\frac{d}{dt}[J(y+t\eta)]\Big|_{t=0} = \int_a^b [F_y(x,y,y')\eta + F_z(x,y,y')\eta']\ dx,$$

from which the expression in (4.13) follows.

Example 4.1.4 (A Sturm–Liouville Problem)**.** Let $p\in C([x_o,x_1],\mathbb{R})$ and $q\in C([x_o,x_1],\mathbb{R})$ be functions satisfying $p(x)\geqslant 0$ and $q(x)\geqslant 0$ for $x_o\leqslant x\leqslant x_1$. Define $J\colon C^1([x_o,x_1],\mathbb{R})\to\mathbb{R}$ by

$$J(y) = \int_{x_o}^{x_1} [p(x)(y'(x))^2 + q(x)(y(x))^2]\ dx, \qquad \text{for } y\in C^1([x_o,x_1],\mathbb{R}). \tag{4.17}$$

We show that J is Gâteaux differentiable at any $y\in V=C^1([x_o,x_1],\mathbb{R})$ in the direction of any $\eta\in V_o=C^1_o([x_o,x_1],\mathbb{R})$. To do this, observe that the functional J in (4.17) is of the form

$$J(y) = \int_{x_o}^{x_1} F(x,y(x),y'(x))\ dx, \qquad \text{for } y\in C^1([x_o,x_1],\mathbb{R}),$$

where

$$F(x,y,z) = p(x)z^2 + q(x)y^2, \qquad \text{for } x\in[x_o,x_1], y\in\mathbb{R}, z\in\mathbb{R}. \tag{4.18}$$

Since we are assuming that the functions p and q are continuous on $[x_o, x_1]$, it follows that the function $F\colon [x_o, x_1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined in (4.18) is continuous on $[x_o, x_1]\times\mathbb{R}\times\mathbb{R}$ with continuous partial derivatives

$$F_y(x, y, z) = 2q(x)y \quad \text{and} \quad F_z(x, y, z) = 2p(x)z,$$

for $(x, y, z) \in [x_o, x_1]\times\mathbb{R}\times\mathbb{R}$. Consequently, by the result of Example and (4.13), we conclude that the functional J defined in (4.17) is Gâteaux differentiable at every $y \in C^1([x_o, x_1], \mathbb{R})$ with Gâteaux derivative

$$dJ(y, \eta) = \int_{x_o}^{x_1} [2q(x)y(x)\eta(x) + 2p(x)y'(x)\eta'(x)]\ dx, \tag{4.19}$$

for every direction $\eta \in C_o^1([x_o, x_1], \mathbb{R})$.

4.2 A Minimization Problem

Let V denote a normed linear space and $J\colon V \to \mathbb{R}$ a functional defined on V. For a given nonempty subset $\mathcal{A}$ of V, we consider the problem of finding an element u of $\mathcal{A}$ for which $J(u)$ is the smallest possible among all values of $J(v)$ for v in $\mathcal{A}$. We write:

$$J(u) \leqslant J(v), \quad \text{for all } v \in \mathcal{A}. \tag{4.20}$$

We call $\mathcal{A}$ the class of **admissible vectors** for the minimization problem in (4.20).

In addition, suppose that there exists a nontrivial subspace V_o of V with the property that: for every $u \in \mathcal{A}$ and every $v \in V_o$ there exists $\delta > 0$ such that

$$|t| < \delta \Rightarrow u + tv \in \mathcal{A}. \tag{4.21}$$

We will refer to V_o as the space of **admissible directions**.

Suppose we have found a solution $u \in \mathcal{A}$ of the minimization problem in (4.20). Assume further that J is Gâtaeux differentiable at u along any direction $v \in V_o$,

Let v be any direction in V_o. It follows from (4.21) and (4.20) that there exists $\delta > 0$ such that

$$J(u + tv) \geqslant J(u), \quad \text{for } |t| < \delta,$$

or

$$J(u + tv) - J(u) \geqslant 0, \quad \text{for } |t| < \delta. \tag{4.22}$$

Dividing on both sides of the inequality in (4.22) by $t > 0$ we obtain that

$$\frac{J(u + tv) - J(u)}{t} \geqslant 0, \quad \text{for } 0 < t < \delta. \tag{4.23}$$

Thus, letting $t \to 0^+$ in (4.23) and using the definition of the Gâteaux derivative of J in (4.7), we get that

$$dJ(u; v) \geqslant 0, \tag{4.24}$$

since we are assuming that J is Gâteaux differentiable at u.

Similarly, dividing on both sides of the inequality in (4.22) by $t < 0$ we obtain that

$$\frac{J(u + tv) - J(u)}{t} \leqslant 0, \quad \text{for } -\delta < t < 0. \tag{4.25}$$

Letting $t \to 0^-$ in (4.25), using the assumption that J is Gâteaux differentiable at u, we have that

$$dJ(u; v) \leqslant 0. \tag{4.26}$$

Combining (4.24) and (4.26), we obtain the result that, if J is Gâteaux differentiable at u, and u is a minimizer of J over $\mathcal{A}$, then

$$dJ(u; v) = 0, \quad \text{for all } v \in V_o. \tag{4.27}$$

The condition in (4.27) is a necessary condition for u to be a minimizer of J over $\mathcal{A}$ in the case in which J is Gâteaux differentiable at u along any direction $v \in V_o$.

Remark 4.2.1. In view of the expression in (4.13) derived in Example 4.1.3, we note that the necessary condition in (4.27), in conjunction with the Fundamental Lemma 3 (Lemma 3.2.9), was used in Section 3.3 to derive the Euler–Lagrange equation.

4.3 Convex Functionals

Many of the functionals discussed in the examples in these notes so far are convex. In this section we present the definitions of convex and strictly convex functionals and discuss a few of their properties.

Definition 4.3.1 (Convex Functionals)**.** Let V denote a linear space, V_o a nontrivial subspace of V, and $\mathcal{A}$ a given nonempty subset of V. Let $J\colon V \to \mathbb{R}$ be a functional defined on V. Suppose that J is Gâteaux differentiable at every $u \in \mathcal{A}$ in any direction $v \in V_o$. The functional J is said to be **convex** on $\mathcal{A}$ if

$$J(u + v) \geqslant J(u) + dJ(u; v) \tag{4.28}$$

for all $u \in \mathcal{A}$ and $v \in V_o$ such that $u + v \in \mathcal{A}$.

A Gâteaux differentiable functional $J\colon V \to \mathbb{R}$ is said to be **strictly convex** in $\mathcal{A}$ if it is convex in $\mathcal{A}$, and

$$J(u+v) = J(u)+dJ(u; v), \quad \text{for } u \in \mathcal{A}, v \in V_o \text{ with } u+v \in \mathcal{A}, \text{ iff } v = 0. \tag{4.29}$$

Example 4.3.2 (The Dirichlet Integral, Revisited)**.** Let $V = C^1(\overline{\Omega}, \mathbb{R})$ and $V_o = C^1(\overline{\Omega}, \mathbb{R})$. Let U denote an open subset in $\mathbb{R}^n$ that contains $\overline{\Omega}$ and let $g \in C(U, \mathbb{R})$. Defined

$$\mathcal{A} = \{u \in C^1(\overline{\Omega}, \mathbb{R}) \mid u = g \text{ on } \partial\Omega\}; \tag{4.30}$$

that is, $\mathcal{A}$ is the class of C^1 functions $u \colon \overline{\Omega} \to \mathbb{R}$ that take on the values of g on the boundary of Ω.

Define $J \colon V \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \; dx, \qquad \text{for all } u \in C^1(\overline{\Omega}, \mathbb{R}). \tag{4.31}$$

We showed in Example 4.1.2 that J is Gâteaux differentiable at every $u \in C^1(\overline{\Omega}, \mathbb{R})$ in the direction of every $v \in C^1_c(\overline{\Omega}, \mathbb{R})$, with Gâteaux derivative given by

$$dJ(u; v) = \int_\Omega \nabla u \cdot \nabla v \; dx, \qquad \text{for } u \in C^1(\overline{\Omega}, \mathbb{R}) \text{ and } v \in C^1_c(\overline{\Omega}, \mathbb{R}). \tag{4.32}$$

We will first show that J is convex in $\mathcal{A}$, where $\mathcal{A}$ is defined in (4.30). Thus, let $u \in \mathcal{A}$ and $v \in C^1_c(\overline{\Omega}, \mathbb{R})$. Then, $u + v \in \mathcal{A}$, since v vanishes on $\partial\Omega$.

Compute

$$J(u+v) = \frac{1}{2} \int_\Omega |\nabla(u+v)|^2 \; dx = J(u) + dJ(u; v) + \frac{1}{2} \int_\Omega |\nabla v|^2 \; dx. \tag{4.33}$$

Thus,

$$J(u+v) \geqslant J(u) + dJ(u; v)$$

for all $u \in \mathcal{A}$ and $v \in C^1_c(\overline{\Omega}, \mathbb{R})$. Consequently, J is convex in $\mathcal{A}$.

Next, we show that J is strictly convex.

From (4.33) we get that

$$J(u+v) = J(u) + dJ(u; v) + \frac{1}{2} \int_\Omega |\nabla v|^2 \; dx,$$

for all $u \in \mathcal{A}$ and $v \in C^1_c(\overline{\Omega}, \mathbb{R})$. Consequently,

$$J(u+v) = J(u) + dJ(u; v)$$

if and only if

$$\int_\Omega |\nabla v|^2 \; dx = 0.$$

Thus, since $v \in C^1(\overline{\Omega}, \mathbb{R})$, $\nabla u = 0$ in Ω, and therefore v is constant on connected components of Ω. Hence, since $v = 0$ on $\partial\Omega$, it follows that $v(x) = 0$ for all $x \in \overline{\Omega}$. We conclude therefore that the Dirichlet integral functional J defined in (4.31) is strictly convex in $\mathcal{A}$.

Example 4.3.3 (A Sturm–Liouville Problem, Revisited)**.** Let $p\colon [x_o, x_1] \to \mathbb{R}$ and $q\colon [x_o, x_1] \to \mathbb{R}$ be coninuous functions satisfying $p(x) > 0$ and $q(x) > 0$ for $x_o \leqslant x \leqslant x_1$. Define $J\colon C^1([x_o, x_1], \mathbb{R}) \to \mathbb{R}$ by

$$J(y) = \int_{x_o}^{x_1} [p(x)(y'(x))^2 + q(x)(y(x))^2]\ dx, \quad \text{for } y \in C^1([x_o, x_1], \mathbb{R}). \tag{4.34}$$

Consider the problem of minimizing J over the class

$$\mathcal{A} = \{y \in C^1([x_o, x_1], \mathbb{R}) \mid y(x_o) = y_o \text{ and } y(x_1) = y_1\}, \tag{4.35}$$

for given real numbers y_o and y_1.

In Example 4.1.4 we showed that the functional $J\colon C^1([x_o, x_1], \mathbb{R}) \to \mathbb{R}$ given in (4.34) is Gâteaux differentiable at every $y \in C^1([x_o, x_1], \mathbb{R})$ with Gâteaux derivative given by (4.19); namely,

$$dJ(y, \eta) = 2\int_{x_o}^{x_1} [q(x)y(x)\eta(x) + p(x)y'(x)\eta'(x)]\ dx, \quad \text{for } y \in V,\ \eta \in V_o, \tag{4.36}$$

where $V = C^1([x_o, x_1], \mathbb{R})$ and $V_o = C^1_o([x_o, x_1], \mathbb{R})$.

In this example we show that J is strictly convex in $\mathcal{A}$ given in (4.35).

We first show that J is convex.

Let $y \in \mathcal{A}$ and $\eta \in V_o$. Then, $y + \eta \in \mathcal{A}$, given that $\eta(x_o) = \eta(x_1) = 0$ and the definition of $\mathcal{A}$ in (4.35).

Compute

$$J(y + \eta) = \int_{x_o}^{x_1} [p(x)(y'(x) + \eta'(x))^2 + q(x)(y(x) + \eta(x))^2]\ dx,$$

or, expanding the integrand,

$$\begin{aligned} J(y + \eta) \quad = \quad & \int_{x_o}^{x_1} [p(x)(y'(x))^2 + 2p(x)y'(x)\eta'(x) + p(x)(\eta'(x))^2]\ dx \\ & + \int_{x_o}^{x_1} [q(x)(y(x))^2 + 2q(x)y(x)\eta(x) + q(x)(\eta(x))^2]\ dx, \end{aligned}$$

which we can rewrite as

$$\begin{aligned} J(y + \eta) \quad = \quad & \int_{x_o}^{x_1} [p(x)(y'(x))^2 + q(x)(y(x))^2]\ dx \\ & +2\int_{x_o}^{x_1} [p(x)y'(x)\eta'(x) + q(x)y(x)\eta(x)]\ dx \\ & + \int_{x_o}^{x_1} [p(x)(\eta'(x))^2 + q(x)(\eta(x))^2]\ dx; \end{aligned}$$

so that, in view of (4.34) and (4.36),

$$J(y + \eta) = J(y) + dJ(y; \eta) + \int_{x_o}^{x_1} [p(x)(\eta'(x))^2 + q(x)(\eta(x))^2]\ dx. \tag{4.37}$$

Since we are assuming in this example that $p(x) > 0$ and $q(x) > 0$ for all $x \in [x_o, x_1]$, it follows from (4.37) that

$$J(y+\eta) \geqslant J(y) + dJ(y;\eta), \quad \text{for } y \in \mathcal{A} \text{ and } \eta \in V_o,$$

which shows that J is convex in $\mathcal{A}$.

Next, observe that, in view of (4.37),

$$J(y+\eta) = J(y) + dJ(y;\eta), \quad \text{for } y \in \mathcal{A} \text{ and } \eta \in V_o$$

if and only if

$$\int_{x_o}^{x_1} [p(x)(\eta'(x))^2 + q(x)(\eta(x))^2]\ dx = 0. \tag{4.38}$$

Now, it follows from (4.38) and the assumption that $p > 0$ and $q > 0$ on $[x_o, x_1]$ that

$$\int_{x_o}^{x_1} p(x)(\eta'(x))^2\ dx = 0 \quad \text{and} \quad \int_{x_o}^{x_1} q(x)(\eta(x))^2\ dx = 0,$$

from which we obtain that $\eta(x) = 0$ for all $x \in [x_o, x_1]$, since η is continuous on $[x_o, x_1]$. Hence, J is strictly convex in $\mathcal{A}$.

The functional in (4.34) is an example of a general class of functionals of the form

$$J(y) = \int_a^b F(x, y(x), y'(x))\ dx, \quad \text{for } y \in C^1([s,b],\mathbb{R}),$$

where $F\colon [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function with continuous partial derivatives derivatives $F_y(x,y,z)$ and $F_z(x,y,z)$ in $[a,b]\times\mathbb{R}\times\mathbb{R}$. In the next example, we present conditions that will guarantee that functionals of this type are convex, or strictly convex.

Example 4.3.4. In Example 4.1.3 we saw that the functional $J\colon V \to \mathbb{R}$ given by

$$J(y) = \int_a^b F(x, y(x), y'(x))\ dx, \quad \text{for } y \in V, \tag{4.39}$$

where $V = C^1([a,b],\mathbb{R}) \to \mathbb{R}$, and the function

$$F\colon [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$$

is a continuous function of three variables $(x,y,x) \in [a,b]\times\mathbb{R}\times\mathbb{R}$ with continuous partial derivatives with respect to y and with respect to z, F_y and F_z, respectively, on $[a,b]\times\mathbb{R}\times\mathbb{R}$, is Gâteaux differentiable at every $y \in V$ in the direction of $\eta \in V_o$, where $V_o = C_o^1([a,b],\mathbb{R})$. We saw in Example 4.1.3 that

$$dJ(y,\eta) = \int_a^b [F_y(x,y,y')\eta + F_z(x,y,y')\eta']\ dx, \quad \text{for } y \in V \text{ and } \eta \in V_o. \tag{4.40}$$

In this example we find conditions on the function F that will guarantee that the functional J given in (4.39) is convex or strictly convex.

In view of 4.40, according to Definition 4.3.1, the functional J defined in (4.39) is convex in

$$\mathcal{A} = \{y \in C^1([a,b],\mathbb{R}) \mid y(a) = y_o,\ y(b) = y_1\}, \tag{4.41}$$

provided that

$$\int_a^b F(x, y{+}v, y'{+}v')\,\mathrm{d}x \geqslant \int_a^b F(x,y,y')\,\mathrm{d}x + \int_a^b [F_y(x,y,y')v + F_z(x,y,y')v']\,\mathrm{d}x$$

for all $y \in \mathcal{A}$ and $v \in V_o$. This inequality will follow, for example, if the function F satisfies

$$F(x, y+v, z+w) \geqslant F(x,y,z) + F_y(x,y,z)v + F_z(x,y,z)w, \tag{4.42}$$

for all (x,y,z) and $(x, y{+}v, z{+}w)$ where F is defined. Furthermore, the equality

$$J(u+v) = J(u) + dJ(u;v)$$

holds true if and only if equality in (4.42) holds true; and this is the case if and only if $v = 0$ or $w = 0$. In this latter case we get that J is also strictly convex.

Example 4.3.5 (A Sturm–Liouville Problem, Revisited Once Again)**.** Let $p \in C([x_o,x_1],\mathbb{R})$ and $q \in C([x_o,x_1],\mathbb{R})$ be such that $p(x) > 0$ and $q(x) > 0$ for $x_o \leqslant x \leqslant x_1$. Define $J\colon C^1([x_o,x_1],\mathbb{R}) \to \mathbb{R}$ by

$$J(y) = \int_{x_o}^{x_1} [p(x)(y'(x))^2 + q(x)(y(x))^2]\ dx, \qquad \text{for } y \in C^1([x_o,x_1],\mathbb{R}). \tag{4.43}$$

Thus, this functional corresponds to the function $F\colon [x_o,x_1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by

$$F(x,y,z) = p(x)z^2 + q(x)y^2, \qquad \text{for } x \in [x_o,x_1],\ y \in \mathbb{R} \text{ and } z \in \mathbb{R};$$

so that

$$F_y(x,y,z) = 2q(x)y \quad \text{ and } \quad F_z(x,y,z) = 2p(x)z,$$

for $(x,y,z) \in [x_o,x_1]\times\mathbb{R}\times\mathbb{R}$. Thus, the condition in (4.42) for the functional in (4.42) reads

$$p(x)(z+w)^2 + q(x)(y+v)^2 \geqslant p(x)z^2 + q(x)y^2 + 2q(x)yv + 2p(x)zw \tag{4.44}$$

To show that (4.44) holds true, expand the term on the left–hand side to get

$$\begin{aligned} p(x)(z+w)^2 + q(x)(y+v)^2 &= p(x)(z^2+2zw+w^2) \\ &\qquad +q(x)(y^2+2yv+v^2) \\ &= p(x)z^2 + 2p(x)zw + p(x)w^2 \\ &\qquad +q(x)y^2 + 2q(x)yv + q(x)v^2, \end{aligned}$$

which we can rewrite as

$$\begin{aligned}
p(x)(z+w)^2 + q(x)(y+v)^2 &= p(x)z^2 + q(x)y^2 \\
&\quad +2q(x)yv + 2p(x)zw \qquad (4.45) \\
&\quad +p(x)w^2 + q(x)v^2.
\end{aligned}$$

Since we are assuming that $p > 0$ and $q > 0$ on $[x_o, x_1]$, we see that (4.44) follows from (4.45). Thus, the functional J given in (4.43) is convex.

To see that J is strictly convex,assume that equality holds in (4.44); so that,

$$p(x)(z+w)^2 + q(x)(y+v)^2 = p(x)z^2 + q(x)y^2 + 2q(x)yv + 2p(x)zw.$$

It then follows from (4.45) that

$$p(x)w^2 + q(x)v^2 = 0, \quad \text{for all } x \in [x_o, x_1].$$

Consequently, since we are assuming that $p(x) > 0$ and $q(x) > 0$ for all $x \in [x_o, x_1]$, we get that

$$w = v = 0.$$

Hence, the functional J given in (4.43) is strictly convex.

Example 4.3.6. Let $p\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Suppose $p(x) > 0$ for all $x \in [a,b]$ and define

$$J(y) = \int_a^b p(x)\sqrt{1+(y')^2}\,\mathrm{d}x \quad \text{for} \quad y \in C^1([a,b],\mathbb{R}). \tag{4.46}$$

We consider the question of whether J is convex in the class

$$\mathcal{A} = \{y \in C^1([a,b],\mathbb{R}) \mid y(a) = y_o \text{ and } y(b) = y_1\} \tag{4.47}$$

for given real numbers y_o and y_1.

Set $V_o = \{v \in C^1([a,b],\mathbb{R}) \mid v(a) = 0 \text{ and } v(b) = 0\}$. Then, V_o is a nontrivial subspace of $C^1([a,b],\mathbb{R})$.

In this case, $F(x,y,z) = p(x)\sqrt{1+z^2}$ for $x \in [a,b]$ and all real values for y and z. Observe that $F_y \equiv 0$ and $F_z(x,y,z) = \dfrac{p(x)z}{\sqrt{1+z^2}}$ are continuous, and the inequality (4.42) for this case reads

$$p(x)\sqrt{1+(z+w)^2} \geqslant p(x)\sqrt{1+z^2} + \frac{p(x)zw}{\sqrt{1+z^2}},$$

which, by virtue of the assumption that $p > 0$ on $[a,b]$, is equivalent to

$$\sqrt{1+(z+w)^2} \geqslant \sqrt{1+z^2} + \frac{zw}{\sqrt{1+z^2}}. \tag{4.48}$$

The fact that the inequality in (4.46) holds true for all $z, w \in \mathbb{R}$, with equality iff $w = 0$, is a consequence of the Cauchy–Schwarz inequality in $\mathbb{R}^2$ applied to the vectors $\overrightarrow{A} = (1, z)$ and $\overrightarrow{B} = (1, z + w)$; see Problem 3 in Assignment #5.

It follows from the inequality in (4.48) that the functional in (4.46) is convex in $\mathcal{A}$.

To show that the functional J given in (4.46) is strictly convex, first rewrite the inequality in (4.48) as

$$\sqrt{1 + (z + w)^2} - \sqrt{1 + z^2} - \frac{zw}{\sqrt{1 + z^2}} \geqslant 0, \quad \text{for } z, w \in \mathbb{R}, \tag{4.49}$$

and note that equality in (4.49) holds if and only if $w = 0$.

It follows from what we have just shown that

$$J(y + v) \geqslant J(y) + dJ(y; v), \quad \text{for } y \in \mathcal{A} \text{ and } v \in V_o, \tag{4.50}$$

where, by virtue of the result in Example 4.1.3,

$$dJ(y; v) = \int_a^b p(x) \frac{y'(x)v'(x)}{\sqrt{1 + (y'(x))^2}}\, dx, \quad \text{for } y \in \mathcal{A} \text{ and } v \in V_o. \tag{4.51}$$

Thus, equality in (4.50) holds if and only if

$$\int_a^b p(x)\sqrt{1 + (y' + v')^2}\, dx = \int_a^b p(x)\sqrt{1 + (y')^2}\, dx + \int_a^b p(x) \frac{y'v'}{\sqrt{1 + (y')^2}}\, dx,$$

where we have used (4.51). So, equality in (4.50) holds if and only if

$$\int_a^b p(x) \left(\sqrt{1 + (y' + v')^2} - \sqrt{1 + (y')^2} - \frac{y'v'}{\sqrt{1 + (y')^2}} \right) dx = 0. \tag{4.52}$$

It follows from (4.52), the inequality in (4.49), the assumption that $p(x) > 0$ for all $x \in [a, b]$, and the assumptions that p, y' and v' are continuous, that

$$\sqrt{1 + (y'(x) + v'(x))^2} - \sqrt{1 + (y'(x))^2} - \frac{y'(x)v'(x)}{\sqrt{1 + (y'(x))^2}} = 0, \tag{4.53}$$

for all $x \in [a, b]$, where we have used the Basic Lemma I (see Lemma 3.2.5 on page 29 in these notes). Hence, since equality in (4.48) holds if and only if $w = 0$, we obtain from (4.53) that

$$v'(x) = 0, \quad \text{for all } x \in [a, b].$$

Thus, v is constant on $[a, b]$. Therefore, since $v(a) = 0$, it follows that $v(x) = 0$ for all $x \in [a, b]$. We have therefore demonstrated that he functional J defined in (4.46) is strictly convex in $\mathcal{A}$ given in (4.47).

We note that in the previous example (Example 4.3.6), the corresponding function, $F(x, y, z) = p(x)\sqrt{1 + z^2}$, to the functional J does not depend explicitly on y. In the next example we consider a general class of functionals of this type.

Example 4.3.7. Let I denote an open interval of real numbers (we note that I could be the entire real line). We consider function $F\colon [a, b] \times I \to \mathbb{R}$; that is, F is a function of two variables (x, z), where $x \in [a, b]$ and $z \in I$. We assume that F is continuous on $[a, b] \times I$. Suppose also that the second partial derivative of F with respect to z, $F_{zz}(x, z)$, exists and is continuous on $[a, b] \times I$, and that

$$F_{zz}(x, z) > 0, \qquad \text{for } (x, z) \in [a, b] \times I, \tag{4.54}$$

except possibly at finitely many values of z in I.

Let $V = C^1([a, b], \mathbb{R})$ and define the functional

$$J(y) = \int_a^b F(x, y'(x))\ \mathrm{d}x, \quad \text{for } y \in V. \tag{4.55}$$

In this example we show that, if (4.54) holds true, then the functional J defined in (4.55) is strictly convex in

$$\mathcal{A} = \{y \in C^1([a, b], \mathbb{R}) \mid y(a) = y_o \text{ and } y(b) = y_1\} \tag{4.56}$$

for given real numbers y_o and y_1.

According to (4.42) in Example 4.3.6, we need to show that

$$F(x, z + w) \geqslant F(x, z) + F_z(x, z)w, \tag{4.57}$$

for all $x \in [a, b]$, $z \in I$ and $w \in \mathbb{R}$ such that $z + w \in I$, since $F_y = 0$ in this case.

Fix $x \in [a, b]$, $z \in I$ and $w \in \mathbb{R}$ such that $z + w \in I$, and put

$$g(t) = F(x, z + tw), \qquad \text{for all } t \in [0, 1].$$

Then, g is C^2 in $(0, 1)$ and, integrating by parts,

$$\begin{aligned}
\int_0^1 (1 - t)g''(t)\ \mathrm{d}t &= (1 - t)g'(t)\Big|_0^1 + \int_0^1 g'(t)\ dt \\
&= -g'(0) + g(1) - g(0) \\
&= -F_z(x, z)w + F(x, z + w) - F(x, z),
\end{aligned}$$

from which it follows that

$$F(x, z + w) = F(x, z) + F_z(x, z)w + \int_0^1 (1 - t)F_{zz}(x, z + tw)w^2\ dt, \tag{4.58}$$

and so

$$F(x, z + w) \geqslant F(x, z) + F_z(x, z)w, \tag{4.59}$$

which is the inequality in (4.57).

Next, assume that equality holds in (4.59). It then follows from (4.58) that

$$\int_0^1 (1-t)F_{zz}(x, z+tw)w^2\ dt = 0,$$

from which we get that $w = 0$, in view of the assumption in (4.54). We have therefore shown that equality in (4.59) holds true if and only if $w = 0$.

To show that the functional J in (4.55) is strictly convex, use the inequality in (4.59) to see that, for $y \in V$ and $\eta \in V_o = C_o^1([a,b],\mathbb{R})$,

$$\begin{aligned} J(y+\eta) &= \int_a^b F(x, y'(x)+\eta'(x))\ dx \\ &\geqslant \int_a^b F(x, y'(x))\ dx + \int_a^b F_z(x, y'(x))\eta'(x)\ dx; \end{aligned}$$

so that, using the result in Example 4.1.3 and the definition of J in (4.55),

$$J(y+\eta) \geqslant J(y) + dJ(y;\eta) \quad \text{ for } y \in V \text{ and } \eta \in V_o. \tag{4.60}$$

Thus, the functional J defined in (4.55) is convex in $\mathcal{A}$, where $\mathcal{A}$ is given in (4.56).

To show that J is strictly convex, assume that equality hods in the inequality in (4.60) holds true; so that

$$J(y+\eta) = J(y) + dJ(y;\eta) \quad \text{ for some } y \in V \text{ and } \eta \in V_o,$$

or, using the the definition of J in (4.55) and the result in Example 4.1.3,

$$\int_a^b F(x, y'(x)+\eta'(x))\ dx = \int_a^b F(x, y'(x))\ dx + \int_a^b F_z(x, y'(x))\eta'(x)\ dx,$$

or

$$\int_a^b [F(x, y'(x)+\eta'(x)) - F(x, y'(x)) - F_z(x, y'(x))\eta'(x)]\ dx = 0. \tag{4.61}$$

It follows from the inequality in (4.59) that the integrand in (4.61) is nonnegative on $[a,b]$; hence, since y', η', F and F_z are continuous functions, it follows from (4.61) that

$$F(x, y'(x)+\eta'(x)) - F(x, y'(x)) - F_z(x, y'(x))\eta'(x) = 0, \quad \text{for } x \in [a,b]. \tag{4.62}$$

by virtue of the Basic Lemma I (see Lemma 3.2.5 on page 29 in these notes). Thus, since equality in (4.57) holds true if an only if $w = 0$, it follows from (4.62) that

$$\eta'(x) = 0, \quad \text{ for all } x \in (a,b),$$

from which we get that $\eta(x) = c$ for all $x \in [a,b]$, where c is a constant. Thus, since $\eta \in V_o$, it follows that $\eta(a) = 0$ and, therefore $c = 0$; so that, $\eta(x) = 0$ for all $x \in [a,b]$. We have therefore shown that equality in (4.60) holds true if and only if $\eta(x) = 0$ for all $x \in [a,b]$. Hence, the functional J defined in (4.55) is strictly convex in $\mathcal{A}$, where $\mathcal{A}$ is given in (4.56).

4.4 Convex Minimization Theorem

The importance of knowing that a given Gâteaux differentiable functional

$$J\colon V \to \mathbb{R}$$

is convex is that, once a vector u in $\mathcal{A}$ satisfying the condition

$$dJ(u; v) = 0, \quad \text{for all } v \in V_o \text{ with } u + v \in \mathcal{A},$$

is found, then we can conclude that u is a minimizer of J over the class $\mathcal{A}$. Furthermore, if we know that J is strictly convex, then we can conclude that there exists a unique minimizer of J over $\mathcal{A}$. This is the essence of the convex minimization theorem discussed in this section. A special version of this theorem is presented as Theorem 3.5 on page 57 in [Tro83].

Theorem 4.4.1 (Convex Minimization Theorem)**.** Let V denote a linear space and V_o a nontrivial subspace of V. Let $\mathcal{A}$ be a nonempty subset of V. Assume that a functional $J\colon V \to \mathbb{R}$ is Gâteaux differentiable at every $u \in \mathcal{A}$ in any direction $v \in V_o$ such that $u + v \in \mathcal{A}$. Assume also that J is convex in $\mathcal{A}$. Suppose there exists $u_o \in \mathcal{A}$ such that $u - u_o \in V_o$ for all $u \in \mathcal{A}$ and

$$dJ(u_o; v) = 0 \quad \text{for } \ v \in V_o \text{ such that } u_o + v \in \mathcal{A}.$$

Then,

$$J(u_o) \leqslant J(u) \quad \text{for all } u \in \mathcal{A}; \tag{4.63}$$

that is, u_o is a minimizer of J over $\mathcal{A}$. Moreover, if J is strictly convex in $\mathcal{A}$, then J can have at most one minimizer over $\mathcal{A}$.

Proof: Let $u_o \in \mathcal{A}$ be such that $u - u_o \in V_o$ for all $u \in \mathcal{A}$ and

$$dJ(u_o; v) = 0 \quad \text{for all } v \in V \text{ such that } u_o + v \in \mathcal{A}. \tag{4.64}$$

Given $u \in \mathcal{A}$, put $v = u - u_o$. Then, $v \in V_o$ and $u_o + v \in \mathcal{A}$, since $u_o + v = u \in \mathcal{A}$. Consequently, by virtue of (4.64),

$$dJ(u_o; v) = 0, \qquad \text{where } v = u - u_o. \tag{4.65}$$

Now, since we are assuming that J is convex in $\mathcal{A}$, it follows that

$$J(u) = J(u_o + v) \geqslant J(u_o) + dJ(u_o; v);$$

so that, in view of (4.65),

$$J(u) \geqslant J(u_o), \qquad \text{for all } u \in \mathcal{A},$$

which is the assertion in (4.63).

Assume further that J is strictly convex in $\mathcal{A}$, and let u_1 and u_2 be two minimizers of J over $\mathcal{A}$ such that $u - u_1 \in V_o$ and $u - u_2 \in V_o$ for all $u \in \mathcal{A}$. Then,

$$J(u_1) = J(u_2), \tag{4.66}$$

since $J(u_1) \leqslant J(u_2)$, given that $u_2 \in \mathcal{A}$ and u_1 is a minimizer of J over $\mathcal{A}$; and $J(u_2) \leqslant J(u_1)$, given that $u_1 \in \mathcal{A}$ and u_2 is a minimizer of J over $\mathcal{A}$.

It is also the case that

$$dJ(u_1; v) = 0, \quad \text{for any } v \in V_o \text{ with } u_1 + v \in \mathcal{A}, \tag{4.67}$$

by the results in Section 4.2 in these notes, since u_1 is a minimizer of J over $\mathcal{A}$.

Next, put $v = u_2 - u_1$; so that, $v \in V_o$ and $u_1 + v = u_2 \in \mathcal{A}$. We then have that

$$J(u_2) = J(u_1 + v);$$

so that, in view of (4.66),

$$J(u_1 + v) = J(u_1),$$

and, using (4.67),

$$J(u_1 + v) = J(u_1) + dJ(u_1; v). \tag{4.68}$$

Thus, since J is strictly convex, it follows from (4.68) that $v = 0$ (see (4.29) in Definition 4.3.1); that is, $u_2 - u_1 = 0$ or $u_1 = u_2$. Hence, if J is strictly convex in $\mathcal{A}$, then J can have at most one minimizer over $\mathcal{A}$. ■

As an application of the convex minimization theorem, we revisit the Brachistochrone problem in the following example.

Example 4.4.2 (The Brachistochrone Problem, Revisited One More Time)**.** We consider again the functional $J\colon \mathcal{A} \to \mathbb{R}$ given in Example 3.3.1,

$$J(y) = \int_0^{x_1} \frac{\sqrt{1 + (y'(x))^2}}{\sqrt{y_o - y(x)}}\, dx, \qquad \text{for } y \in \mathcal{A}, \tag{4.69}$$

where where $\mathcal{A}$ is the class

$$\mathcal{A} = \{y \in V \mid y(0) = y_o,\ y(x_1) = y_1,\ \text{and } y(x) < y_o \text{ for } 0 < x < x_1\}, \tag{4.70}$$

where $V = C^1([0, x_1], \mathbb{R})$.

We saw in that example that the value $J(y)$ given in (4.69) is proportional to the time of descent of an object from point $P(0, y_o)$ to point $Q(x_1, y_1)$, where $x_1 > 0$ and $y_o > y_1$, along a path that is the graph of a function $y \in C^1([0, x_1], \mathbb{R})$, with $y(0) = y_o$ and $y(x_1) = y_1$, assuming that gravitation is the only force acting on the object; see Figure 4.4.1.

In this example we will see how to rewrite the functional J given in (4.69) and (4.70) in a form for which it will be apparent how to use the convex minimization theorem in Theorem 4.4.1.

First, we use the transformation

$$u(x) = y_o - y(x), \qquad \text{for } x \in [0, x_1], \tag{4.71}$$

which was used in Example 3.3.5 when solving the Euler–Lagrange equation associated with the functional J given in (4.69) and (4.70).

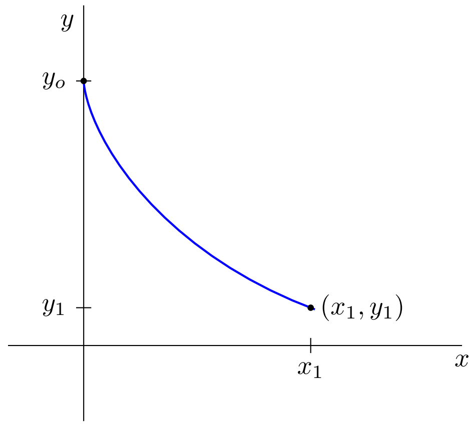

Figure 4.4.1: Curve connecting P to Q

Making the change of variables given in (4.71) in the integral on the right–hand side of (4.69), we obtain

$$J(u) = \int_0^{x_1} \frac{\sqrt{1 + (u'(x))^2}}{\sqrt{u(x)}}\, dx, \quad \text{for } u \in \mathcal{A}, \tag{4.72}$$

where the class $\mathcal{A}'$ is the class $\mathcal{A}$ in (4.2) written in terms of u; so that,

$$\mathcal{A}' = \{u \in V \mid u(0) = 0,\ u(x_1) = y_o - y_1,\ \text{and } u(x) > 0 \text{ for } 0 < x < x_1\},$$

and where we have used the boundary conditions defining the class $\mathcal{A}$ in (4.70).

We note that the graph of u given in (4.71) is as shown in Figure 4.4.2.

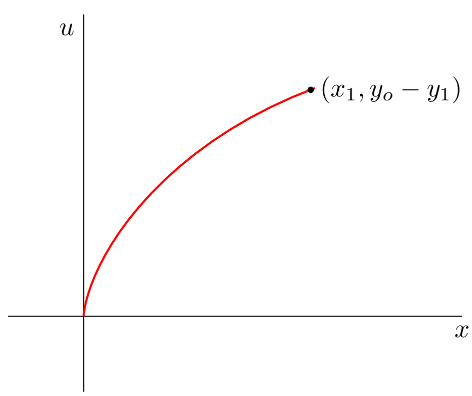

Figure 4.4.2: Graph of u

We assume that a typical function in the class $\mathcal{A}'$ has a graph as shown in Figure 4.4.2; so that, we may assume that the function $u\colon [0,x_1] \to [0,\infty)$ has an inverse. Thus, we can solve for x in terms of $u \in [0, y_o - y_1]$ to get $x = x(u)$. Hence, we can take u to be the independent variable and x the dependent variable. We can therefore make another change of variables in the integral defining J in (4.72) in terms of the function $x\colon [0, y_o - y_1] \to [0, x_1]$. To do this, we use the Chain Rule to compute

$$u'(x) = \frac{1}{x'(u)}, \qquad \text{for } 0 < x < x_1, \tag{4.73}$$

and

$$dx = x'(u)\; du.$$

We can therefore rewrite the integral expression for J in (4.72) as

$$J(x) = \int_0^{y_o - y_1} \frac{\sqrt{1 + \frac{1}{(x'(u))^2}}}{\sqrt{u}}\; x'(u)\; du,$$

or

$$J(x) = \int_0^{y_o - y_1} \frac{1}{\sqrt{u}} \sqrt{1 + (x'(u))^2}\;\; du, \qquad \text{for } x \in \mathcal{A}'', \tag{4.74}$$

where $\mathcal{A}''$ is given by

$$\mathcal{A}'' = \{x \in C^1([0, y_o - y_1], \mathbb{R}) \mid x(0) = 0,\; x(y_o - y_1) = x_1\}. \tag{4.75}$$

Note that the functional J defined in (4.74) and (4.75) is given in terms of an improper integral, since the integrand is not defined at $u = 0$. Thus, we need to see that the integral on the right–hand side of (4.74) converges. To do this, let $\varepsilon > 0$ and consider the integral

$$\int_\varepsilon^{y_o - y_1} \frac{1}{\sqrt{u}} \sqrt{1 + (x'(u))^2}\;\; du, \qquad \text{for } x \in C^1([0, y_o - y_1], \mathbb{R}). \tag{4.76}$$

Since $x \in C^1([0, y_o - y_1], \mathbb{R})$, the derivative $x'(u)$ is bounded; so that, there exists a constant $M > 0$ such that

$$|x'(u)| \leqslant M, \qquad \text{for all } u \in [0, y_o - y_1].$$

Consequently, the integral in (4.76) can be estimated as follows

$$\int_\varepsilon^{y_o - y_1} \frac{1}{\sqrt{u}} \sqrt{1 + (x'(u))^2}\;\; du \leqslant \sqrt{1 + M^2} \int_\varepsilon^{y_o - y_1} \frac{1}{\sqrt{u}}\; du,$$

or

$$\int_\varepsilon^{y_o - y_1} \frac{1}{\sqrt{u}} \sqrt{1 + (x'(u))^2}\;\; du \leqslant 2\sqrt{1 + M^2}(\sqrt{y_o - y_1} - \sqrt{\varepsilon});$$

from which we get

$$\int_{\varepsilon}^{y_o-y_1} \frac{1}{\sqrt{u}}\sqrt{1+(x'(u))^2}\ du < 2\sqrt{1+M^2}\sqrt{y_o-y_1}, \qquad \text{for all } \varepsilon > 0. \quad (4.77)$$

It follows from the estimate in (4.77), and the fact that the integrand of the integrals in (4.76) is positive for $u > 0$, that the integrals in(4.76) define a bounded and monotone function of ε that increases as ε decreases to 0. Consequently,

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{y_o-y_1} \frac{1}{\sqrt{u}}\sqrt{1+(x'(u))^2}\ du \quad (4.78)$$

exists. Thus, the limit in (4.78) defines the improper integral on the right hand side of (4.74) giving the value $J(x)$.

The functional in (4.74) is of the form

$$J(x) = \int_a^b F(u, x(u), x'(u))\ du, \qquad \text{for } x \in C^1([a,b],\mathbb{R}). \quad (4.79)$$

The function F in (4.79) is a continuous function of three variables, (u, x, z), where $u \in [a, b]$ and $x, z \in \mathbb{R}$. We will also assume that the partial derivatives

$$F_x(u,x,z) \quad \text{and} \quad F_z(u,x,z), \qquad \text{for } (u,z,x) \in (a,b)\times\mathbb{R}\times\mathbb{R},$$

exist and are continuous on $(a, b) \times \mathbb{R} \times \mathbb{R}$. Indeed, in this example, F is of the form

$$F(u,x,z) = p(u)\sqrt{1+z^2}, \qquad \text{for } u \in (0, y_o - y_1) \quad \text{and } z \in \mathbb{R}, \quad (4.80)$$

where

$$p(u) = \frac{1}{\sqrt{u}}, \qquad \text{for } u > 0. \quad (4.81)$$

This is the situation that we had in Example 4.3.6 on page 60 in these notes for the case in which u is playing the role of x, and x is playing the role of y. We showed in that example that the functional J is strictly convex (see also the result of Example 4.3.7 on page 62 in these notes).

We can therefore apply the convex minimization theorem (Theorem 4.4.1 on page 64 in these notes) to conclude that, if $x \in \mathcal{A}''$, where $\mathcal{A}''$ is given in (4.75), satisfies the equation

$$dJ(x;\eta) = 0, \qquad \text{for all } \eta \in C_o^1([0, y_o - y_1], \mathbb{R}), \quad (4.82)$$

then x is the unique minimizer of J in $\mathcal{A}''$.

A function $x \in \mathcal{A}''$ satisfies the necessary condition in (4.82) if and only if x is a solution of the Euler–Lagrange equation associated with the functional J given in (4.74) and (4.75), which is

$$\frac{d}{du}\left[F_z(u, x(u), x'(u))\right] = F_x(u, x(u), x'(u)), \qquad \text{for } 0 < u \leqslant y_o - y_1. \quad (4.83)$$

In this example, the function F is given by (4.80) and (4.81); so that,

$$F_x(u,x,z) = 0, \qquad \text{for } (u,x,z) \in (0, y_o - y_1) \times \mathbb{R} \times \mathbb{R},$$

and

$$F_z(u,x,z) = \frac{p(u)z}{\sqrt{1+z^2}}, \qquad \text{for } (u,x,z) \in (0, y_o - y_1) \times \mathbb{R} \times \mathbb{R}.$$

Thus, substituting into the equation in (4.83) we obtain that the Euler–Lagrange equation associate with the functional in (4.74) is

$$\frac{d}{du}\left[\frac{p(u)x'(u)}{\sqrt{1+(x'(u))^2}}\right] = 0, \qquad \text{for } 0 < u \leqslant y_o - y_1. \tag{4.84}$$

The differential equation in (4.84) can be integrated to yield

$$\frac{p(u)x'(u)}{\sqrt{1+(x'(u))^2}} = c_1, \qquad \text{for } 0 < u \leqslant y_o - y_1, \tag{4.85}$$

for some constant of integration c_1.

We note that the constant c_1 in (4.85) cannot be 0 if $x \in \mathcal{A}''$. Otherwise, since $p(u) > 0$ for all $u \in (0, y_o - y_1)$, we would have that $x'(u) = 0$ for $0 < u < y_o - y_1$, which implies that x is constant. Thus, since $u(0) = 0$, it would follow that $x(y_o - y_1) = 0$, which contradicts the assumption that $x(y_o - y_1) = x_1 > 0$ in the definition of $\mathcal{A}''$ in (4.75).

Next, use the definition of $p(u)$ in (4.81) and the identity in (4.73) to rewrite the equation in (4.85) as

$$(u')^2 = \frac{C-u}{u}, \qquad \text{for } 0 < x < x_1, \tag{4.86}$$

where we have set $C = \dfrac{1}{c_1^2}$.

The differential equation in (4.86) is the equation in (3.95) that we derived in Example 3.3.5 when solving the Euler–Lagrange equation associated with the functional J in (4.69) and (4.70). In that example we solved the equation in (4.86) and obtained parametric equations for a cycloid (see the equations in (3.108)). We also showed in Example 3.3.5 that there is cycloid parametrized by the equations in (3.108) that goes through the points $(0,0)$ and $(x_1, y_o - y_1)$ in the xu–plane. We have therefore shown that the section of the cycloid connecting the point $P(0, y_o)$ to the point (x_1, y_1), where $y_o > y_1 > 0$ and $x_1 > 0$, minimizes the time of descent of an object going from point P to point Q under the sole action of gravity. Furthermore, it is the only curve connecting P to Q with that property. This is a consequence of the convex minimization theorem.

Chapter 5

Optimization Problems with Constraints

Let V be a normed linear space and V_o a nontrivial subspace of V. Let $J\colon V \to \mathbb{R}$ be a functional that is Gâteaux differentiable at every $u \in V$ in the direction of every $v \in V_o$. In many applications we would like to optimize (maximize or minimize) J over a class of admissible vectors $\mathcal{A}$ that are also in a subset of V that is defined as the level set (or sets) of another Gâteaux differentiable functional (or functionals) $K\colon V \to \mathbb{R}$. In this chapter, we discuss how to obtain necessary conditions for a given vector $u \in \mathcal{A}$ to be an optimizer of J over a constraint of the form

$$K^{-1}(c) = \{v \in V \mid K(v) = k_o\}, \tag{5.1}$$

for some real number k_o. The level set $K^{-1}(k_o)$ is an example of a constraint. In some applications there can be several constraints, and sometimes constraints might come in the form of inequalities.

An example of an optimization problem with constraint might take the form of: Find $u \in K^{-1}(k_o) \cap \mathcal{A}$ such that

$$J(u) = \max_{v \in K^{-1}(k_o)} J(v). \tag{5.2}$$

In the following section, we present a classical example of an optimization problem of the type in (5.2) and (5.1).

5.1 Queen Dido's Problem

Let ℓ denote a given positive number and consider all curves in the xy–plane that are the graph of a function $y \in C^1([0,\ell],\mathbb{R})$ such that $y(x) \geqslant 0$ for all $x \in [0,b]$, $y(0) = 0$, $y(b) = 0$ for some $b \in (a,\ell)$, and the arc–length of the curve from $(0,0)$ to $(b,0)$ equal to ℓ; that is,

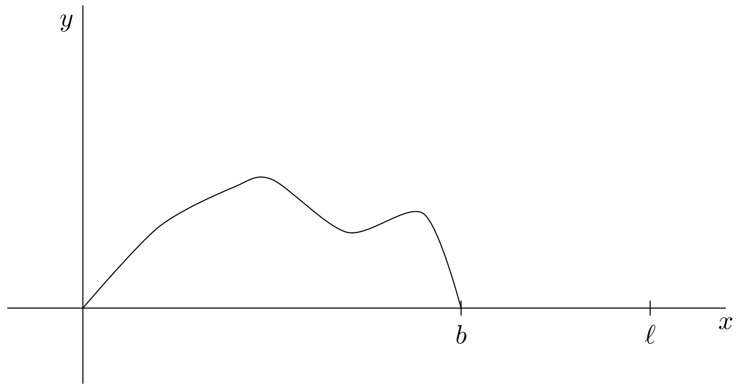

Figure 5.1.1: Graph of $y(x)$ for $0 \leqslant x \leqslant b$

$$\int_0^b \sqrt{1+(y'(x))^2}\ dx = \ell. \tag{5.3}$$

Figure 5.1.1 shows one of those curves (the graph of the corresponding function $y \in C^1([0,b],\mathbb{R})$ is shown over the interval $[0,b]$). We will denote the class of these curves by $\mathcal{A}$. Setting $V_o = C^1_o([0,b],\mathbb{R})$ and

$$K(y) = \int_0^b \sqrt{1+(y'(x))^2}\ dx, \qquad \text{for } y \in C^1([0,b],\mathbb{R}), \tag{5.4}$$

we can express the class $\mathcal{A}$ as

$$\mathcal{A} = \{y \in V_o \mid y(x) \geqslant 0 \text{ for } x \in [0,b], \text{ and } K(y) = \ell\}. \tag{5.5}$$

Note that K in (5.4) defines a functional $K\colon V \to \mathbb{R}$ where $V = C^1([0,b],\mathbb{R})$. This is the arc–length functional that we have encountered previously in these notes. The expression

$$K(y) = \ell, \qquad \text{for } y \in \mathcal{A}, \tag{5.6}$$

in the definition of $\mathcal{A}$ in (5.5) defines a constraint.

We would like to find, if possible, the curve in $\mathcal{A}$ for which the area enclosed by it and the positive x–axis is the largest possible; that is,

$$\int_0^b y(x)\ dx, \qquad \text{for } y \in \mathcal{A}$$

is the largest possible among all functions $y \in \mathcal{A}$.

Defining the functional $J\colon V \to \mathbb{R}$ by

$$J(y) = \int_0^b y(x)\ dx, \qquad \text{for } y \in V, \tag{5.7}$$

we can restate the problem as

Problem 5.1.1 (Queen Dido's Problem)**.** Let $J\colon V \to \mathbb{R}$ be the area functional defined in (5.7), $K\colon V \to \mathbb{R}$ the arc–length functional defined in (5.4), and $\mathcal{A}$ be the class of admissible functions defined in (5.5) in terms of the constraint $K(y) = \ell$ in (5.6). If possible, find $y \in \mathcal{A}$ such that

$$J(y) \geqslant J(v), \quad \text{for all } v \in \mathcal{A}.$$

To make the dependence of the constraint in (5.6) more explicit, we may also write the problem as: Find $y \in \mathcal{A}$ such that

$$J(y) \geqslant J(v), \quad \text{for } v \in V_o \text{ with } v(x) \geqslant 0 \text{ for } x \in [0, b] \text{ and } K(v) = \ell. \tag{5.8}$$

In the following section we will see how we can approach this kind of problems in a general setting.

5.2 Euler–Lagrange Multiplier Theorem

Let V denote a normed linear space with norm $\| \cdot \|$, and let V_o be a nontrivial subspace of V. Let $J\colon V \to \mathbb{R}$ and $K\colon V \to \mathbb{R}$ be functionals that are Gâteaux differentiable at every $u \in V$ in the direction of every $v \in V_o$. We would like to obtain necessary conditions for a given vector $u \in V$ to be an optimizer (a maximizer or a minimizer) of J subject to a constraint of the form

$$K^{-1}(c) = \{v \in V \mid K(v) = c\}, \tag{5.9}$$

where c is a given real number; it is assumed that the level set $K^{-1}(c)$ in (5.9) is nonempty. Thus, we would like to find conditions satisfied by $u_o \in V$ such that

$$K(u_o) = c \tag{5.10}$$

and

$$J(u_o) \leqslant J(v) \text{ (or } J(u_o) \geqslant J(v)) \text{ for all } v \in V \text{ such that } K(v) = c. \tag{5.11}$$

We also assume that the Gâteaux derivatives of J and K, $dJ(u; v)$ and $dK(u; v)$, respectively, are **weakly continuous** in u.

Definition 5.2.1 (Weak Continuity)**.** Let V denote a normed linear space with norm $\| \cdot \|$ and V_o a nontrivial subspace of V. Let $J\colon V \to \mathbb{R}$ be Gâteaux differentiable at every $u \in V$ in the direction of every $v \in V_o$. We say that the Gâteaux derivative of J, $dJ(u; v)$, is weakly continuous at $u_o \in V$ if and only if

$$\lim_{u \to u_o} dJ(u; v) = dJ(u_o; v), \quad \text{for every } v \in V_o; \tag{5.12}$$

that is, for each $v \in V_o$, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|u - u_o\| < \delta \Rightarrow |dJ(u; v) - dJ(u_o; v)| < \varepsilon.$$

Example 5.2.2. Let

$$V = \left\{ y \in C^1([0,1], \mathbb{R}) \;\Big|\; \int_0^1 (y(x))^2 \, dx < \infty \text{ and } \int_0^1 (y'(x))^2 \, dx < \infty \right\},$$

and define a norm on V by

$$\|y\| = \sqrt{\int_0^1 (y(x))^2 \, dx + \int_0^1 (y'(x))^2 \, \mathrm{d}x}, \quad \text{ for all } y \in V. \tag{5.13}$$

Define $J \colon V \to \mathbb{R}$ by

$$J(y) = \frac{1}{2} \int_0^1 (y'(x))^2 \, dx \quad \text{for all } y \in V. \tag{5.14}$$

Then, J is Gâteaux differentiable at every $y \in V$ with Gâteaux derivative at $y \in V$ in the direction of $v \in V$ given by

$$dJ(y; v) = \int_0^1 y'(x) v'(x) \, dx, \quad \text{ for } y, v \in V. \tag{5.15}$$

To see that the Gâteaux derivative of J given in (5.15) is weakly continuous at every $y \in V$, use (5.15) to compute, for $u \in V$,

$$\begin{aligned} dJ(u; v) - dJ(y; v) &= \int_0^1 u'(x) v'(x) \, dx - \int_0^1 y'(x) v'(x) \, dx \\ &= \int_0^1 [u'(x) - y'(x)] v'(x) \, dx; \end{aligned}$$

so that, taking absolute values on both sides,

$$|dJ(u; v) - dJ(y; v)| \leqslant \int_0^1 |u'(x) - y'(x)| \, |v'(x)| \, dx. \tag{5.16}$$

Then, applying the Cauchy–Schwarz inequality on the right–hand side of (5.16),

$$|dJ(u; v) - dJ(y; v)| \leqslant \sqrt{\int_0^1 |u'(x) - y'(x)|^2 \, dx} \cdot \sqrt{\int_0^1 |v'(x)|^2 \, dx};$$

so that, using the definition of norm in V given in (5.13),

$$|dJ(u; v) - dJ(y; v)| \leqslant \|u - y\| \cdot \|v\|. \tag{5.17}$$

Thus, given any $\varepsilon > 0$ and assuming that $v \neq 0$, we see from (5.17) that, setting

$$\delta = \frac{\varepsilon}{\|v\|},$$

$$\|u - y\| < \delta \Rightarrow |dJ(u; v) - dJ(y; v)| < \varepsilon.$$

We therefore conclude that

$$\lim_{u \to y} dJ(u; v) = dJ(y; v), \quad \text{ for all } v \in V.$$

Example 5.2.3. Let $V = C^1([a,b],\mathbb{R})$ and $V_o = C_o^1([a,b],\mathbb{R})$. We endow the space V with the norm

$$\|y\| = \max_{x\in[a,b]} |y(x)| + \max_{x\in[a,b]} |y'(x)|, \qquad \text{for all } y \in V. \tag{5.18}$$

Assume that $p \in C([a,b],\mathbb{R})$ is such that $p(x) \geqslant 0$, for all $x \in [a,b]$, and define the functional $J\colon V \to \mathbb{R}$ by

$$J(y) = \int_a^b p(x)\sqrt{1+(y'(x))^2}\; dx, \qquad \text{for all } y \in V. \tag{5.19}$$

The functional in (5.19) can be shown to be Gâteaux differentiable using the result in Example 4.1.3 and to have Gâteaux derivative given by

$$dJ(y;\eta) = \int_a^b \frac{p(x)y'(x)\eta'(x)}{\sqrt{1+(y'(x))^2}}\; dx, \qquad \text{for all } y \in V \text{ and } \eta \in V_o. \tag{5.20}$$

In this example we show that the Gâteaux derivative of J given in (5.20) is weakly continuous in $y \in V$; that is, we show that

$$\lim_{\|v-y\|\to 0} |dJ(v;\eta) - dJ(y;\eta)| = 0 \qquad \text{for all } \eta \in V_o. \tag{5.21}$$

Let $v, y \in V$ and use (5.20) to compute

$$dJ(v;\eta) - dJ(y;\eta) = \int_a^b p(x)\left(\frac{v'(x)}{\sqrt{1+(v'(x))^2}} - \frac{y'(x)}{\sqrt{1+(y'(x))^2}}\right)\eta'(x)\; dx,$$

which we can write as

$$dJ(v;\eta) - dJ(y;\eta) = \int_a^b p(x)\left(f(v'(x)) - f(y'(x))\right)\eta'(x)\; dx, \tag{5.22}$$

where

$$f(z) = \frac{z}{\sqrt{1+z^2}}, \qquad \text{for } z \in \mathbb{R}. \tag{5.23}$$

Next, take absolute value on both sides of the equation in (5.22) and estimate the integral on the right–hand side to get

$$|dJ(v;\eta) - dJ(y;\eta)| \leqslant M\|\eta\| \int_a^b |f(v'(x)) - f(y'(x))|\; dx, \tag{5.24}$$

where

$$M = \max_{x\in[a,b]} p(x)$$

and $\|eta\|$ is the norm of η given by the formula in (5.18).

Observe that the derivative of the function f given in (5.23) is

$$f'(z) = \frac{1}{(1+z^2)^{3/2}}, \qquad \text{for all } z \in \mathbb{R};$$

so that,

$$0 < f'(z) \leqslant 1, \qquad \text{for } z \in \mathbb{R}. \tag{5.25}$$

Now, it follows from the mean–value theorem and the left–most inequality in (5.25) that

$$|f(w) - f(z)| = f'(c)|w - z|, \qquad \text{for } w, z \in \mathbb{R}, \tag{5.26}$$

where c is some value between z and w. Thus, using the right–most inequality in (5.25), we obtain from (5.26) that

$$|f(w) - f(z)| \leqslant |w - z|, \qquad \text{for } w, z \in \mathbb{R}. \tag{5.27}$$

Next, apply the inequality in (5.26) to the integrand on the right–hand side of (5.24) to get

$$|dJ(v;\eta) - dJ(y;\eta)| \leqslant M\|\eta\| \int_a^b |v'(x) - y'(x)| \; dx. \tag{5.28}$$

Then, using the definition of the norm $\|\cdot\|$ in (5.18), we get from (5.28) that

$$|dJ(v;\eta) - dJ(y;\eta)| \leqslant M\|\eta\| \int_a^b \|v - y\| \; dx. \tag{5.29}$$

Evaluating the integral on the right–hand side of (5.29) then yields

$$|dJ(v;\eta) - dJ(y;\eta)| \leqslant M\|\eta\|(b-a)\|v - y\|. \tag{5.30}$$

It follows from (5.30) and the squeeze theorem that

$$\lim_{\|v-y\|\to 0} |dJ(v;\eta) - dJ(y;\eta)| = 0, \qquad \text{for all } \eta \in V_o,$$

which is the assertion in (5.21). Hence, the Gâteaux derivative of J is weakly continuous in $y \in V$.

The following theorem will be helpful in obtaining necessary conditions for solutions of the general constrained optimization problem in (5.10) and (5.11). It is a generalization of the Lagrange multiplier theorem in Euclidean space.

Theorem 5.2.4 (Euler–Lagrange Multiplier Theorem)**.** Let V denote a normed linear space and V_o a nontrivial subspace of V. Let $J\colon V \to \mathbb{R}$ and $K\colon V \to \mathbb{R}$ be functionals that are Gâteaux differentiable at every $u \in V$ in the direction of every $v \in V_o$. Suppose there exists $u_o \in V$ such that

$$K(u_o) = c, \tag{5.31}$$

for some real number c, and

$$J(u_o) \leqslant J(v) \text{ (or } J(u_o) \geqslant J(v)\text{) for all } v \in V \text{ such that } K(v) = c. \tag{5.32}$$

Suppose also that the Gâteaux derivatives, $dJ(u;v)$ and $dK(u;v)$, of J and K, respectively, are weakly continuous in u for all $u \in V$. Then, either

$$dK(u_o;v) = 0, \quad \text{for all } v \in V_o, \tag{5.33}$$

or there exists a real number μ such that

$$dJ(u_o;v) = \mu\, dK(u_o;v), \quad \text{for all } v \in V_o. \tag{5.34}$$

Remark 5.2.5. A proof of a slightly more general version of Theorem 5.2.4 can be found in [Smi74, pp. 72–77].

Remark 5.2.6. The scalar μ in (5.34) is called and Euler–Lagrange multiplier.

Remark 5.2.7. In practice, when solving the constrained optimization problem in (5.31) and (5.32), we solve (if possible) the equations in (5.31), (5.33) and (5.34), simultaneously, to find a candidate, u_o, for the solution of the optimization problem. We will get to see an instance of this approach in the following example.

Example 5.2.8 (Queen Dido's Problem, Revisited)**.** For the problem introduced in Section 5.1, we were given a fixed positive number ℓ, and we defined $V = C^1([0,b],\mathbb{R})$, $V_o = C^1_o([0,b],\mathbb{R})$, and

$$\mathcal{A} = \{y \in V_o \mid y(x) \geqslant 0 \text{ for } x \in [0,b], \text{ and } K(y) = \ell\} \tag{5.35}$$

where $K\colon V \to \mathbb{R}$ is the arc–length functional

$$K(y) = \int_a^b \sqrt{1+(y'(x))^2}\; dx, \quad \text{for } y \in V. \tag{5.36}$$

We would like to maximize the area functional $J\colon V \to \mathbb{R}$ given by

$$J(y) = \int_0^b y(x)\; dx, \quad \text{for } y \in V, \tag{5.37}$$

subject to the constraint $K(y) = \ell$.

Eying to apply the Euler–Lagrange Multiplier Theorem, we compute the Gâteaux derivatives of the functionals in (5.36) and (5.37) to get

$$dK(y;\eta) = \int_a^b \frac{y'(x)\eta'(x)}{\sqrt{1+(y'(x))^2}}\; dx, \quad \text{for } y \in V \text{ and } \eta \in V_o, \tag{5.38}$$

and

$$dJ(y;\eta) = \int_0^b \eta(x)\; dx, \quad \text{for } y \in V \text{ and } \eta \in V_o. \tag{5.39}$$

Endowing the space V with the norm

$$\|y\| = \max_{0\leqslant x\leqslant b} |y(x)| + \max_{0\leqslant x\leqslant b} |y'(x)|, \quad \text{for all } y \in V,$$

we can show that the Gâteaux derivative of K given in (5.38) is weakly continuous (see Problem 4 in Assignment #6).

To see that the Gâteaux derivative of J in (5.39) is also weakly continuous, observe that, for all u and y in V,

$$dJ(u;\eta) - dJ(y;\eta) = 0, \qquad \text{for all } \eta \in V_o,$$

in view of (5.39).

Thus, we can apply the Euler–Lagrange multiplier theorem to the optimization problem: Find $y \in \mathcal{A}$, where $\mathcal{A}$ is given in (5.35), such that

$$J(y) = \max_{u \in \mathcal{A}} J(u). \tag{5.40}$$

We obtain that necessary conditions for $y \in \mathcal{A}$ to be a candidate for a solution of the problem in (5.40) are

$$\int_0^b \sqrt{1 + (y'(x))^2}\; dx = \ell; \tag{5.41}$$

$$\int_a^b \frac{y'(x)\eta'(x)}{\sqrt{1 + (y'(x))^2}}\; dx = 0, \qquad \text{for all } \eta \in V_o; \tag{5.42}$$

or, there exists a multiplier $\mu \in \mathbb{R}$ such that

$$\int_0^b \eta(x)\; dx = \mu \int_a^b \frac{y'(x)\eta'(x)}{\sqrt{1 + (y'(x))^2}}\; dx, \qquad \text{for all } \eta \in V_o, \tag{5.43}$$

where we have used (5.31), (5.33) and (5.34) in the conclusion of Theorem 5.2.4.

We first consider the case in which (5.42) holds true. In this case, the Second Fundamental Lemma of the Calculus of Variations (Lemma 3.2.8 on page 31 in these notes) implies that

$$\frac{y'(x)}{\sqrt{1 + (y'(x))^2}} = C, \qquad \text{for all } x \in [0, b], \tag{5.44}$$

for some constant C.

Solving the equation in (5.44) for $y'(x)$ yields the differential equation

$$y'(x) = c_1, \qquad \text{for all } x \in [0, b], \tag{5.45}$$

for some constant c_1. Solving the equation in (5.45) in turn yields

$$y(x) = c_1 x + c_2, \qquad \text{for all } x \in [0, b], \tag{5.46}$$

for some constants c_1 and c_2. Then, using the requirement that $y \in \mathcal{A}$, so that $y \in V_o$, we obtain from (5.46) that

$$c_1 = c_2 = 0;$$

Thus,

$$y(x) = 0, \qquad \text{for all } x \in [0, b], \tag{5.47}$$

is a candidate for an optimizer of J over $\mathcal{A}$, provided that (5.41) holds true.

We get from (5.41) and (5.47) that

$$\int_0^b \sqrt{1 + 0^2}\; dx = \ell,$$

from which we get that

$$b = \ell;$$

this shows a connection between the length of the curve, ℓ, and the end–point, b, of the interval $[0, b]$ determined by the constraint in (5.41).

We have shown that the function $y \in C^1_o([0, b], \mathbb{R})$, where $b = \ell$, given in (5.47) is a candidate for an optimizer of J defined in (5.37) over the class $\mathcal{A}$ given in (5.35). Since in this case $J(y) = 0$, for the function y given in (5.47), y is actually a minimizer of J over $\mathcal{A}$, and not a maximizer. Thus, we turn to the second alternative in (5.43), which we can rewrite as

$$\int_0^b \left(\eta(x) - \frac{\mu y'(x)}{\sqrt{1 + (y'(x))^2}}\; \eta'(x) \right) dx = 0, \qquad \text{for all } \eta \in V_o. \tag{5.48}$$

Now, it follows from (5.48) and the Third Fundamental Lemma (Lemma 3.2.9 on page 32 of theses notes) that y must be a solution of the differential equation

$$-\frac{d}{dx}\left[\frac{\mu y'(x)}{\sqrt{1 + (y'(x))^2}} \right] = 1, \qquad \text{for } 0 < x < b. \tag{5.49}$$

We see from (5.49) that $\mu \neq 0$, since $1 \neq 0$. We can therefore rewrite the equation in (5.49) as

$$\frac{d}{dx}\left[\frac{y'(x)}{\sqrt{1 + (y'(x))^2}} \right] = -\frac{1}{\mu}, \qquad \text{for } 0 < x < b. \tag{5.50}$$

Integration the equation in (5.50) yields

$$\frac{y'(x)}{\sqrt{1 + (y'(x))^2}} = -\frac{x}{\mu} + c_1, \qquad \text{for } 0 < x < b, \tag{5.51}$$

for some constant of integration c_1.

Next, solve the equation in (5.51) for $y'(x)$ to obtain the differential equation

$$\frac{dy}{dx} = \pm \frac{x - \mu c_1}{\sqrt{\mu^2 - (x - \mu c_1)^2}}, \qquad \text{for } 0 < x < b, \tag{5.52}$$

The differential equation in (5.52) can be integrated to yield

$$y = \mp \sqrt{\mu^2 - (x - \mu c_1)^2} + c_2, \tag{5.53}$$

for a constant of integration c_2.

Observe that the equation in (5.53) can be written as

$$(x - \mu c_1)^2 + (y - c_2)^2 = \mu^2, \tag{5.54}$$

which is the equation of a circle of radius μ (here we are taking $\mu > 0$) and center at $(\mu c_1, c_2)$. Thus, the graph of a $y \in \mathcal{A}$ for which the area, $J(y)$, under it and above the x–axis is the largest possible must be a semicircle of radius μ and centered at $(\mu c_1, 0)$; so that, $c_2 = 0$. We then get from (5.53) that

$$y(x) = \sqrt{\mu^2 - (x - \mu c_1)^2}, \qquad \text{for } 0 < x < b, \tag{5.55}$$

where we have taken the positive solution in (5.53) to ensure that $y(x) \geqslant 0$ for all $x \in [0, b]$, according to the definition of $\mathcal{A}$ in (5.35); so that, the graph of y is the upper semicircle. We are also assuming that $c_1 > 0$. Furthermore,

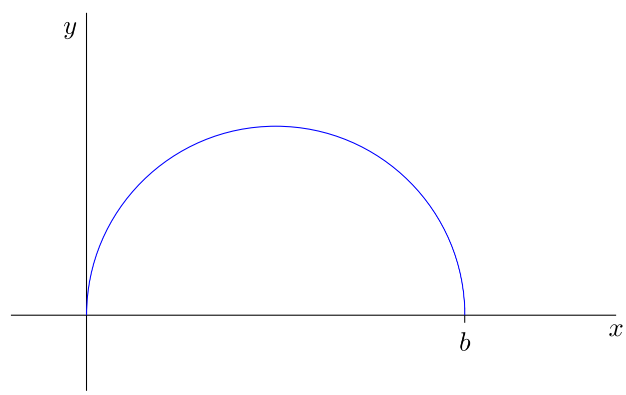

Figure 5.2.2: Graph of optimal solution $y(x)$ for $0 \leqslant x \leqslant b$

the condition $y(0) = 0$ in the definition of $\mathcal{A}$ in (5.35) implies from (5.55) that $c_1 = 1$; so that, (5.55) now reads

$$y(x) = \sqrt{\mu^2 - (x - \mu)^2}, \qquad \text{for } 0 < x < b; \tag{5.56}$$

thus, the graph of y is a semicircle of radius $\mu > 0$ centered at $(\mu, 0)$; this is pictured in Figure 5.2.2, where $b = 2\mu$. Hence, according to the definition of $\mathcal{A}$ in (5.36), we also obtain an expression for μ in terms of b:

$$\mu = \frac{b}{2}. \tag{5.57}$$

Finally, since $K(y) = \ell$, according to the definition of $\mathcal{A}$ in (5.36), it must also be the case that

$$\pi\mu = \ell, \tag{5.58}$$

given that $\pi\mu$ is the arc–length of the semicircle of radius μ pictured in Figure 5.2.2. Combining (5.57) and (5.58) we see that

$$b = \frac{2\ell}{\pi},$$

which gives the connection between b and ℓ imposed by the constraint in (5.41).

Remark 5.2.9. It is important to keep in mind that the condition for an optimizer that we obtained in Example 5.2.8 was obtained under the assumption that an optimizer exists. This is the main assumption in the statement of the Euler–Lagrange multiplier theorem. We have not proved that an optimizer for J exists in $\mathcal{A}$. What we did prove is that, if a solution $y \in \mathcal{A}$ of the optimization problem

$$J(y) = \max_{v\in\mathcal{A}} J(v), \qquad \text{subjec to } K(y) = \ell,$$

where $\mathcal{A}$ is as given in (5.35) and J as defined in (5.37), exists, then the graph of $y(x)$, for $0 \leqslant x \leqslant b$, must be a semicircle.

Example 5.2.10. In this example we consider a general class of problems that can be formulated as follows: Let a and b be real numbers with $a < b$ and define $V = C^1([a,b],\mathbb{R})$. The space V is a normed linear space with norm

$$\|y\| = \max_{a\leqslant c\leqslant b} |y(x)| + \max_{a\leqslant c\leqslant b} |y'(x)|, \qquad \text{for all } y \in V. \tag{5.59}$$

Put $V_o = C^1([a,b],\mathbb{R})$ and define

$$\mathcal{A} = \{y \in V \mid y(a) = y_o \text{ and } y(b) = y_1\}, \tag{5.60}$$

for given real numbers y_o and y_1.

Let $F\colon [a,b]\times\mathbb{R}\times\mathbb{R}$ and $G\colon [a,b]\times\mathbb{R}\times\mathbb{R}$ be continuous functions with continuous partial derivatives

$$F_y(x,y,z),\ F_z(x,y,z),\ G_y(x,y,z) \text{ and } G_z(x,y,z), \tag{5.61}$$

for $(x,y,z) \in [a,b]\times\mathbb{R}\times\mathbb{R}$. Define functionals $J\colon V\to\mathbb{R}$ and $K\colon V\to\mathbb{R}$ by

$$J(y) = \int_a^b F(x,y(x),y'(x))\ dx, \qquad \text{for } y \in V, \tag{5.62}$$

and

$$K(y) = \int_a^b G(x,y(x),y'(x))\ dx, \qquad \text{for } y \in V, \tag{5.63}$$

respectively.

We consider the problem of finding an optimizer $y \in \mathcal{A}$, of J over the class $\mathcal{A}$, where $\mathcal{A}$ is given in (5.60) subject to the constraint

$$K(y) = c, \tag{5.64}$$

for some given constant c.

To apply the Euler–Lagrange multiplier theorem (Theorem 5.2.4 on page 76 in these notes), we need to consider the Gâteaux derivatives of the functionals J and K given in (5.62) and (5.63), respectively:

$$dJ(y,\eta) = \int_a^b [F_y(x,y,y')\eta + F_z(x,y,y')\eta']\, dx, \quad \text{for } y \in V \text{ and } \eta \in V_o, \tag{5.65}$$

and

$$dK(y,\eta) = \int_a^b [G_y(x,y,y')\eta + G_z(x,y,y')\eta']\, dx, \quad \text{for } y \in V \text{ and } \eta \in V_o, \tag{5.66}$$

where we have written y for $y(x)$, y' for $y'(x)$, η for $\eta(x)$, and η' for $\eta'(x)$ in the integrands in (5.65) and (5.66). The assumption that the partial derivatives in (5.61) are continuous for $(x,y,z) \in [a,b] \times \mathbb{R} \times \mathbb{R}$ will allow us to show that the Gâteaux derivatives of J and K in J in (5.65) and (5.66), respectively, are weakly continuous in V with respect to the norm $\|\cdot\|$ defined in (5.59). This fact is proved in Appendix C starting on page 159 of these notes. Thus, we can apply Theorem 5.2.4 to obtain that, if $(x,y) \in \mathcal{A}$, where $\mathcal{A}$ is given in (5.60), is an optimizer of J over $\mathcal{A}$ subject to the constraint in (5.64), then, either

$$\int_a^b [G_y(x,y,y')\eta + G_z(x,y,y')\eta']\, dx = 0, \quad \text{for all } \eta \in V_o, \tag{5.67}$$

or there exists a multiplier $\mu \in \mathbb{R}$ such that

$$\int_a^b [F_y(x,y,y')\eta + F_z(x,y,y')\eta']\, dx = \mu \int_a^b [G_y(x,y,y')\eta + G_z(x,y,y')\eta']\, dx,$$

for all $\eta \in V_o$, which we can rewrite as

$$\int_a^b [H_y(x,y,y')\eta + H_z(x,y,y')\eta']\, dx = 0, \quad \text{for all } \eta \in V_o, \tag{5.68}$$

where $H\colon [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$H(x,y,z) = F(x,y,z) - \mu G(x,y,z), \quad \text{for } (x,y,z) \in [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}. \tag{5.69}$$

Thus, the conditions in (5.67), and (5.68) and (5.69) are necessary conditions for $(x,y) \in \mathcal{A}$ being an optimizer of J over $\mathcal{A}$ subject to the constraint in (5.64). These conditions in turn yield the following Euler–Lagrange equations by virtue of the third fundamental lemma in the Calculus of Variations (Lemma 3.2.9): Either

$$\frac{d}{dx}[G_z(x,y,y')] = G_y(x,y,y'), \tag{5.70}$$

or there exists $\mu \in \mathbb{R}$ such that

$$\frac{d}{dx}[H_z(x,y,y')] = H_y(x,y,y'), \tag{5.71}$$

where H is given in (5.69).

In the following example we present an application of the equations in (5.70), (5.71) and (5.69).

Example 5.2.11. For given $b > 0$, put $V = C^1([0,b],\mathbb{R})$ and $V_o = C_o^1([0,b],\mathbb{R})$. Define functionals $J\colon V \to \mathbb{R}$ and $K\colon V \to \mathbb{R}$ by

$$J(y) = \int_0^b \sqrt{1+(y'(x))^2}\, dx, \quad \text{for all } y \in V, \tag{5.72}$$

and

$$K(y) = \int_0^b y(x)\, dx, \quad \text{for all } y \in V, \tag{5.73}$$

respectively. Let

$$\mathcal{A} = \{y \in V_o \mid y(x) \geqslant 0\}. \tag{5.74}$$

We consider the following constraint optimization problem:

Problem 5.2.12. Minimize $J(y)$ for $y \in \mathcal{A}$ subject to the constraint

$$K(y) = a, \tag{5.75}$$

for some $a > 0$.

The Gâteaux derivatives of the functionals J and K defined in (5.72) and (5.74), respectively, are

$$dJ(y;\eta) = \int_0^b \frac{y'(x)}{\sqrt{1+(y'(x))^2}}\, \eta'(x)\, dx, \quad \text{for } y \in V \text{ and } \eta \in V_o, \tag{5.76}$$

and

$$dK(y;\eta) = \int_a^b \eta(x)\, dx, \quad \text{for } y \in V \text{ and } \eta \in V_o, \tag{5.77}$$

respectively. In Example 5.2.8 we saw that the Gâteaux derivatives in (5.77) and (5.76) are weakly continuous. Thus, the Euler–Lagrange multiplier theorem applies. Hence, if $y \in \mathcal{A}$ is an optimizer for J over $\mathcal{A}$ subject to the constraint in (5.75), then either y must solve the Euler–Lagrange equation in (5.70), or there exists a multiplier $\mu \in \mathbb{R}$ such that

$$H(x,y,z) = F(x,y,z) - \mu F(x,y,z), \quad \text{for } x \in [0,b]\times\mathbb{R}\times\mathbb{R},$$

solves the Euler–Lagrange equation in (5.71), where

$$F(x,y,z) = \sqrt{1+z^2}, \quad \text{for } x \in [0,b]\times\mathbb{R}\times\mathbb{R},$$

and

$$G(x,y,z) = y, \quad \text{for } x \in [0,b]\times\mathbb{R}\times\mathbb{R}.$$

We then have that

$$F_y(x,y,z) = 0 \quad \text{and} \quad F_z(x,y,z) = \frac{z}{\sqrt{1+z^2}}, \quad \text{for } x \in [0,b]\times\mathbb{R}\times\mathbb{R},$$

and

$$G_y(x,y,z) = 1 \quad \text{and} \quad G_z(x,y,z) = 0, \quad \text{for } x \in [0,b] \times \mathbb{R} \times \mathbb{R}.$$

The differential equation in (5.70) then reads: $0 = 1$, which is impossible; hence, there must exist an Euler–Lagrange multiplier $\mu \in \mathbb{R}$ such that y solves the differential equation

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right] = \mu,$$

which yields

$$\frac{y'}{\sqrt{1+(y')^2}} = \mu x + c_1, \tag{5.78}$$

for some constant c_1.

We consider two cases in (5.78): either $\mu = 0$, or $\mu \neq 0$.

If $\mu = 0$ in (5.78), we obtain that

$$y' = c_2,$$

for some constant c_2, which yields the general solution

$$y(x) = c_2 x + c_3,$$

for another constant c_3. The assumption that $y \in V_o$ (see the definition of $\mathcal{A}$ in (5.74)) then implies that $y(x) = 0$ for all $x \in [0,b]$; however, since y also must satisfy the constraint in (5.75), we get from the definition of K in (5.73) that $a = 0$, which is impossible since we are assuming that $a > 0$. Hence, it must be the case that $\mu \neq 0$.

Thus, assume that $\mu \neq 0$ and solve the equation in (5.78) for y' to obtain the differential equation

$$\frac{dy}{dx} = \pm \frac{\mu x + c_1}{\sqrt{1-(\mu x + c_1)^2}}, \quad \text{for } 0 < x < b,$$

which can be integrated to yield

$$y = \mp \frac{1}{\mu}\sqrt{1-(\mu x + c_1)^2} + c_2, \quad \text{for } 0 \leqslant x \leqslant b, \tag{5.79}$$

for some constant of integration c_2.

It follows from the expression in (5.79) that

$$(\mu x + c_1)^2 + (\mu y - \mu c_2)^2 = 1,$$

which we can rewrite as

$$\left(x + \frac{c_1}{\mu}\right)^2 + (y - c_2)^2 = \frac{1}{\mu^2}. \tag{5.80}$$

Observe that the expression in (5.80) is the equation of a circle in the xy–plane of radius $1/\mu$ (here, we are taking $\mu > 0$) and center at

$$\left(-\frac{c_1}{\mu}, c_2\right).$$

The boundary conditions in the definition of $\mathcal{A}$ in (5.74) imply that

$$y(0) = 0 \quad \text{and} \quad y(b) = 0.$$

Using these conditions in (5.80) we obtain that

$$\frac{c_1}{\mu} = -\frac{b}{2}.$$

Thus, we can rewrite the equation in (5.80) as

$$\left(x - \frac{b}{2}\right)^2 + (y - c_2)^2 = \frac{1}{\mu^2}, \tag{5.81}$$

which is the equation of a circle in the xy–plane of radius $1/\mu$ and center at

$$\left(\frac{b}{2}, c_2\right).$$

Thus, according to (5.81), a solution $y \in \mathcal{A}$ of the constrained optimization Problem 5.2.12 has graph along an arc of the circle connecting the point $(0, 0)$ to the point $(b, 0)$. The value of c_2 can be determined by the condition in (5.75). We'll get back to the solution Problem 5.2.12 in the next section dealing with an isoperimetric problem

5.3 An Isoperimetric Problem

In this section we discuss an example in the same class of problems as Queen Dido's Problem presented in Example 5.2.8.

Problem 5.3.1 (An Isoperimetric Problem)**.** Out of all smooth, simple, closed curves in the plane of a fixed perimeter, ℓ, find one that encloses the largest possible area.

In what follows, we shall define several of the terms that appear in the statement of Problem 5.3.1.

Let $V = C^1([0, 1], \mathbb{R}^2)$; so that, the elements of V are vector–valued functions

$$(x, y) \colon [0, 1] \to \mathbb{R}^2,$$

whose values are denoted by $(x(t), y(t))$, for $t \in [0, 1]$, where the functions $x \colon [0, 1] \to \mathbb{R}$ and $y \colon [0, 1] \to \mathbb{R}$ are differentiable functions of $t \in (0, 1)$, with continuous derivatives $\dot{x}$ and $\dot{y}$ (the dot on top of a variable name indicates derivative with respect to t). Note that V is a linear space with sum and scalar multiplication defined by

$$(x(t), y(t)) + (u(t), v(t)) = (x(t) + u(t), y(t)) + v(t)), \quad \text{for all } t \in [0,1],$$

for $(x, y) \in V$ and $(u, v) \in V$, and

$$c(x(t), y(t)) = (cx(t), cy(t))), \quad \text{for all } t \in [0,1],$$

for $c \in \mathbb{R}$ and $(x, y) \in V$.

We can also endow V with a norm $\|(\cdot,\cdot)\|$ defined by

$$\|(x, y)\| = \max_{0 \leqslant t \leqslant 1} |x(t)| + \max_{0 \leqslant t \leqslant 1} |y(t)| + \max_{0 \leqslant t \leqslant 1} |\dot{x}(t)| + \max_{0 \leqslant t \leqslant 1} |\dot{y}(t)|, \tag{5.82}$$

for $(x, y) \in V$ (see Problem 2 in Assignment #7).

Notation 5.3.2. We will denote and element (x, y) in V by a single symbol $\sigma \in V$; so that,

$$\sigma(t) = (x(t), y(t)), \quad \text{for } t \in [0,1].$$

The derivative of $\sigma\colon [0,1] \to \mathbb{R}^2$ will be denoted by

$$\dot{\sigma}(t) = (\dot{x}(t), \dot{y}(t)), \quad \text{for } t \in (0,1),$$

and $\dot{\sigma}(t)$ is tangent to the curve traced by σ at the point $\sigma(t)$, provided that $\dot{\sigma}(t) \neq (0,0)$.

Definition 5.3.3 (Smooth, simple, closed curve)**.** A plane curve parametrized by a map $\sigma \in V$ is said to be a smooth, simple, closed curve if

$$\sigma(0) = \sigma(1);$$

the map $\sigma\colon [0,1) \to \mathbb{R}^2$ is one–to–one; and

$$\dot{\sigma}(t) \neq (0,0), \quad \text{for all } t \in [0,1]. \tag{5.83}$$

Remark 5.3.4. A continuous, simple, closed curve in the plane is also called a Jordan curve. The Jordan curve theorem states that any continuous, simple, closed curve in the plane separates the plane into two disjoint, connected, regions: a bounded region (we shall refer to this region as the region enclosed by the curve) and an unbounded region (the region outside the curve).

We denote by $\mathcal{A}$ the class of smooth, simple, closed curves in the plane. According the definition of a smooth, simple, closed curve in Definition 5.3.3, we can identify $\mathcal{A}$ with the class of functions $\sigma \in V$ such that $\sigma\colon [0,1] \to \mathbb{R}^2$ satisfies the conditions in Definition 5.3.3. We will also assume that the paths in $\mathcal{A}$ induce a counterclockwise (or positive) orientation on the curve that σ parametrizes. Thus, for each $\sigma = (x, y) \in \mathcal{A}$, we can compute the area of the region enclosed by σ by using the formula (B.35) derived in Appendix B.2 using the divergence theorem. Denoting the area enclosed by $(x, y) \in \mathcal{A}$ by $J((x, y))$ we have that

$$J((x,y)) = \frac{1}{2}\oint_{\partial\Omega} (xdy - ydx), \tag{5.84}$$

where Ω is the region enclosed by the path $(x, y) \in \mathcal{A}$. Expressing the line integral in (5.84) in terms of the parametrization $(x, y)\colon [0, 1] \to \mathbb{R}^2$ of $\partial\Omega$, we have that

$$J((x,y)) = \frac{1}{2}\int_0^1 (x(t)\dot{y}(t) - y(t)\dot{x}(t))\; dt, \quad \text{for } (x,y) \in \mathcal{A}. \tag{5.85}$$

We note that the functional $J\colon V \to \mathbb{R}$ defined in (5.85) defines a a functional on V that, when restricted to $\mathcal{A}$, gives the area of the region enclosed by $\sigma = (x, y) \in \mathcal{A}$. The arc–length of any curve $(x, y) \in V$ is given by

$$K((x,y)) = \int_0^1 \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\; dt, \quad \text{for } (x,y) \in V. \tag{5.86}$$

We can then restate Problem 5.3.1 as follows:

Problem 5.3.5 (An Isoperimetric Problem, Restated). Find $(x, y) \in \mathcal{A}$ such that

$$J((x,y)) = \max_{(u,v)\in\mathcal{A}} J((u,v)), \quad \text{subject to} \quad K((x,y)) = \ell. \tag{5.87}$$

Thus, the isoperimetric problem in Problem 5.3.1 is a constrained optimization problem. Thus, we may attempt to use the Euler–Lagrange multiplier theorem to obtain necessary conditions for a solution of the problem.

Observe that the functionals in (5.85) and (5.86) are of the form

$$J((x,y)) = \int_a^b F(t, x(t), y(t), \dot{x}(t), \dot{y}(t))\; dt, \quad \text{for } (x,y) \in C^1([a,b], \mathbb{R}^2), \tag{5.88}$$

and

$$K((x,y)) = \int_a^b G(t, x(t), y(t), \dot{x}(t), \dot{y}(t))\; dt, \quad \text{for } (x,y) \in C^1([a,b], \mathbb{R}^2), \tag{5.89}$$

where $F\colon [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ and $G\colon [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions of the variables $(t, x, y, p, q) \in [a, b] \times \mathbb{R}^4$, that have continuous partial derivatives

$$F_x(t,x,y,p,q),\; F_y(t,x,y,p,q),\; F_p(t,x,y,p,q),\; F_q(t,x,y,p,q),$$

and

$$G_x(t,x,y,p,q),\; G_y(t,x,y,p,q),\; G_p(t,x,y,p,q),\; G_q(t,x,y,p,q),$$

for $(t, x, y, p, q) \in [a, b] \times \mathbb{R}^4$ (with possible exceptions). Indeed, for the functionals in (5.85) and (5.86), $[a, b] = [0, 1]$,

$$F(t,x,y,p,q) = \frac{1}{2}xq - \frac{1}{2}yp, \quad \text{for } (t,x,y,p,q) \in [0,1] \times \mathbb{R}^4, \tag{5.90}$$

and

$$G(t,x,y,p,q) = \sqrt{p^2+q^2}, \quad \text{for } (t,x,y,p,q) \in [0,1]\times\mathbb{R}^4, \tag{5.91}$$

with $p^2+q^2 \neq 0$. We note that for the functions F and G defined in (5.90) and (5.91), respectively,

$$\begin{aligned} F_x(t,x,y,p,q) &= \frac{1}{2}q, \quad & F_y(t,x,y,p,q) &= -\frac{1}{2}p, \\ F_p(t,x,y,p,q) &= -\frac{1}{2}y, \quad & F_q(t,x,y,p,q) &= \frac{1}{2}x, \end{aligned} \tag{5.92}$$

which are continuous functions for $(t,x,y,p,q) \in [0,1]\times\mathbb{R}^4$, and

$$\begin{aligned} G_x(t,x,y,p,q) &= 0, & G_y(t,x,y,p,q) &= 0, \\ G_p(t,x,y,p,q) &= \frac{p}{\sqrt{p^2+q^2}}, \quad & G_q(t,x,y,p,q) &= \frac{q}{\sqrt{p^2+q^2}}, \end{aligned} \tag{5.93}$$

which are continuous as long as $p^2+q^2\neq 0$.

Using V to denote $C^1([a,b],\mathbb{R}^2)$, momentarily, V_o to denote $C^1_o([a,b],\mathbb{R}^2)$, and $\mathcal{A}$ to denote

$$\{(x,y)\in V \mid (x(a),y(a)) = (x_o,y_o) \text{ and } (x(b),y(b)) = (x_1,y_1)\},$$

for given $(x_o,y_o)\in\mathbb{R}^2$ and $(x_1,y_1)\in\mathbb{R}^2$, we can use the assumptions that the partial derivatives F_x, F_y, F_p, F_q, G_x, G_y, G_p and G_q are continuous to show the the functionals $J\colon V\to\mathbb{R}$ and $K\colon V\to\mathbb{R}$ defined in (5.88) and (5.89), respectively, are Gâteaux differentiable at $(x,y)\in V$ in the direction of $(\eta_1,\eta_2)\in V_o$, with Gâteaux derivatives given by

$$dJ((x,y);(\eta_1,\eta_2)) = \int_a^b [F_x\eta_1 + F_y\eta_2 + F_p\dot\eta_1 + F_q\dot\eta_2]\; dt, \tag{5.94}$$

for all $(x,y)\in V$ and $(\eta_1,\eta_2)\in V_o$, where we have written

$$F_x \text{ for } F_x(t,x(t),y(t),\dot x(t),\dot y(t)), \text{ for } t\in[a.b];$$

$$F_y \text{ for } F_y(t,x(t),y(t),\dot x(t),\dot y(t)), \text{ for } t\in[a.b];$$

$$F_p \text{ for } F_p(t,x(t),y(t),\dot x(t),\dot y(t)), \text{ for } t\in[a.b];$$

$$F_q \text{ for } F_q(t,x(t),y(t),\dot x(t),\dot y(t)), \text{ for } t\in[a.b];$$

and η_1, $\dot\eta_1$, η_2 and $\dot\eta_2$ for $\eta_1(t)$, $\dot\eta_1(t)$, $\eta_2(t)$ and $\dot\eta_2(t)$, for $t\in[a,b]$, respectively, (see Problem 3 in Assignment #7); similarly, we have that

$$dK((x,y);(\eta_1,\eta_2)) = \int_a^b [G_x\eta_1 + G_y\eta_2 + G_p\dot\eta_1 + G_q\dot\eta_2]\; dt, \tag{5.95}$$

The isoperimetric problem in Problem 5.3.5 is then a special case of the optimization problem:

$$\text{optimize } J((x,y)) \text{ over } \mathcal{A} \text{ subject to } K((x,y)) = c, \tag{5.96}$$

for some constant c, where J and K are given in (5.88) and (5.89), respectively. We can use the Euler–Lagrange multiplier theorem (Theorem 5.2.4 on page 76 in these notes) to obtain necessary conditions for the solvability of the variational problem in (5.96), provided we can show that the Gâteaux derivatives of J and K in (5.94) and (5.95), respectively, are weakly continuous; this can be shown using the arguments in Appendix C. We therefore obtain that, if $(x,y) \in \mathcal{A}$ is an optimizer of J over $\mathcal{A}$ subject to the constraint $K((x,y)) = c$, then either

$$\int_a^b [G_x\eta_1 + G_y\eta_2 + G_p\dot{\eta}_1 + G_q\dot{\eta}_2]\ dt = 0, \quad \text{for all } (\eta_1,\eta_2) \in V_o, \tag{5.97}$$

or there exists a multiplier $\mu \in \mathbb{R}$ such that

$$\int_a^b [F_x\eta_1 + F_y\eta_2 + F_p\dot{\eta}_1 + F_q\dot{\eta}_2]\ dt = \mu \int_a^b [G_x\eta_1 + G_y\eta_2 + G_p\dot{\eta}_1 + G_q\dot{\eta}_2]\ dt,$$

for all $(\eta_1,\eta_2) \in V_o$, or, setting

$$H(t,x,y,p,q) = F(t,x,y,p,q) - \mu G(t,x,y,p,q), \tag{5.98}$$

for $(t,x,y,p,q) \in [a,b] \times \mathbb{R}^4$,

$$\int_a^b [H_x\eta_1 + H_y\eta_2 + H_p\dot{\eta}_1 + H_q\dot{\eta}_2]\ dt = 0, \quad \text{for all } (\eta_1,\eta_2) \in V_o. \tag{5.99}$$

Now, taking $\eta_2(t) = 0$ for all $t \in [a,b]$ we obtain from (5.97) that

$$\int_a^b [G_x\eta_1 + G_p\dot{\eta}_1]\ dt = 0, \quad \text{for all } \eta_1 \in C_o^1[a,b]. \tag{5.100}$$

It follows from (5.100) and the third fundamental lemma in the Calculus of Variations (Lemma 3.2.9 on page 32 in these notes) that $G_p(t,x(t),y(t),\dot{x}(t),\dot{y}(t))$ is a differentiable function of t with derivative

$$\frac{d}{dt}[G_p(t,x,y,\dot{x},\dot{y})] = G_x(t,x,y,\dot{x},\dot{y}), \quad \text{for } t \in (a,b).$$

Similarly, taking $\eta_1(t) = 0$ for all $t \in [a,b]$ in (5.97) and applying the third fundamental lemma of the Calculus of Variations, we obtain the differential equation

$$\frac{d}{dt}[G_q(t,x,y,\dot{x},\dot{y})] = G_y(t,x,y,\dot{x},\dot{y}).$$

We have therefore shown that the condition in (5.97) implies that $(x, y) \in \mathcal{A}$ must solve the system of differential equations

$$\begin{cases} \dfrac{d}{dt}[G_p(t,x,y,\dot{x},\dot{y})] &= G_x(t,x,y,\dot{x},\dot{y}); \\ \\ \dfrac{d}{dt}[G_q(t,x,y,\dot{x},\dot{y})] &= G_y(t,x,y,\dot{x},\dot{y}). \end{cases} \tag{5.101}$$

Similarly, we obtain from (5.100) the system of differential equations

$$\begin{cases} \dfrac{d}{dt}[H_p(t,x,y,\dot{x},\dot{y})] &= H_x(t,x,y,\dot{x},\dot{y}); \\ \\ \dfrac{d}{dt}[H_q(t,x,y,\dot{x},\dot{y})] &= H_y(t,x,y,\dot{x},\dot{y}), \end{cases} \tag{5.102}$$

where $H = F - \mu G$ is given in (5.98).

Hence, if (x, y) is a solution of the constrained optimization problem in (5.96), where $J \colon V \to \mathbb{R}$ and $K \colon V \to \mathbb{R}$ are as given in (5.88) and (5.89), respectively, then, either (x, y) solves the system of Euler–Lagrange equations in (5.101), or there exists an Euler–Lagrange multiplier $\mu \in \mathbb{R}$ such that (x, y) solves the system of Euler–Lagrange equations in (5.102), where H is as given in (5.98). We next apply this reasoning to the isoperimetric problem stated in Problem 5.3.5.

In the case of the constrained optimization problem in Problem 5.3.5, F and G are given by (5.90) and (5.91), respectively, and their partial derivatives are given in (5.92) and (5.93), respectively. Thus, if (x, y) is a simple, closed curve of arc–length ℓ that encloses the largest possible area, then either (x, y) solves the system

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) &= 0; \\ \\ \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) &= 0, \end{cases} \tag{5.103}$$

or there exists a multiplier, μ, such that (x, y) solves the system of differential equations

$$\begin{cases} \dfrac{d}{dt}\left(-\dfrac{1}{2}y - \mu\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) &= \dfrac{1}{2}\dot{y}; \\ \\ \dfrac{d}{dt}\left(\dfrac{1}{2}x - \mu\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) &= -\dfrac{1}{2}\dot{x}. \end{cases} \tag{5.104}$$

Let $(x, y) \in \mathcal{A}$, where $\mathcal{A}$ is the class of smooth, simple, closed curves in the plane, be a solution of the isoperimetric problem in Problem 5.87. suppose also that (x, y) solves the system of Euler–Lagrange equations in (5.103). We then have that

$$\begin{cases} \dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} &= c_1; \\[2ex] \dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} &= d_1, \end{cases} \tag{5.105}$$

for constants c_1 and d_1. Note that, in view of (5.83) in Definition 5.3.3, which is part of the definition of $\mathcal{A}$, the denominators in the expressions in (5.105) are not zero. Also, it follows from (5.105) that

$$c_1^2 + d_1^2 = 1;$$

consequently, c_1 and d_1 cannot both be zero.

Assume that $c_1 = 0$. In this case, it follows from the first equation in (5.105) that $\dot{x} = 0$; from which we get that $x(t) = c_2$ for all $t \in [0, 1]$, which implies that the simple, closed curve $(x(t), y(t)$, for $t \in [0, 1]$, lies on the line $x = c_2$; this is impossible.

Suppose next that $c_1 \neq 0$ in (5.105) and divide the first expression in (5.105) into the second to obtain

$$\frac{\dot{y}}{\dot{x}} = \frac{d_1}{c_1};$$

so that, by virtue of the Chain Rule,

$$\frac{dy}{dx} = c_3,$$

for some constant c_3. We therefore get that

$$y = c_3 x + c_4,$$

for constants c_3 and c_4; so that, the simple closed $(x(t), y(t))$, for $t \in [0, 1]$, again lies on a straight line, which is impossible.

Thus, the second alternative in the Euler–Lagrange multiplier theorem must hold true for a solution $(x, y) \in \mathcal{A}$ of the constrained optimization problem in (5.87). Therefore, there exists a multiplier $\mu \in \mathbb{R}$ for which the system of Euler–Lagrange equations in (5.104) holds true.

Rewrite the equations in (5.104) as

$$\begin{cases} -\dfrac{1}{2}\dot{y} - \mu \dfrac{d}{dt}\left(\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) &= \dfrac{1}{2}\dot{y}; \\[2ex] \dfrac{1}{2}\dot{x} - \mu \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) &= -\dfrac{1}{2}\dot{x}, \end{cases}$$

which can in turn be written as

$$\begin{cases} \dot{y} + \mu \dfrac{d}{dt}\left(\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) & = & 0; \\[2ex] \dot{x} - \mu \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) & = & 0, \end{cases}$$

or

$$\begin{cases} \dfrac{d}{dt}\left(y + \mu \dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) & = & 0; \\[2ex] \dfrac{d}{dt}\left(x - \mu \dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) & = & 0. \end{cases} \tag{5.106}$$

The equations in (5.106) can be integrated to yield

$$\begin{cases} y + \mu \dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} & = & c_2; \\[2ex] x - \mu \dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} & = & c_1, \end{cases}$$

for constants c_1 and c_2, from which we get that

$$\begin{cases} y - c_2 & = & -\mu \dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}; \\[2ex] x - c_1 & = & \mu \dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}. \end{cases} \tag{5.107}$$

It follows from the equations in (5.107) that

$$(x - c_1)^2 + (y - c_2)^2 = \mu^2, \tag{5.108}$$

which is the equation of a circle of radius μ (we are taking $\mu > 0$) and center at (c_1, c_2).

We have therefore shown that if $(x, y) \in \mathcal{A}$ is a solution of the isoperimetric problem in (5.87), then the simple, closed curve $(x(t), y(t))$, for $t \in [0, 1]$, must lie on some circle of radius μ (see equation (5.108)). Since $K((x, y)) = \ell$, it follows that

$$2\pi\mu = \ell,$$

from which we get that

$$\mu = \frac{\ell}{2\pi}. \tag{5.109}$$

Remark 5.3.6. We have not proved yet that a circle of radius μ given in (5.109) is the smooth, simple, closed curve of arc–length ℓ that encloses the largest possible area. What we proved is that, if such a curve exists, then it must be a circle of radius μ given in (5.109). We will prove the existence of an area maximizing closed curve of a fixed perimeter in the next section in these notes.

5.4 The Isoperimetric Theorem

Let C denote any simple, closed curve in the xy–plane of perimeter ℓ. Assume the curve C encloses a region, R, in the plane of area A. The Isoperimetric Theorem states that the inequality

$$4\pi A \leqslant \ell^2 \tag{5.110}$$

must hold true. Furthermore, equality in (5.110) holds true if and only if C is a circle.

The goal of this section is to give proof of this result, known as the **isoperimetric theorem**, for the case in which C is assumed to be smooth. We will also derive some consequences of the isoperimetric theorem. In particular, we will complete the solution of the isoperimetric problem discussed in the previous section. We will also provide a complete solution of Queen Dido's problem discussed in Section 5.1 and in Example 5.1.1. Recall that, in those examples, we assumed that an area maximizing curve of a fixed perimeter existed, and then derived consequences of that assumption to show that the curve had to be a circle.

The inequality in (5.110) is known in the literature as the **isoperimetric inequality**. Thus, we will derive the isoperimetric inequality for any smooth, simple, closed curve in the plane; we will also prove that equality occurs when the curve is a circle.

5.4.1 Proof of the Isoperimetric Theorem

We will state the two–dimensional isoperimetric theorem in the form in which it will be proved in this section. The statement is actually more general; in particular, the curve does not need to be smooth; it can be piecewise smooth, or continuous and rectifiable (see [Cou62, pp. 113-114] for a proof of this fact).

Theorem 5.4.1 (Isoperimetric theorem)**.** Let C be a smooth, simple, closed curve in the xy–plane of perimeter ℓ. Assume the curve C encloses a region, R, in the xy–plane of area A. Then,

$$4\pi A \leqslant \ell^2. \tag{5.111}$$

Equality in (5.111) holds true if and only if C is a circle.

Proofs of the isoperimetric inequality in (5.111) may be found in several places in the literature. We give here a proof presented in [HLP34] based on Wirtinger's inequality:

Theorem 5.4.2 (Wirtinger's inequality)**.** Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a 2π–periodic function with mean 0; so that,

$$f(t + 2\pi) = f(t), \quad \text{for all } t \in \mathbb{R},$$

and

$$\int_0^{2\pi} f(t)\ dt = 0.$$

Then,

$$\int_0^{2\pi} (f(t))^2\ dt \leqslant \int_0^{2\pi} (f'(t))^2\ dt. \tag{5.112}$$

Equality in (5.112) occurs if and only if

$$f(t) = a\cos t + b\sin t, \quad \text{for all } t \in \mathbb{R}, \tag{5.113}$$

for some real constants a and b.

A proof of Wirtinger's inequality may also be found in [HLP34]; we have also presented one in Appendix A.3 in these notes.

We note that the expression in (5.113) can also be written as

$$f(t) = r\cos(t - t_o), \quad \text{for all } t \in \mathbb{R}, \tag{5.114}$$

for some real constants r and t_o.

Proof of Theorem 5.4.1: We assume that C is parametrized by a smooth map $(u, v)\colon [0, \ell] \to \mathbb{R}^2$, where s denotes the arc–length along the curve measured from a point (u_o, v_o) on the curve, and C is traversed in the counterclockwise sense. We then have that

$$(u(0), v(0)) = (u(\ell), v(\ell)) \tag{5.115}$$

and

$$(u'(s))^2 + (u'(s))^2 = 1, \quad \text{for } s \in [0, \ell]. \tag{5.116}$$

Making the change of variables

$$t = \frac{2\pi}{\ell}s, \quad \text{for } s \in [0, \ell],$$

we obtain a reparametrization of the map (u, v) to yield the path

$$(x, y)\colon [0, 2\pi] \to \mathbb{R}^2$$

given by

$$(x(t), y(t)) = \left(u\left(\frac{\ell}{2\pi}t\right), v\left(\frac{\ell}{2\pi}t\right)\right), \quad \text{for } 0 \leqslant t \leqslant 2\pi. \tag{5.117}$$

It follow from (5.115) and (5.117) that

$$(x(0), y(0)) = (x(2\pi), y(2\pi)).$$

We can therefore extend x and y to 2π–periodic functions over all of $\mathbb{R}$.

Furthermore, by taking

$$x(t) - \frac{1}{2\pi}\int_0^{2\pi} x(t)\ dt,$$

instead of $x(t)$, if necessary, we may assume that

$$\int_0^{2\pi} x(t)\ dt = 0; \tag{5.118}$$

i.e., we may assume that x has zero mean over $[0, 2\pi]$. This amounts to translating the curve C appropriately.

We also have from (5.117) and (5.116) that

$$(x'(t))^2 + (y'(t))^2 = \frac{\ell^2}{4\pi^2}, \qquad \text{for } t \in [0, 2\pi],$$

where we have used the Chain Rule. Consequently,

$$\int_0^{2\pi} \left[(x'(t))^2 + (y'(t))^2\right]\ dt = \frac{\ell^2}{2\pi},$$

from which we get that

$$\ell^2 = 2\pi \int_0^{2\pi} \left[(x'(t))^2 + (y'(t))^2\right]\ dt. \tag{5.119}$$

On the other hand, letting A denote the area of the region enclosed by the curve, an application of the divergence theorem in two dimensions (see Appendix B.2) yields

$$A = \int_0^{2\pi} x(t)y'(t)\ dt,$$

from which we get that

$$4\pi A = 2\pi \int_0^{2\pi} 2x(t)y'(t)\ dt. \tag{5.120}$$

Combining (5.119) and (5.120) we get

$$\ell^2 - 4\pi A = 2\pi \int_0^{2\pi} \left[(x'(t))^2 + (y'(t))^2 - 2x(t)y'(t)\right]\ dt; \tag{5.121}$$

so that, completing the square in the last two terms of the integrand in (5.121),

$$\ell^2 - 4\pi A = 2\pi \int_0^{2\pi} \left[(x'(t))^2 - (x(t))^2 + (y'(t) - x(t))^2\right]\ dt,$$

which we can rewrite as

$$\ell^2 - 4\pi A = 2\pi\left(\int_0^{2\pi} (x'(t))^2\ dt - \int_0^{2\pi} (x(t))^2\ dt\right) + 2\pi\int_0^{2\pi} (y'(t) - x(t))^2\ dt, \tag{5.122}$$

Now, since $x\colon \mathbb{R} \to \mathbb{R}$ is 2π–periodic with mean 0, it follows from Wirtinger's Inequality that

$$\int_0^{2\pi} (x(t))^2\ dt \leqslant \int_0^{2\pi} (x'(t))^2\ dt, \tag{5.123}$$

with equality if and only if

$$x(t) = r\cos(t - t_o), \qquad \text{for all } t \in \mathbb{R}, \tag{5.124}$$

for some real constants r and t_o, according to (5.114). It then follows from (5.122) and (5.123) that

$$\ell^2 - 4\pi A \geqslant 0,$$

from which the isoperimetric inequality in (5.111) follows.

It follows from (5.122) that, if equality in (5.111) holds true, then

$$\left(\int_0^{2\pi} (x'(t))^2\ dt - \int_0^{2\pi} (x(t))^2\ dt\right) + \int_0^{2\pi} (y'(t) - x(t))^2\ dt = 0,$$

from which we get that

$$\int_0^{2\pi} (y'(t) - x(t))^2\ dt = 0, \tag{5.125}$$

and

$$\int_0^{2\pi} (x'(t))^2\ dt - \int_0^{2\pi} (x(t))^2\ dt = 0. \tag{5.126}$$

The Basic Lemma 1 on page 29 in these notes can now be applied to (5.125) to yield the differential equation

$$y'(t) = x(t), \qquad \text{for all } t \in (0, 2\pi). \tag{5.127}$$

The expression in (5.126) implies that equality in (5.123) occurs; hence, (5.124) must hold true; so that,

$$x(t) = r\cos(t - t_o), \qquad \text{for all } t \in \mathbb{R}, \tag{5.128}$$

for some real constants r and t_o.

It follows from (5.127) and (5.128) that

$$y(t) = r\sin(t - t_o) + c_2, \qquad \text{for all } t \in \mathbb{R}, \tag{5.129}$$

for some real constant c_2.

The expressions in (5.128) and (5.129) are the parametric equations of a circle of radius r centered at $(0, c_2)$ in the xy–plane. We have therefore shown that, if equality holds true in (5.111), then C must be a circle of radius

$$r = \frac{\ell}{2\pi}. \tag{5.130}$$

Conversely, if C is a circle of perimeter ℓ and area A, then its radius is given by (5.130) and its area by

$$A = \pi \left(\frac{\ell}{2\pi}\right)^2,$$

or

$$A = \frac{\ell^2}{4\pi};$$

so that, equality in (5.110) holds true. ■

5.4.2 Some Consequences of the Isoperimetric Theorem

In the previous section we proved the isoperimetric theorem for the case in which the simple, closed curve C was assumed to be smooth. The theorem is true for piece–wise smooth, simple, closed curves and, more generally, for continuous, rectifiable, simple, closed curves of perimeter ℓ enclosing a region of area A. In this section, we assume the general version of the isoperimetric theorem to be true, and derive a few of its consequences.

Example 5.4.3 (The Isoperimetric Problem, Revisited)**.** In Example 5.3.1 we showed that if there exists a smooth, simple, closed curve, C, of perimeter ℓ in the xy–plane for which the area of the enclosed region is the largest possible among all smooth, simple, closed curves of perimeter ℓ, then it had to be a circle. In this example, we use the isoperimetric theorem (Theorem 5.4.1) to prove that

(i) the set of areas of regions enclosed by simple, closed curves in the plane of perimeter ℓ is bounded above by

$$\frac{\ell^2}{4\pi}; \tag{5.131}$$

(ii) the upper bound in (5.131) is attained when the curve is a circle of radius given in (5.130).

To prove the assertion in (i), let C be any simple, closed curve in the xy–plane of perimeter ℓ. Let a denote the area of the region enclosed by C. Then, by the isoperimetric inequality in (5.111),

$$4\pi a \leqslant \ell^2, \tag{5.132}$$

from which we get that

$$a \leqslant \frac{\ell^2}{4\pi}, \tag{5.133}$$

which proves the assertion in (i).

To prove the assertion in (ii), note that equality in (5.133) implies equality in the isoperimetric inequality in (5.132). Consequently, by the isoperimetric theorem, the curve must be a circle.

Example 5.4.4 (Queen Dido's Problem, Revisited One More Time)**.** In Section 5.1 we introduced Queen Dido's problem: Find $y \in C^1_o([0,b],\mathbb{R})$ such that $y(x) \geqslant 0$ for all $x \in [0,b]$, the arclength along the graph of $y = y(x)$, for $0 \leqslant x \leqslant b$, is ℓ, the area under the graph of $y = y(x)$, for $0 \leqslant x \leqslant b$, is the largest possible; see a sketch of $y = y(x)$ in Figure 5.4.3. In this example we use the isoperimetric

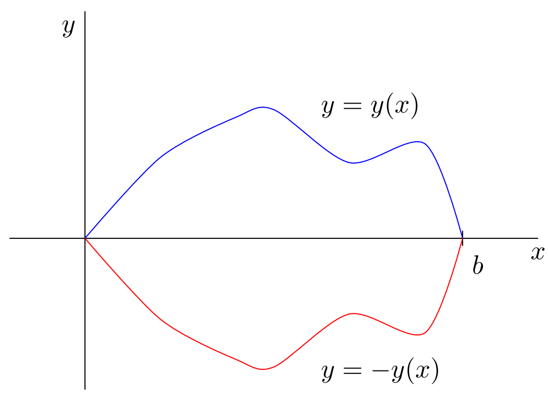

Figure 5.4.3: Queen Dido's Problem

theorem (see Theorem 5.4.1 on page 93 in these notes) to solve this problem.

Figure 5.4.3 also shows the reflection of the graph of $y = y(x)$ on the x–axis; namely, the graph of $y = -y(x)$. The union of the graphs of $y = y(x)$ and $y = -y(x)$ forms a simple, closed curve in the xy–plane that is piece–wise C^1. The arclength of the closed curve is 2ℓ. We can therefore apply the isoperimetric theorem to obtain

$$4\pi a \leqslant (2\ell)^2,$$

or

$$a \leqslant \frac{1}{\pi}\ell^2, \tag{5.134}$$

where a denotes the area enclosed by the curve.

Equality in (5.134) holds true if and only if the curve is a circle. This is shown in Figure 5.4.4. The figure shows a circle with a diameter along the x–axis and endpoints at $(0,0)$ and $(b,0)$. This is in accordance with the boundary conditions imposed on y by the requirement that $y \in C^1_o([0,b],\mathbb{R})$.

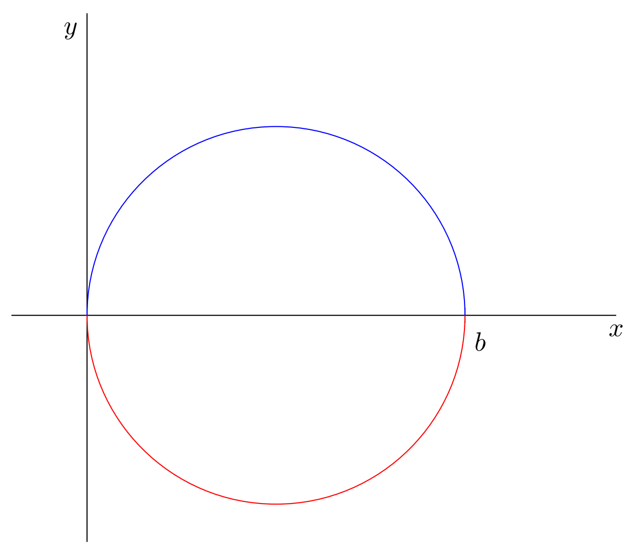

Figure 5.4.4: Solution of Queen Dido's Problem

Chapter 6

The Variational Approach

In the previous sections in these notes, we have focussed mostly on indirect methods in the Calculus of Variations: Given a functional $J: V \to \mathbb{R}$ defined on a normed linear space V, we assume that $u \in \mathcal{A}$, where $\mathcal{A}$ is a class of admissible vectors in V, is either a local minimizer or a local maximizer of J in $\mathcal{A}$. We then proceeded to find conditions that u must satisfy. It the case in which the functional J is Gâteaux differentiable at u, one of those conditions translates into

$$dJ(u; v) = 0, \quad \text{for all } v \in V_o, \tag{6.1}$$

where V_o is a nontrivial subspace of V with the property that $u + tv \in \mathcal{A}$, for $v \in V_o$, and $t \in \mathbb{R}$ with $|t|$ sufficiently small. In the indirect methods, we look for solutions of the equation in (6.1), which in some cases leads to the Euler–Lagrange equations, to obtain candidates for a local minimizer or maximizer of J. Further study of the functional will allow as to conclude that solutions of (6.1) are optimizers of the functional.

In this chapter, we reverse the procedure outlined above. The focus will be on the equation in (6.1). This equation might take the form of a boundary value problem for a differential equation, or a system of differential equations. We are interested in proving that the boundary value problem has solutions. Solutions of the problem in (6.1) correspond to optimizers of J over some class $\mathcal{A}$. Thus, to prove that the equation in (6.1) has solutions, we will show that the functional J has optimizers. In some cases, the existence of the optimizers can be shown directly. We will illustrate this procedure, which we shall refer to as the variational approach, in the next section involving a finite dimensional example. In subsequent sections we illustrate the vcariational approach in the context of the Dirichlet Principle.

6.1 A Finite Dimensional Example

Let $V = \mathbb{R}^n$, n–dimensional Euclidean space, and A denote an $n \times n$ symmetric matrix with real entries. We consider the problem of solving the linear equation

$$Au = b, \tag{6.2}$$

for a given vector b in V.

The variational approach for solving the problem in (6.2) consists of finding a functional $J \colon V \to \mathbb{R}$ that is Gâteaux differentiable and for which solutions of the problem

$$dJ(u; v) = 0, \quad \text{for all } v \in V, \tag{6.3}$$

yield solutions of the problem in (6.2). Solutions of the problem in (6.3) are called **critical points** of J.

Definition 6.1.1 (Critical Points)**.** Let V denote any vector space and $J \colon V \to \mathbb{R}$ a functional that is Gâteaux differentiable for every u in V in the direction of any v in V. A vector u_o in V is said to be a critical point of J if and only if

$$dJ(u_o; v) = 0, \quad \text{for all } v \in V.$$

For the problem in (6.2), where A is an $n \times n$ symmetric matrix with real entries, a functional $J \colon V \to \mathbb{R}$ whose critical points yield solutions of (6.2), is given by

$$J(u) = \frac{1}{2}\langle Au, u\rangle - \langle b, u\rangle, \quad \text{for all } u \in V, \tag{6.4}$$

where $\langle v, w\rangle$ denotes the Euclidean inner product, or dot–product, of vectors v and w in $\mathbb{R}^n$. Indeed, since A is symmetric, it can be shown that the functional $J \colon V \to \mathbb{R}$ defined in (6.4) is Gâteaux differentiable with derivative given by

$$dJ(u; v) = \langle Au, v\rangle - \langle b, v\rangle, \quad \text{for all } u, v \in V. \tag{6.5}$$

Suppose that $u_o \in V$ is a critical point of the functional J given in (6.4). Then, according to (6.5) and Definition 6.1.1,

$$\langle Au_o, v\rangle - \langle b, v\rangle, \quad \text{for all } v \in V,$$

or

$$\langle Au_o - b, v\rangle = 0, \quad \text{for all } v \in V. \tag{6.6}$$

Next, take $v = Au_o - b$ in (6.6) to get

$$\langle Au_o - b, Au_o - b\rangle = 0,$$

or

$$\|Au_o - b\|^2 = 0,$$

from which we get that

$$Au_o = b.$$

We have therefore shown that, if $u_o \in V$ is a critical point of $J\colon V \to \mathbb{R}$ given in (6.4), then u_o is a solution of the equation in (6.2). Thus, to find solutions of the equation in (6.2), we can look for critical points of the functional J defined (6.4). This is the essence of the variational approach for the equation in (6.2).

Critical points of a Gâtaeux differentiable functional $J\colon V \to \mathbb{R}$ come in several flavors. Local, or global, minima or maxima, and saddle points yield critical points. For instance, if the matrix A in (6.2) is assumed to be positive definite, in addition to being symmetric, the functional J given in (6.4) can be shown to be strictly convex. In this case, the functional J has a unique global minimizer; this follows from the convex minimization theorem proved in Section 4.4 in these notes. Thus, if the matrix A in (6.2) is symmetric and positive definite, then the equation in (6.2) has a unique solution that can be obtained as the minimizer of the functional J given in (6.4).

6.2 The Dirichlet Principle

Let U denote an open subset of $\mathbb{R}^2$ and Ω an open, bounded subset of U such that $\overline{\Omega} \subset U$. Let $g \in C(U,\mathbb{R})$ be given. We assume that $\partial\Omega$ is a piece–wise smooth, simple, closed curve, with Ω as the region enclosed by the curve. In Section 2.2 we encountered the following boundary value problem

$$\left\{\begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega; \\ u & = & g & \text{on } \partial\Omega, \end{array}\right. \tag{6.7}$$

where

$$\Delta u = u_{xx} + u_{yy}$$

is the two–dimensional Laplacian of u (we are assuming that $u \in C^2(U,\mathbb{R})$). The boundary value problem in (2.34) is called the Dirichlet problem for Laplace's equation over Ω with boundary data g. A function $u \in C^2(\Omega,\mathbb{R})$ satisfying the differential equation in (6.7); namely,

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad \text{for } (x,y) \in \Omega, . \tag{6.8}$$

is said to be **harmonic** in Ω.

We would like to prove that the boundary value problem in (6.7) has a solution; that is, there exists a harmonic function in Ω that takes on values given by g on the boundary of Ω; that is,

$$u(x,y) = g(x,y), \quad \text{for all } (x,y) \in \partial\Omega.$$

In Section 2.2 we also saw that there is a functional associated with the boundary value problem in (6.7) in the sense that the differential equation in (6.7) is the Euler–Lagrange equation corresponding to the functional, and the boundary condition in (6.7) determines the class of admissible functions over which J is being optimized.

Indeed, let $V = C^1(\overline{\Omega}, \mathbb{R})$ and define $J\colon V \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\iint_\Omega |\nabla u|^2 \, dxdy, \qquad \text{for all } u \in C^1(\overline{\Omega}, \mathbb{R}). \tag{6.9}$$

We showed in Example 4.1.2 that J is Gâteaux differentiable at every $u \in V$ in the direction of every $v \in V_o$, where $V_o = C^1_o(\overline{\Omega}, \mathbb{R})$, with Gâteaux derivative given by

$$dJ(u; v) = \iint_\Omega \nabla u \cdot \nabla v \, dxdy, \qquad \text{for } u \in V \text{ and } v \in V_o. \tag{6.10}$$

Define the class of admissible functions

$$\mathcal{A} = \{u \in V \mid u = g \text{ on } \partial\Omega\}, \tag{6.11}$$

and observe that, since $J\colon V \to \mathbb{R}$ defined in (6.9) is nonnegative, J is bounded from below over $\mathcal{A}$; thus,

$$\inf_{v \in \mathcal{A}} J(v) \tag{6.12}$$

exists. The Dirichlet principle asserts that the infimum of J over $\mathcal{A}$ in (6.12) is attained by some $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ such that $u \in \mathcal{A}$; so that,

$$J(u) = \inf_{v \in \mathcal{A}} J(v).$$

This proves the existence of a solution of the Dirichlet problem in (6.7).

The Dirichlet principle was assumed to be true by Riemann in his work on complex functions in the 1850s, and he used it to prove the existence of harmonic functions taking on prescribed values on the boundary of domains in the complex plane. Indeed, it was Riemann who coined the term "Dirichlet principle" in reference to the fact that he learned it from lectures given by Dirichlet at the University of Gottingen, Germany (see [Cou50]). However, in 1869, Weierstrass pointed our that there was a flaw in the principle stemming from the fact that, in general, the infimum in (6.12) might not be attained by a function in the class $C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R}) \cap \mathcal{A}$. In fact, it is possible to come up with examples of functionals that are bounded below in some class of admissible functions, $\mathcal{A}$, but the infimum is not attained in that class (see [Cou50]).

In 1900, the German mathematician David Hilbert was able to justify Riemann's use of the Dirichlet principle in some cases by proving directly that the Dirichlet integral in (6.9) actually attains its minimum. We will hint as to how this can be done in the following section in the case of Poisson's equation.

6.3 A Minimization Problem: Direct Approach

As in the previous section, let U denote an open subset of $\mathbb{R}^2$ and Ω an open, bounded subset of U such that $\overline{\Omega} \subset U$. We assume that $\partial\Omega$ is a piece–wise smooth, simple, closed curve, with Ω as the region enclosed by the curve.

Let $f\colon \Omega \to \mathbb{R}$ be a square integrable function in Ω; i.e.,

$$\iint_{\Omega} (f(x,y))^2 \; dxdy < \infty. \tag{6.13}$$

Remark 6.3.1. At this point in these notes we understand the integral in (6.13) to be the Riemann integral of f^2 over Ω. In subsequent sections, we will need to view the integral in (6.13) as the Lebesgue integral.

We consider the boundary value problem

$$\begin{cases} -\Delta u(x,y) &= f(x,y), \quad \text{for } (x,y) \in \Omega; \\ \quad u(x,y) &= 0, \quad\quad\quad \text{for all } (x,y) \in \partial\Omega. \end{cases} \tag{6.14}$$

The problem in (6.14) is called the Dirichlet problem for Poisson's equation.

We will denote the space of real–valued, square integrable functions in Ω by $L^2(\Omega)$; thus,

$$L^2(\Omega) = \left\{ f\colon \Omega \to \mathbb{R} \;\Big|\; \iint_{\Omega} (f(x,y))^2 \; dxdy < \infty \right\}.$$

We can make $L^2(\Omega)$ into a normed, linear space with norm $\|\cdot\|_{L^2(\Omega)}\colon L^2(\Omega) \to \mathbb{R}$ given by

$$\|f\|_{L^2(\Omega)} = \sqrt{\iint_{\Omega} (f(x,y))^2 \; dxdy}, \qquad \text{for all } f \in L^2(\Omega).$$

We note that, if $f \in C(\overline{\Omega}, \mathbb{R})$, then $f \in L^2(\Omega)$. This is true because we are assuming that Ω is bounded; therefore, $\overline{\Omega}$ is compact. Thus, there exists $M > 0$ such that

$$|f(x,y)| \leqslant M, \qquad \text{for all } (x,y) \in \overline{\Omega},$$

by virtue of the assumption that f is continuous on $\overline{\Omega}$. Consequently,

$$\iint_{\Omega} (f(x,y))^2 \; dxdy \leqslant \iint_{\Omega} M^2 \; dxdy = M^2 \text{area}(\Omega) < \infty.$$

The problem we would like to discuss in the next couple of sections is the following: Given $f \in C(\overline{\Omega}, \mathbb{R})$, is there a function $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ satisfying the differential equation and boundary condition in (6.14)? To answer this question, we will use the variational approach in the sense that we will realize a solution of the boundary value problem in (6.14) as a minimizer of a functional $J\colon V_o \to \mathbb{R}$ is some vector space V_o to be defined shortly. We will then prove directly that the functional attains a minimum at a function u in some class of functions to be defined later in this section.

The vector space V_o will be a subspace of $C^1(U, \mathbb{R})$ consisting of functions $u\colon U \to \mathbb{R}$ with continuous partial derivatives whose support is contained in Ω. We will therefore begin with the definition of the support of a function.

Definition 6.3.2 (Support)**.** For a function $u\colon U \to \mathbb{R}$, the support of u, denoted Support(u), is the closure of the set where u is not zero; that is,

$$\text{Support}(u) = \overline{\{(x,y) \in U \mid u(x,y) \neq 0\}}.$$

Thus, the support of a function is closed, by definition. If Support(u) is also bounded, then Support(u) is compact. In particular, if Support$(u) \subset \Omega$, where Ω is bounded, then Support(u) is compact.

We define V_o to be the space of functions $u \in C^1(U,\mathbb{R})$ that have (compact) support contained in Ω; i.e.,

$$V_o = \{u \in C^1(U,\mathbb{R}) \mid \text{Support}(u) \subset \Omega\}. \tag{6.15}$$

The space V_o in (6.15) is also denoted by $C_c^1(\Omega)$, where the subscript c indicates that the functions in $C_c^1(\Omega)$ have compact support contained in Ω.

In the space V_o in (6.15) we have the following important inequality:

Proposition 6.3.3 (Poincaré Inequality)**.** Let Ω be an open, bounded subset of $\mathbb{R}^2$. There exists a constant, $c(\Omega) > 0$, depending only on Ω, such that

$$\iint_\Omega (u(x,y))^2\ dxdy \leqslant c(\Omega) \iint_\Omega |\nabla u(x,y)|^2\ dxdy, \quad \text{for all } u \in C_c^1(\Omega). \tag{6.16}$$

For a derivation of the Poincaré inequality, see Appendix A.2.

We will define a norm in V_o as follows:

$$\|u\| = \sqrt{\iint_\Omega |\nabla u(x,y)|^2\ dxdy}, \quad \text{for all } u \in V_o. \tag{6.17}$$

We can use the Poincaré inequality in (6.16) to verify that the function $\|\cdot\|\colon V_o \to \mathbb{R}$ defined in (6.17) is positive definite. Indeed, if $\|u\| = 0$, we have that

$$\iint_\Omega |\nabla u(x,y)|^2\ dxdy = 0.$$

It then follows from (6.16) that

$$\iint_\Omega (u(x,y))^2\ dxdy = 0,$$

from which we get that $u(x,y) = 0$, for all $(x,y) \in \Omega$, by the basic lemma I.

To establish the triangle inequality for $\|\cdot\|$ defined in (6.17), compute, for

$u, v \in V_o$,

$$\begin{aligned}
\|u+v\|^2 &= \iint_\Omega |\nabla(u+v)|^2 \; dxdy \\
&= \iint_\Omega |\nabla u + \nabla v|^2 \; dxdy \\
&= \iint_\Omega (\nabla u + \nabla v)\cdot(\nabla u + \nabla v) \; dxdy \\
&= \iint_\Omega (\nabla u \cdot \nabla u + 2\nabla u \cdot \nabla v + \nabla v \cdot \nabla v) \; dxdy;
\end{aligned}$$

so that

$$\|u+v\|^2 = \|u\|^2 + 2\iint_\Omega \nabla u \cdot \nabla v \; dxdy + \|v\|^2. \qquad (6.18)$$

Next, use the Cauchy–Schwarz inequality for the Euclidean inner product, and the Cauchy–Schwarz inequality for functions in $L^2(\Omega)$ (see Appendix A.1), to get that

$$\begin{aligned}
\left|\iint_\Omega \nabla u \cdot \nabla v \; dxdy\right| &\leqslant \iint_\Omega |\nabla u \cdot \nabla v| \; dxdy \\
&\leqslant \iint_\Omega |\nabla u||\nabla v| \; dxdy \\
&\leqslant \sqrt{\iint_\Omega |\nabla u|^2 \; dxdy}\sqrt{\iint_\Omega |\nabla v|^2 \; dxdy};
\end{aligned}$$

so that, in view of (6.17),

$$\left|\iint_\Omega \nabla u \cdot \nabla v \; dxdy\right| \leqslant \|u\|\|v\|. \qquad (6.19)$$

Combining (6.18) and (6.19) we then obtain that

$$\|u+v\|^2 \leqslant \|u\|^2 + 2\|u\|\|v\| + \|v\|^2,$$

or

$$\|u+v\|^2 \leqslant (\|u\| + \|v\|)^2,$$

from which we get that

$$\|u+v\| \leqslant \|u\| + \|v\|, \quad \text{for } u, v \in V_o, \qquad (6.20)$$

which is the triangle inequality.

We obtain from (6.20) that

$$|\|v\| - \|u\|| \leqslant \|v - u\|, \quad \text{for } u, v \in V_o, \qquad (6.21)$$

which shows that the map $\|\cdot\|\colon V_o \to \mathbb{R}$ is a continuous function on V_o.

In addition to being a normed, linear space, the space V_o is a real, inner–product space with (real) inner–product, $\langle \cdot, \cdot \rangle\colon V_o \times V_o \to \mathbb{R}$, given by

$$\langle u, v \rangle = \iint_\Omega \nabla u(x,y) \cdot \nabla v(x,y)\ dxdy, \quad \text{for all } u, v \in V_o; \tag{6.22}$$

that is, the function $\langle \cdot, \cdot \rangle\colon V_o \times V_o \to \mathbb{R}$ satisfies the following three properties:

(i) Positive definiteness: $\langle u, u \rangle \geqslant 0$ for all $u \in V_o$ and $\langle u, u \rangle = 0$ if and only if $u = 0$.

(ii) Symmetry: $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V_o$.

(iii) Bi–linearity: $\langle u, c_1 v_1 + c_2 v_2 \rangle = c_1 \langle u, v_1 \rangle + c_2 \langle u, v_2 \rangle$ for all $u, v_1, v_2 \in V_o$ and $c_1, c_2 \in \mathbb{R}$.

Positive definiteness follows from the fact that

$$\langle u, u \rangle = \|u\|^2, \quad \text{for } u \in V_o, \tag{6.23}$$

according to the definitions in (6.22) and (6.17).

Symmetry and bi–linearity follow from the symmetry and bi–linearity of the Euclidean inner product.

We also have the Cauchy–Schwarz inequality established in (6.19),

$$|\langle u, v \rangle| \leqslant \|u\|\|v\|, \quad \text{for all } u, v \in V_o. \tag{6.24}$$

Given a function $f \in L^2(\Omega)$, we define a functional $J\colon V_o \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \iint_\Omega |\nabla u|^2\ dxdy - \iint_\Omega fu\ dxdy, \quad \text{for all } u \in V_o. \tag{6.25}$$

Note that, by virtue of the Cauchy–Schwarz inequality (see Appendix A.1),

$$\left| \iint_\Omega fu\ dxdy \right| \leqslant \|f\|_{L^2(\Omega)} \sqrt{\iint_\Omega u^2\ dxdy}, \tag{6.26}$$

which is finite, since $f \in L^2(\Omega)$ and $u \in C_c^1(\Omega)$. Consequently, the functional J given in (6.25) is well–defined. We can also show that J is Gâteaux differentiable at every $u \in V_o$ in the direction of every $v \in V_o$ with

$$dJ(u; v) = \iint_\Omega \nabla u \cdot \nabla v\ dxdy - \iint_\Omega fv\ dxdy, \quad \text{for } u, v \in V_o. \tag{6.27}$$

(see Example 4.1.2 on page 50 in these notes).

It is also the case that $J\colon V_o \to \mathbb{R}$ is continuous at every $u \in V_o$; that is, given $u \in V_o$,

$$\lim_{v \to u} J(v) = J(u), \tag{6.28}$$

where (6.28) is interpreted as follows: Given $u \in V_o$, for every $\varepsilon > 0$ there exists $\delta > 0$ (where δ might depends on u, as well as ε) such that

$$\|v - u\| < \delta \Rightarrow |J(v) - J(u)| < \varepsilon. \tag{6.29}$$

We will establish the assertion in (6.29) for the functional J defined in (6.25), for given $\varepsilon > 0$ and an appropriate choice of δ.

First, using (6.25) and the definition of the norm $\|\cdot\|$ in (6.17), we can write

$$J(v) = \frac{1}{2}\|v\|^2 - \iint_\Omega fv \; dxdy, \qquad \text{for all } v \in V_o, \tag{6.30}$$

and

$$J(u) = \frac{1}{2}\|u\|^2 - \iint_\Omega fu \; dxdy, \qquad \text{for all } u \in V_o. \tag{6.31}$$

Subtracting (6.31) from (6.30) we then have that

$$J(v) - J(u) = \frac{1}{2}\left(\|v\|^2 - \|u\|^2\right) + \iint_\Omega f(u - v) \; dxdy. \tag{6.32}$$

Next, use the Cauchy–Schwarz inequality and Poincaré inequality to estimate the right–most term in (6.32) to get

$$\begin{aligned}\left|\iint_\Omega f(u-v)\; dxdy\right| &\leqslant \sqrt{\iint_\Omega f^2 \; dxdy}\sqrt{\iint_\Omega (u-v)^2 \; dxdy} \\ &\leqslant \sqrt{c}\|f\|_{L^2(\Omega)}\sqrt{\iint_\Omega |\nabla(u-v)|^2 \; dxdy},\end{aligned}$$

where c is the constant $c(\Omega)$ in (6.16); we therefore have that

$$\left|\iint_\Omega f(u-v)\; dxdy\right| \leqslant \sqrt{c}\|f\|_{L^2(\Omega)}\|v - u\|, \tag{6.33}$$

in view of the definition of the norm $\|\cdot\|$ in (6.17).

To estimate the first term on the right–hand side of (6.32), observe that, by the properties of the inner–product, $\langle\cdot,\cdot\rangle$, defined in (6.22),

$$\|v\|^2 - \|u\|^2 = \langle v + u, v - u\rangle,$$

where we have also used (6.23). Consequently, applying the Cauchy–Scwarz inequality (see (6.24)),

$$\left|\|v\|^2 - \|u\|^2\right| \leqslant \|v + u\|\|v - u\|;$$

so that, by the triangle inequality,

$$\left|\|v\|^2 - \|u\|^2\right| \leqslant (\|v\| + \|u\|)\|v - u\|,$$

and, using the triangle inequality again,

$$\left|\|v\|^2 - \|u\|^2\right| \leqslant (\|v - u\| + 2\|u\|)\|v - u\|. \tag{6.34}$$

Then, choosing $v \in V_o$ so that

$$\|v - u\| \leqslant 1, \tag{6.35}$$

we get from (6.34) that

$$\left|\|v\|^2 - \|u\|^2\right| \leqslant (1 + 2\|u\|)\|v - u\|, \quad \text{for } \|v - u\| \leqslant 1. \tag{6.36}$$

Next, combine the estimates in (6.33) and (6.36), and use the triangle inequality in (6.32), we obtain from (6.32) that

$$|J(v) - J(u)| \leqslant \left[\frac{1}{2}(1 + 2\|u\|) + \sqrt{c}\|f\|_{L^2(\Omega)}\right]\|v - u\|, \tag{6.37}$$

where v satisfies the estimate in (6.35). Thus, given any $\varepsilon > 0$, we can choose $\delta > 0$ so that

$$\delta = \min\left\{1, \frac{2\varepsilon}{1 + 2\|u\| + 2\sqrt{c}\|f\|_{L^2(\Omega)}}\right\} \tag{6.38}$$

Hence, if $v \in V_o$ is such that $\|v - u\| < \delta$, where δ is given by (6.38), then (6.35) holds true, from which we get that (6.37), which in turn implies that

$$|J(v) - J(u)| < \varepsilon.$$

We have therefore established that (6.29) holds true for the choice of δ in (6.38). Consequently, J is continuous at $u \in V_o$. Since this holds true for every $u \in V_o$, we can say that J is continuous on V_o.

We will next show that the functional $J\colon V_o \to \mathbb{R}$ defined in (6.25) is bounded from below in V_o.

For $v \in V_o$ use (6.30) and the estimate in (6.26) to get

$$J(v) \geqslant \frac{1}{2}\|v\|^2 - \|f\|_{L^2(\Omega)}\sqrt{\iint_\Omega v^2\ dxdy};$$

so that, using the Poincaré inequality (6.16),

$$J(v) \geqslant \frac{1}{2}\|v\|^2 - \sqrt{c}\|f\|_{L^2(\Omega)}\sqrt{\iint_\Omega |\nabla v|^2\ dxdy},$$

where we have written c for $c(\Omega)$; thus, in view of the definition of the norm $\|\cdot\|$ in (6.17),

$$J(v) \geqslant \frac{1}{2}\|v\|^2 - \sqrt{c}\|f\|_{L^2(\Omega)}\|v\|, \quad \text{for all } v \in V_o,$$

which we can rewrite as

$$J(v) \geqslant \frac{1}{2}\left(\|v\|^2 - 2\sqrt{c}\|f\|_{L^2(\Omega)}\|v\|\right), \quad \text{for all } v \in V_o. \tag{6.39}$$

Completing the square in the expression in parentheses on the right–hand side of (6.39) yields

$$J(v) \geqslant \frac{1}{2}\left(\|v\| - \sqrt{c}\|f\|_{L^2(\Omega)}\right)^2 - \frac{c}{2}\|f\|^2_{L^2(\Omega)}, \quad \text{for all } v \in V_o,$$

from which it follows that

$$J(v) \geqslant -\frac{c}{2}\|f\|^2_{L^2(\Omega)}, \quad \text{for all } v \in V_o, \tag{6.40}$$

and therefore $J(v)$ is bounded from below in V_o by $-\dfrac{c}{2}\|f\|^2_{L^2(\Omega)}$.

Consequently, $\inf\limits_{v \in V_o} J(v)$ exists.

In this section we would like to answer the question of whether there exists $u \in V_o$ such that

$$J(u) = \inf_{v \in V_o} J(v). \tag{6.41}$$

In other words, is the infimum of J over V_o attained in V_o?

We will see in this section that, in general, the answer to the question posed above is no. However, if the class of admissible functions over which the infimum is taken is enlarged, we will be able to answer the question in the affirmative.

We first note that, by the definition of $\inf\limits_{v \in V_o} J(v)$, there exists a sequence of functions (u_m) in V_o such that

$$\lim_{m \to \infty} J(u_m) = \inf_{v \in V_o} J(v). \tag{6.42}$$

Indeed, setting

$$k = \inf_{v \in V_o} J(v), \tag{6.43}$$

for every $m \in \mathbb{N}$, there exists $u_m \in V_o$ such that

$$k \leqslant J(u_m) < k + \frac{1}{m}. \tag{6.44}$$

We then see that (6.42) follows from (6.44), (6.43) and the squeeze theorem.

Definition 6.3.4 (Minimizing Sequence)**.** A sequence (u_m) in V_o satisfying (6.42) is called a **minimizing sequence** for J over V_o.

We have therefore shown the existence of a minimizing sequence for J over V_o. We will next show that the minimizing sequence (u_m) is also a Cauchy sequence.

Use the properties of the inner product in (6.22) to compute, for $u, v \in V_o$,

$$\|u - v\|^2 = \|u\|^2 - 2\langle u, v\rangle + \|v\|^2 \tag{6.45}$$

and

$$\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2. \tag{6.46}$$

Adding the expressions in (6.45) and (6.46) we then obtain that

$$\|u-v\|^2 + \|u+v\|^2 = 2\|u\|^2 + 2\|v\|^2,$$

from which we get that

$$\|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 - \|u+v\|^2. \tag{6.47}$$

Divide on both sides of (6.47) by 4 to get

$$\frac{1}{4}\|u-v\|^2 = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \left\|\frac{1}{2}u + \frac{1}{2}v\right\|^2. \tag{6.48}$$

Next, use the expressions in (6.30) and (6.31) to rewrite (6.48) as

$$\frac{1}{4}\|u-v\|^2 = J(u) + \iint_\Omega fu\ dxdy + J(v) + \iint_\Omega fv\ dxdy - \left\|\frac{1}{2}u + \frac{1}{2}v\right\|^2, \tag{6.49}$$

where, by the same token,

$$\left\|\frac{1}{2}u + \frac{1}{2}v\right\|^2 = 2J\left(\frac{1}{2}u + \frac{1}{2}v\right) + 2\iint_\Omega f\left(\frac{1}{2}u + \frac{1}{2}v\right)\ dxdy,$$

or

$$\left\|\frac{1}{2}u + \frac{1}{2}v\right\|^2 = 2J\left(\frac{1}{2}u + \frac{1}{2}v\right) + \iint_\Omega f\,(u+v)\ dxdy. \tag{6.50}$$

Combining (6.49) and (6.50) then yields that

$$\frac{1}{4}\|u-v\|^2 = J(u) + J(v) - 2J\left(\frac{1}{2}u + \frac{1}{2}v\right), \qquad \text{for } u, v \in V_o. \tag{6.51}$$

Next, let (u_m) be a minimizing sequence for J in V_o; so that, according to (6.42) and (6.43),

$$\lim_{m\to\infty} J(u_m) = k, \tag{6.52}$$

where

$$k = \inf_{v\in V_o} J(v). \tag{6.53}$$

Applying the identity in (6.51) with u_m in place of u and u_n in place of v, we obtain that

$$\frac{1}{4}\|u_m - u_n\|^2 = J(u_m) + J(u_n) - 2J\left(\frac{1}{2}u_m + \frac{1}{2}u_n\right), \qquad \text{for } m, n \in \mathbb{N}. \tag{6.54}$$

Note that, by virtue of (6.53),

$$J\left(\frac{1}{2}u_m + \frac{1}{2}u_n\right) \geqslant k, \qquad \text{for } m, n \in \mathbb{N},$$

given that $\dfrac{1}{2}u_m + \dfrac{1}{2}u_n \in V_o$, since V_o is a linear space, Thus, using (6.54) we obtain that

$$\frac{1}{4}\|u_m - u_n\|^2 \leqslant J(u_m) + J(u_n) - 2k, \quad \text{for } m, n \in \mathbb{N}. \tag{6.55}$$

We therefore obtain from (6.52), (6.55) and the squeeze theorem that

$$\|u_m - u_n\| \to 0, \quad \text{as } m, n \to \infty. \tag{6.56}$$

It follows from (6.56) that the minimizing sequence (u_m) is a Cauchy sequence in V_o, with respect to the norm $\|\cdot\|$ given in (6.17). We note at this time that the space V_o is not a complete metric space with respect to the norm $\|\cdot\|$ defined in (6.17). Thus, we cannot conclude that the sequence (u_m) converges in V_o. However, we can consider the completion of V_o with respect to the metric induced by the norm $\|\cdot\|$. We will identify the completion of V_o with respect to the norm $\|\cdot\|$ with a subspace of $L^2(\Omega)$, denoted by $H_o^1(\Omega)$, to be discussed in the next section. It suffices to say (in what remains of this section) that V_o is a subspace of $H_o^1(\Omega)$, and that the norm $\|\cdot\|$ in V_o can be extended to a norm in $H_o^1(\Omega)$, which we will still denote by $\|\cdot\|$, as follows: For any $u \in H_o^1(\Omega)$, there exists a Cauchy sequence (φ_m) in V_o associated with u [1]; we can then define $\|u\|$ by

$$\|u\| = \lim_{m\to\infty} \|\varphi_m\|. \tag{6.57}$$

To ascertain that the norm given in (6.57) is well defined, we need to show that that the limit on the right–hand side of (6.57) exists and that it is independent of the choice of representative of the class corresponding to u. Indeed, it follows from the triangle inequality that

$$|\|\varphi_m\| - \|\varphi_n\|| \leqslant \|\varphi_m - \varphi_n\|, \quad \text{for all } m, n \in \mathbb{N};$$

so that, the sequence of real numbers $(\|\varphi_m\|)$ is a Cauchy sequence. Consequently, since $\mathbb{R}$ is a complete metric space, the limit on the right–hand side of (6.57) exists. By the same token, if (ψ_m) is another representative of the $u \in H_o^1(\Omega)$,

$$|\|\varphi_m\| - \|\psi_n\|| \leqslant \|\varphi_m - \psi_n\|, \quad \text{for all } m, n \in \mathbb{N}, \tag{6.58}$$

where (ψ_m) is a Cauchy sequence in V_o with the property that

$$\lim_{m\to\infty} \|\varphi_m - \psi_m\| = 0.$$

[1]The completion of V_o is made up of equivalence classes of Cauchy sequences under the equivalence relation: $(\varphi_m) \sim (\psi_m)$ if and only if

$$\lim_{m\to\infty} \|\varphi_m - \psi_m\| = 0.$$

Thus, to each element in the completion of V_o, there corresponds a representative Cauchy sequence in V_o.

Consequently, it follows from (6.58) and the squeeze theorem that

$$\lim_{m\to\infty} |\|\varphi_m\| - \|\psi_n\|| = 0.$$

or

$$\lim_{m\to\infty} \|\varphi_m\| = \lim_{m\to\infty} \|\psi_n\|,$$

which shows that the limit on the right–hand side of (6.57) is independent of the choice of representative of the class $u \in H_o^1(\Omega)$.

In addition, every $u \in H_o^1(\Omega)$ has the property that there exists a sequence (φ_m) in V_o such that

$$\lim_{m\to\infty} \|\varphi_m - u\| = 0. \tag{6.59}$$

Consequently, $H_o^1(\Omega)$ can be identified with the topological closure of V_o with respect to the norm given in (6.17); or, equivalently, V_o is a dense subspace of $H_o^1(\Omega)$; that is,

$$\overline{V_o} = H_o^1(\Omega).$$

This fact will be established in Section 6.4.6 in these notes.

Using this interpretation of $H_o^1(\Omega)$, the functional $J\colon V_o \to \mathbb{R}$ given in (6.31) can be extended to a functional on $H_o^1(\Omega)$, which we will also denote by J, given by

$$J(u) = \frac{1}{2}\|u\|^2 - \iint_\Omega fu\ dxdy, \qquad \text{for all } u \in H_o^1(\Omega). \tag{6.60}$$

To see how to define the functional $J \in H_o^1(\Omega) \to \mathbb{R}$ given in (6.60), let $u \in H_o^1(\Omega)$ and (φ_m) be a sequence in V_o for which (6.59) holds true. Define

$$J(u) = \lim_{m\to\infty} \frac{1}{2}\|\varphi_m\|^2 - \lim_{m\to\infty} \iint_\Omega f\varphi_m\ dxdy. \tag{6.61}$$

To justify the definition of $J\colon H_o^1(\Omega) \to \mathbb{R}$ given in (6.60), we need to show the limits on the right–hand side of (6.61) exist and are independent of the Cauchy sequence used to represent the element $u \in H_o^1(\Omega)$.

The first part of the expression on the right–hand side of (6.61) makes sense in view of the definition of the norm in $H_o^1(\Omega)$ given in (6.57), which is independent of the choice of Cauchy sequence representing u.

To see that the second limit on the right–hand side of the expression in (6.61) exists, compute

$$\begin{aligned} \left|\iint_\Omega f\varphi_m\ dxdy - \iint_\Omega f\varphi_n\ dxdy\right| &= \left|\iint_\Omega f(\varphi_m - \varphi_n)\ dxdy\right| \\ &\leqslant \sqrt{c}\|f\|_{L^2(\Omega)}\|\varphi_m - \varphi_n\|, \end{aligned} \tag{6.62}$$

for all $m, n \in \mathbb{N}$, where we have used the estimate in (6.33).

Now, since (φ_m) is a Cauchy sequence in $H_o^1(\Omega)$, it follows from (6.62) that the sequence of real numbers

$$\left(\iint_\Omega f\varphi_m\ dxdy\right)$$

is a Cauchy sequence in $\mathbb{R}$; thus, since $\mathbb{R}$ is complete, the limit

$$\lim_{m\to\infty} \iint_\Omega f\varphi_m \, dxdy$$

exists. This is the right–most limit in the definition of $J(u)$ in (6.61).

To complete the definition of $J(u)$ in (6.61)), let (ψ_m) be another approximating sequence to u; so that, (ψ_m) is a sequence in V_o such that

$$\lim_{m\to\infty} \|\psi_m - u\| = 0. \tag{6.63}$$

It then follows from (6.59), (6.63) and the triangle inequality that

$$\lim_{m\to\infty} \|\psi_m - \varphi_m\| = 0. \tag{6.64}$$

Similar calculations to those in (6.62) lead to

$$\left| \iint_\Omega f\psi_m \, dxdy - \iint_\Omega f\varphi_m \, dxdy \right| \leqslant \sqrt{c}\|f\|_{L^2(\Omega)}\|\psi_m - \varphi_m\|, \quad \text{for all } m;$$

consequently, in view of (6.64) we get that

$$\lim_{m\to\infty} \iint_\Omega f\psi_m \, dxdy = \lim_{m\to\infty} \iint_\Omega f\varphi_m \, dxdy.$$

Thus, it makes sense to define

$$\iint_\Omega fu \, dxdy = \lim_{m\to\infty} \iint_\Omega f\varphi_m \, dxdy,$$

where (φ_m) is any sequence in V_o for which (6.59) holds true. By the same token, according to the definition of the norm $\|\cdot\|$ in (6.57),

$$\frac{1}{2}\|u\|^2 = \lim_{m\to\infty} \frac{1}{2}\|\varphi_m\|^2,$$

for any any sequence (φ_m) in V_o for which (6.59) holds true.

In addition to being a complete metric space with respect to the norm given in (6.57), $H^1_o(\Omega)$ is also an inner product space with (real) inner product, $\langle\cdot,\cdot\rangle\colon H^1_o(\Omega)\times H^1_o(\Omega)\to\mathbb{R}$, defined by

$$\langle u, v\rangle = \lim_{m\to\infty} \langle \varphi_m, \psi_m\rangle, \quad \text{for } u, v \in H^1_o(\Omega) \tag{6.65}$$

where (φ_m) is any sequence in V_o that approximates u according to (6.59), and (ψ_m) is any sequence in V_o that approximates v according to

$$\lim_{m\to\infty} \|\psi_m - v\| = 0. \tag{6.66}$$

Using the identity

$$\langle \varphi, \psi \rangle = \frac{1}{4}\|\varphi + \psi\|^2 - \frac{1}{4}\|\varphi - \psi\|^2, \quad \text{for all } \varphi, \psi \in V_o,$$

as well as (6.66), (6.59) and the definition of the norm $\|\cdot\| \colon H_o^1(\Omega) \to \mathbb{R}$ given in (6.57), we can show that the definition of the real inner product in $H_o^1(\Omega)$ given in (6.65) is independent of the choices of approximating sequences (φ_m) and (ψ_m) for u and v, respectively.

Hence, $H_o^1(\Omega)$ with the (real) inner product defined in (6.65) is a complete (real) inner product space. The space $H_o^1(\Omega)$ is therefore an example of a (real) Hilbert space.

Using the definition of $J \colon H_o^1(\Omega) \to \mathbb{R}$ given in (6.60) and (6.61); namely,

$$J(u) = \lim_{m\to\infty} J(\varphi_m), \quad \text{for } u \in H_o^1(\Omega), \tag{6.67}$$

for any any sequence (φ_m) in V_o that approximates u in the sense of (6.59), we can establish the following properties of J:

(i) $J \colon H_o^1(\Omega) \to \mathbb{R}$ is continuous;

(ii) J is Gâteaux differentiable at every $u \in H_o^1(\Omega)$ in the direction of every $v \in H_o^1(\Omega)$, with Gâteaux derivative given by

$$dJ(u; v) = \langle u, v \rangle - \iint_\Omega fv\ dxdy, \quad \text{for } u, v \in H_o^1(\Omega). \tag{6.68}$$

(see (6.27) in these notes);

(iii) J is bounded from below in $H_o^1(\Omega)$, with

$$J(v) \geqslant -\frac{c}{2}\|f\|^2_{L^2(\Omega)}, \quad \text{for all } v \in\in H_o^1(\Omega) \tag{6.69}$$

(see the calculations leading to the estimate in (6.40)).

It follows from (6.69) that $\inf\limits_{v\in H_o^1(\Omega)} J(v)$ exists. Put

$$\ell = \inf_{v\in H_o^1(\Omega)} J(v), \tag{6.70}$$

and let (u_m) be a minimizing sequence for J in $H_o^1(\Omega)$; so that, $u_m \in H_o^1(\Omega)$ for all $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} J(u_m) = \ell. \tag{6.71}$$

The estimates leading to (6.56) in these notes can now be used to show that (u_m) is a Cauchy sequence in $H_o^1(\Omega)$. Hence, since $H_o^1(\Omega)$ is a complete, normed, linear space, we conclude that there exists $u \in H_o^1(\Omega)$ such that

$$\lim_{m\to\infty} \|u_m - u\| = 0. \tag{6.72}$$

Hence, combining (6.72), (6.71) and the continuity of $J\colon H_o^1(\Omega) \to \mathbb{R}$, we get that

$$J(u) = \ell,$$

and, in view of (6.70),

$$J(u) = \inf_{v \in H_o^1(\Omega)} J(v). \tag{6.73}$$

We have therefore shown that the minimization problem stated in (6.41) has a solution in $H_o^1(\Omega)$, which is not necessarily an element of V_o.

In the remainder of this section we will explore a few properties of a solution of the minimization problem in (6.73).

First, we note that, if $u \in H_o^1(\Omega)$ is a solution of the minimization problem in (6.73), then

$$\langle u, v\rangle - \iint_\Omega fv \; dxdy = 0, \quad \text{ for } v \in H_o^1(\Omega), \tag{6.74}$$

or

$$\langle u, v\rangle = \iint_\Omega fv \; dxdy, \quad \text{ for } v \in H_o^1(\Omega). \tag{6.75}$$

To see why (6.74) is true, use the result of Section 4.2 to see that, if $u \in H_o^1(\Omega)$ is a minimizer of J in $H_o^1(\Omega)$, then

$$dJ(u; v) = 0, \quad \text{ for all } v \in H_o^1(\Omega), \tag{6.76}$$

(look at the argument leading to (4.27) in Section 4.2 of these notes); so that, (6.74) follows from (6.76) and (6.68).

Secondly, the minimization problem in (6.73) has at most one solution. This follows from the fact that the functional $J\colon H_o^1(\Omega) \to \mathbb{R}$ given in (6.60) is strictly convex (see Example 4.3.2 on page 56 in these notes). Thus, the assertion follows from the convex minimization theorem (Theorem 4.4.1 in these notes).

We end this section with a discussion of the connection between the minimization problem in (6.73) and the boundary value problem in (6.14) discussed at the start of this section. We will continue the discussion in the next section dealing with the concept of weak solutions and the space $H_o^1(\Omega)$, which is an example of a Sobolev space.

Suppose for the moment that, in addition to $u \in H_o^1(\Omega)$ being the solution of the minimization problem in (6.73), u also belongs to the spaces $C^2(U, \mathbb{R})$ and $C(\overline{\Omega}, \mathbb{R})$ and that $u = 0$ on $\partial\Omega$. Assume also that $f \in C(\Omega, \mathbb{R})$, in addition to f being in $L^2(\Omega)$.

From (6.74) we obtain that

$$\iint_\Omega \nabla u \cdot \nabla v \; dxdy - \iint_\Omega fv \; dxdy = 0, \quad \text{ for } v \in C_c^1(\Omega), \tag{6.77}$$

since $V_o = C_c^1(\Omega)$ is a subspace of $H_o^1(\Omega)$. Next, apply Green's identity I (see Appendix B.4) to the left–most integral in (6.77) to obtain

$$-\iint_\Omega \Delta u \; v \; dxdy + \oint_{\partial\Omega} v \frac{\partial u}{\partial n} \; ds - \iint_\Omega fv \; dxdy = 0, \quad \text{ for } v \in C_c^1(\Omega);$$

so that, since v vanishes on a neighborhood of $\partial\Omega$,

$$-\iint_\Omega \Delta u\ v\ dxdy - \iint_\Omega fv\ dxdy = 0, \qquad \text{for } v \in C_c^1(\Omega),$$

or

$$\iint_\Omega (\Delta u + f)v\ dxdy = 0, \qquad \text{for } v \in C_c^1(\Omega). \tag{6.78}$$

Since we are assuming that $u \in C^2(\Omega, \mathbb{R})$ and $f \in C(\Omega, \mathbb{R})$, we can apply the two–dimensional version of the first fundamental lemma of the Calculus of Variations to obtain from (6.78) that

$$\Delta u(x, y) + f(x, y) = 0, \qquad \text{for all } (x, y) \in \Omega.$$

Consequently, u solves the partial differential equation in the boundary value problem in (6.14). We have therefore shown that, if $u \in H_o^1(\Omega)$ is a minimizer of the functional $J\colon H_o^1(\Omega) \to \mathbb{R}$ given in (6.60), and if, in addition $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ with $u = 0$ on $\partial\Omega$, then u is a solution of the boundary value problem in (6.14). In this section we have given the first step in showing that the boundary value problem in (6.14) has a solution by showing that the corresponding functional J in (6.60) has a minimizer in $H_o^1(\Omega)$. In the following sections we embark into completing the existence proof for the boundary value problem in (6.14), under additional assumptions on f and the boundary of the domain Ω.

6.4 Weak Derivatives and the Space $H_o^1(\Omega)$

Let Ω denote an open, connected subset of $\mathbb{R}^2$ with piece–wise smooth boundary $\partial\Omega$. Let V_o be as defined in (6.15); that is, V_o is the space of C^1 functions with compact support in Ω, $u \in C_c^1(\Omega, \mathbb{R})$. Note that if $\varphi \in V_o$, then $\varphi \in L^2(\Omega)$, $\dfrac{\partial\varphi}{\partial x} \in L^2(\Omega)$, and $\dfrac{\partial\varphi}{\partial y} \in L^2(\Omega)$. In section 6.3 we introduced the Sobolev space $H_o^1(\Omega)$ as the completion of the space V_o with respect to the norm $\|\cdot\|\colon V_o \to \mathbb{R}$ given in (6.17); namely,

$$\|u\| = \sqrt{\iint_\Omega |\nabla u|^2\ dxdy}, \qquad \text{for } u \in V_o. \tag{6.79}$$

The space $H_o^1(\Omega)$ is therefore a Hilbert space with inner product given by

$$\langle u, v\rangle = \iint_\Omega \nabla u \cdot \nabla v\ dxdy, \qquad \text{for all } u, v \in H_o^1(\Omega). \tag{6.80}$$

In this section, we present a characterization of the elements of $H_o^1(\Omega)$ based on the concept of weak derivatives. We well see that elements of $H_o^1(\Omega)$ can be realized as weakly differentiable functions whose weak derivatives are in $L^2(\Omega)$.

6.4.1 Weak Derivatives

Definition 6.4.1 (Locally integrable functions)**.** A function $u\colon \Omega \to \mathbb{R}$ is said to be locally integrable in Ω, if

$$\iint_K |u(x,y)|\ dxdy < \infty \tag{6.81}$$

for every compact subset K of Ω. We write $u \in L^1_{\text{loc}}(\Omega)$.

If the condition in (6.81) is replaced by

$$\iint_K |u(x,y)|^2\ dxdy < \infty \tag{6.82}$$

for every compact subset, K, of Ω, we say that $u \in L^2_{\text{loc}}(\Omega)$.

Definition 6.4.2 (Weak Derivatives)**.** A function $u\colon \Omega \to \mathbb{R}$ that is locally integrable is said to have a weak partial derivative with respect to x if there exists a function $v \in L^1_{\text{loc}}(\Omega)$ for which

$$\iint_\Omega u\frac{\partial \varphi}{\partial x}\ dxdy = -\iint_\Omega v\varphi\ dxdy, \quad \text{for all } \varphi \in C^1_c(\Omega). \tag{6.83}$$

We will call the function v in (6.83) the weak partial derivative of u with respect to x and we will denote it by $\dfrac{\partial u}{\partial x}$; thus, according to (6.83) we have that $\dfrac{\partial u}{\partial x} \in L^1_{\text{loc}}(\Omega)$, and

$$\iint_\Omega u\frac{\partial \varphi}{\partial x}\ dxdy = -\iint_\Omega \frac{\partial u}{\partial x}\varphi\ dxdy, \quad \text{for all } \varphi \in C^1_c(\Omega). \tag{6.84}$$

A similar definition can be applied to the weak partial derivative, $\dfrac{\partial u}{\partial y}$, of u with respect to y.

6.4.2 Properties of Weak Derivatives

The definition of the weak partial derivatives given in Definition6.4.2 and the linearity of the integral imply the following linearity properties of the weak partial derivative

Proposition 6.4.3. Let $u, v \in L^1_{\text{loc}}(\Omega)$ have a weak partial derivatives with respect to x, $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$. Then, $u+v$ has a weak partial derivative with respect to x given by

$$\frac{\partial}{\partial x}[u+v] = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}.$$

For $c \in \mathbb{R}$, cv has a weak partial derivative with respect to x given by

$$\frac{\partial}{\partial x}[cu] = c\frac{\partial u}{\partial x}.$$

Similar assertions hold for the weak partial derivative with respect to y.

In the following, we derive other properties of weak partial derivatives and give some examples.

Proposition 6.4.4. Let $u \in L^1_{\text{loc}}(\Omega)$ have a weak partial derivative with respect to x, $\dfrac{\partial u}{\partial x}$. Let $\psi \in C^1(\Omega)$. Then, ψu has a weak partial derivative with respect to x given by

$$\frac{\partial}{\partial x}[\psi u] = \psi \frac{\partial u}{\partial x} + u \frac{\partial \psi}{\partial x}.$$

A similar assertion holds true for the weak partial derivative with respect to y.

Proof: For any $\varphi \in C^1_c(\Omega)$, compute

$$\begin{aligned}
\iint_\Omega \left[\psi \frac{\partial u}{\partial x} + u \frac{\partial \psi}{\partial x}\right] \varphi \; dxdy &= \iint_\Omega \frac{\partial u}{\partial x} \psi\varphi \; dxdy + \iint_\Omega u\varphi \frac{\partial \psi}{\partial x} \; dxdy \\
&= -\iint_\Omega u \frac{\partial}{\partial x}[\psi\varphi] \; dxdy + \int_\Omega u\varphi \frac{\partial \psi}{\partial x} \; dxdy \\
&= -\iint_\Omega \psi u \frac{\partial \varphi}{\partial x} \; dxdy - \iint_\Omega u\varphi \frac{\partial \psi}{\partial x} \; dxdy \\
&\qquad + \iint_\Omega u\varphi \frac{\partial \psi}{\partial x} \; dxdy \\
&= -\iint_\Omega \psi u \frac{\partial \varphi}{\partial x} \; dxdy,
\end{aligned}$$

from which we get that

$$\iint_\Omega \psi u \frac{\partial \varphi}{\partial x} \; dxdy = -\iint_\Omega \left[\psi \frac{\partial u}{\partial x} + u \frac{\partial \psi}{\partial x}\right] \varphi \; dxdy, \quad \text{for all } \varphi \in C^1_c(\Omega).$$

■

6.4.3 Approximation Properties

Let $\psi \in C^1_c(\mathbb{R}^2)$ have support $\overline{B}_1(O)$, the closed disc of radius 1 around the origin in $\mathbb{R}^2$; furthermore, assume that $\psi \geqslant 0$ in $\mathbb{R}^2$ and

$$\iint_{\mathbb{R}^2} \psi(x, y) \; dxdy = 1. \tag{6.85}$$

In the appendix, we show how the function ψ can be constructed.

For each $\varepsilon > 0$, define

$$\psi_\varepsilon(x, y) = \frac{1}{\varepsilon^2} \psi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right), \quad \text{for all } (x, y) \in \mathbb{R}^2. \tag{6.86}$$

By the change of variables theorem, we obtain from (6.86) and (6.85) that

$$\iint_{\mathbb{R}^2} \psi_\varepsilon(x,y)\ dxdy = 1, \quad \text{for all } \varepsilon > 0. \tag{6.87}$$

For each $\varepsilon > 0$, put

$$\Omega_\varepsilon = \{(x\, y) \in \Omega \mid \text{dist}((x,y), \partial\Omega) > \varepsilon\}, \tag{6.88}$$

and define $u_\varepsilon \colon \Omega_\varepsilon \to \mathbb{R}$ by

$$u_\varepsilon(x,y) = \iint_\Omega \psi_\varepsilon(x-\xi, y-\eta) u(\xi,\eta)\ d\xi d\eta, \quad \text{for } (x,y) \in \Omega_\varepsilon. \tag{6.89}$$

The expression on the right–hand side of (6.89) is called the **convolution** of ψ_ε and u, and it is denoted by $\psi_\varepsilon * u$; so that, (6.89) can also be written as

$$u_\varepsilon(x,y) = \psi_\varepsilon * u(x,y), \quad \text{for } (x,y) \in \Omega_\varepsilon. \tag{6.90}$$

Proposition 6.4.5. Let $u \in L^1_{\text{loc}}(\Omega)$ have weak partial derivatives that are in $L^1_{\text{loc}}(\Omega)$. Then, $u_\varepsilon \in C^1(\Omega_\varepsilon)$,

$$\frac{\partial u_\varepsilon}{\partial x} = \psi_\varepsilon * \frac{\partial u}{\partial x}, \tag{6.91}$$

$$\frac{\partial u_\varepsilon}{\partial y} = \psi_\varepsilon * \frac{\partial u}{\partial y}, \tag{6.92}$$

and

$$\frac{\partial u_\varepsilon}{\partial x} \to \frac{\partial u}{\partial x} \ \text{ and } \ \frac{\partial u_\varepsilon}{\partial y} \to \frac{\partial u}{\partial y} \quad \text{in} \quad L^1_{\text{loc}}(\Omega) \quad \text{as } \varepsilon \to 0^+. \tag{6.93}$$

Thus, weak derivatives can be approximated in $L^1_{\text{loc}}(\Omega)$ by C^1 functions.

Proof: The fact that u_ε is C^1 follows from the definition of u_ε in (6.89), the fact that $\psi \in C^1_c(\mathbb{R}^2)$, and differentiation under the integral sign.

To prove (6.91), use (6.89) to compute

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \iint_\Omega \frac{\partial}{\partial x}[\psi_\varepsilon(x-\xi, y-\eta)] u(\xi,\eta)\ d\xi d\eta,$$

and use the definition of weak derivative in (6.84) to get

$$\begin{aligned}
\frac{\partial u_\varepsilon}{\partial x}(x,y) &= \iint_\Omega \frac{\partial}{\partial x}[\psi_\varepsilon(x-\xi, y-\eta)] u(\xi,\eta)\ d\xi d\eta \\
&= -\iint_\Omega \frac{\partial}{\partial \xi}[\psi_\varepsilon(x-\xi, y-\eta)] u(\xi,\eta)\ d\xi d\eta \\
&= \iint_\Omega \psi_\varepsilon(x-\xi, y-\eta) \frac{\partial u}{\partial \xi}(\xi,\eta)\ d\xi d\eta \\
&= \psi_\varepsilon * \frac{\partial u}{\partial x}(x,y),
\end{aligned}$$

which yields (6.91).

For ease of notation, we will denote $\dfrac{\partial u}{\partial x}$ by u_x. We will also stipulate that u_x indicates the partial derivative of u with respect to the first variable. Consequently, we rewrite (6.91) as

$$\frac{\partial u_\varepsilon}{\partial x} = \psi_\varepsilon * u_x, \tag{6.94}$$

where

$$\psi_\varepsilon * u_x(x,y) = \iint_\Omega \psi_\varepsilon(x-\xi, y-\eta) u_x(\xi,\eta)\ d\xi d\eta, \quad \text{for } (x,y) \in \Omega_\varepsilon. \tag{6.95}$$

To prove the first assertion in (6.93) let $(x,y) \in \Omega_\varepsilon$ and use (6.94), (6.95) and (6.87) to compute

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial x}(x,y) - u_x(x,y) &= \iint_\Omega \psi_\varepsilon(x-\xi, y-\eta) u_x(\xi,\eta)\ d\xi d\eta \\ &\quad - u_x(x,y) \int_\Omega \psi_\varepsilon(x-\xi, y-\eta)\ d\xi d\eta; \end{aligned}$$

so that,

$$\frac{\partial u^\varepsilon}{\partial x}(x,y) - u_x(x,y) = \iint_\Omega \psi_\varepsilon(x-\xi, y-\eta) \left[u_x(\xi,\eta) - u_x(x,y)\right]\ d\xi d\eta. \tag{6.96}$$

Next, make the change of variables $(z_1, z_2) = (x,y) - (\xi,\eta)$ in (6.96) to get

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) - u_x(x,y) = \iint_\Omega \psi_\varepsilon(z_1,z_2) \left[u_x(x-z_1, y-z_2) - u_x(x,y)\right]\ dz_1 dz_2,$$

or

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) - u_x(x,y) = \iint_{\overline{B}_\varepsilon(O)} \psi_\varepsilon(z_1,z_2) \left[u_x(x-z_1, y-z_2) - u_x(x,y)\right]\ dz_1 dz_2, \tag{6.97}$$

in view of the definition of ψ_ε in (6.86), where O denotes the origin $(0,0)$ in $\mathbb{R}^2$, and $\overline{B}_\varepsilon(O)$ the closed disc of radius ε centered at O.

Make a further change of variables $(w_1, w_2) = \dfrac{1}{\varepsilon}(z_1, z_2)$ in (6.97) to obtain

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) - u_x(x,y) = \iint_{\overline{B}_1(O)} \psi(w_1,w_2) \left[u_x(x-\varepsilon w_1, y-\varepsilon w_2) - u_x(x,y)\right]\ dw_1 dw_2,$$

from which we get

$$\left|\frac{\partial u_\varepsilon}{\partial x}(x,y) - u_x(x,y)\right| \leqslant \iint_{\overline{B}_1(O)} \psi(w) \left|u_x(x-\varepsilon w_1, y-\varepsilon w_2) - u_x(x,y)\right|\ dw, \tag{6.98}$$

where we have written w for (w_1, w_2) and dw for $dw_1 dw_2$.

Next, let Ω_1 be subset of Ω with compact closure and such that $\overline{\Omega}_1 \subset \Omega_\varepsilon$. Then, integrating on both sides of (6.98) over $\overline{\Omega}_1$ and using Fubini's theorem,

$$\iint_{\overline{\Omega}_1} \left| \frac{\partial u_\varepsilon}{\partial x} - u_x \right| \, dxdy \leqslant \iint_{\overline{B}_1(O)} \psi(w) \iint_{\overline{\Omega}_1} |u_x((x,y)-\varepsilon w) - u_x(x,y)| \, dxdydw. \tag{6.99}$$

Now, since continuous functions with compact support are dense in L^1, given $\delta > 0$, there exists $g \in C(\overline{\Omega}_{1,\varepsilon})$ such that

$$\left\| \frac{\partial u}{\partial x_i} - g \right\|_{L^1(\overline{\Omega}_{1,\varepsilon})} < \frac{\delta}{4}, \tag{6.100}$$

where we define

$$\overline{\Omega}_{1,\varepsilon} = \{ x \in \Omega \mid \operatorname{dist}(x, \overline{\Omega}_1) \leqslant \varepsilon \}.$$

We then obtain from (6.99), (6.100) and the triangle inequality that

$$\iint_{\overline{\Omega}_1} \left| \frac{\partial u_\varepsilon}{\partial x} - u_x \right| \, dxdy \leqslant \iint_{\overline{B}_1(O)} \psi(w) \iint_{\overline{\Omega}_1} |g((x,y) - \varepsilon w) - g(x,y)| \, dxdydw + \frac{\delta}{2}, \tag{6.101}$$

where we have also used (6.85).

Next, use the uniform continuity of g on $\overline{\Omega}_1$ to deduce that there exists $\varepsilon_o > 0$ such that

$$0 < \varepsilon < \varepsilon_o \implies \iint_{\overline{\Omega}_1} |g((x,y) - \varepsilon w) - g(x,y)| \, dxdy < \frac{\delta}{2}, \quad \text{for } w \in \overline{B}_1(O). \tag{6.102}$$

Consequently, combining (6.102) and (6.85) with (6.101), we obtain that

$$0 < \varepsilon < \varepsilon_o \implies \iint_{\overline{\Omega}_1} \left| \frac{\partial u_\varepsilon}{\partial x} - u_x \right| \, dxdy < \delta.$$

We have therefore established the first assertion in (6.93).

The arguments and calculations leading to the proof of (6.91) and the first assertion in (6.93) can be used to prove that if $u \in L^1_{\text{loc}}(\Omega)$ has a weak partial derivative with respect to y, with $\dfrac{\partial u}{\partial y} \in L^1_{\text{loc}}(\Omega)$, then

$$\frac{\partial u_\varepsilon}{\partial y} = \psi_\varepsilon * \frac{\partial u}{\partial y},$$

which is (6.92), and

$$\frac{\partial u_\varepsilon}{\partial y} \to \frac{\partial u}{\partial y} \quad \text{in} \quad L^1_{\text{loc}}(\Omega) \quad \text{as } \varepsilon \to 0^+,$$

which is the second assertion in (6.93). ■

Proposition 6.4.6. A function $u \in L^1_{\text{loc}}(\Omega)$ has a weak partial derivative $\dfrac{\partial u}{\partial x} \in L^1_{\text{loc}}(\Omega)$ if and only if there exists a sequence of functions (u_m) in $C^1(\Omega)$ and a function $v \in L^1_{\text{loc}}(\Omega)$ such that

$$u_m \to u \quad \text{and} \quad \frac{\partial u_m}{\partial x} \to v \quad \text{in } L^1_{\text{loc}}(\Omega) \quad \text{as } m \to \infty. \tag{6.103}$$

A similarly assertion holds true for the weak partial derivative with respect to y.

Proof: Suppose that $u \in L^1_{\text{loc}}(\Omega)$ has a weak partial derivative with respect to x, and $\dfrac{\partial u}{\partial x} \in L^1_{\text{loc}}(\Omega)$. Then, (6.103) follows with $v = \dfrac{\partial u}{\partial x}$ by applying Proposition 6.4.5 with $\varepsilon = \dfrac{1}{m}$, for $m = 1, 2, 3, \ldots$

Conversely, assume that (6.103) holds true and let $\varphi \in C^1_c(\Omega)$. Then, using integration by parts,

$$\iint_\Omega u_m \frac{\partial \varphi}{\partial x}\, dxdy = -\iint_\Omega \frac{\partial u_m}{\partial x}\varphi\, dx, \quad \text{for all } m. \tag{6.104}$$

Let K denote the support of φ and compute

$$\left|\iint_\Omega u_m \frac{\partial \varphi}{\partial x}\, dxdy - \iint_\Omega u \frac{\partial \varphi}{\partial x}\, dxdy\right| \leqslant \iint_\Omega |u_m - u| \left|\frac{\partial \varphi}{\partial x}\right|\, dxdy;$$

so that

$$\left|\iint_\Omega u_m \frac{\partial \varphi}{\partial x}\, dxdy - \int_\Omega u \frac{\partial \varphi}{\partial x}\, dxdy\right| \leqslant \sup_{(x,y)\in K}\left|\frac{\partial \varphi(x,y)}{\partial x}\right| \iint_K |u_m - u|\, dxdy;$$

and, therefore, by virtue of (6.103),

$$\lim_{m\to\infty} \iint_\Omega u_m \frac{\partial \varphi}{\partial x}\, dxdy = \iint_\Omega u \frac{\partial \varphi}{\partial x}\, dxdy, \quad \text{for all } \varphi \in C^1_c(\Omega). \tag{6.105}$$

Similarly, we get from (6.103) that

$$\lim_{m\to\infty} \iint_\Omega \frac{\partial u_m}{\partial x}\varphi\, dxdy = \iint_\Omega v\varphi\, dxdy, \quad \text{for all } \varphi \in C^1_c(\Omega). \tag{6.106}$$

It then follows from (6.104), (6.105) and (6.106) that

$$\iint_\Omega u \frac{\partial \varphi}{\partial x}\, dxdy = -\iint_\Omega v\varphi\, dxdy, \quad \text{for all } \varphi \in C^1_c(\Omega),$$

which shows that u has weak partial derivative with respect to x given by

$$\frac{\partial u}{\partial x} = v.$$

Similar considerations hold true for the weak partial derivative or u with respect to y. ■

6.4.4 A Few Examples

Proposition 6.4.7. Let $u \in L^1_{\text{loc}}(\Omega)$ have a weak partial derivative with respect to x, with $\dfrac{\partial u}{\partial x} \in L^1_{\text{loc}}(\Omega)$. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function with bounded and continuous derivative f'. Then, $f \circ u\colon \Omega \to \mathbb{R}$ has a weak partial derivative with respect to x_i given by

$$\frac{\partial}{\partial x}[f \circ u] = f'(u)\frac{\partial u}{\partial x}. \tag{6.107}$$

An analogous assertion holds true for the weak partial derivative with respect to y.

Proof: Since f' is bounded, we get that

$$|f(s_1) - f(s_2)| \leqslant \|f'\|_{L^\infty}|s_1 - s_2|, \quad \text{for all } s_1, s_2 \in \mathbb{R}; \tag{6.108}$$

thus, f is globally Lipschitz continuous.

By the results in Proposition 6.4.6, there exists a sequence (u_m) of functions in $C^1(\Omega)$ such that

$$u_m \to u \quad \text{and} \quad \frac{\partial u_m}{\partial x} \to \frac{\partial u}{\partial x} \quad \text{in } L^1_{\text{loc}}(\Omega) \quad \text{as } m \to \infty. \tag{6.109}$$

We may assume, passing to a subsequences if necessary, that

$$u_m(x,y) \to u(x,y) \quad \text{as } m \to \infty \quad \text{for a.e. } (x,y) \in \Omega, \tag{6.110}$$

and

$$\frac{\partial u_m}{\partial x}(x,y) \to \frac{\partial u}{\partial x}(x,y) \quad \text{as } m \to \infty \quad \text{for a.e. } (x,y) \in \Omega. \tag{6.111}$$

We may also assume that, for any given $\Omega_1 \subset \Omega$, with $\overline{\Omega}_1 \subset \Omega$ and compact, there exists $Z \in L^1(\overline{\Omega}_1)$ such that $Z \geqslant 0$ a.e. in $\overline{\Omega}_1$, and

$$\left|\frac{\partial u_m}{\partial x}(x,y)\right| \leqslant Z(x,y), \quad \text{for a.e. } (x,y) \in \overline{\Omega}_1. \tag{6.112}$$

Set $v = f \circ u$ and $v_m = f \circ u_m$, for all $m = 1, 2, 3, \ldots$; so that, each $v_m \in C^1(\Omega)$ with

$$\frac{\partial v_m}{\partial x} = f'(u_m)\frac{\partial u_m}{\partial x}, \quad \text{for all } m, \tag{6.113}$$

by the chain rule.

Let $\Omega_1 \subset \Omega$ be such that $\overline{\Omega}_1 \subset \Omega$ and $\overline{\Omega}_1$ is compact, and compute

$$\iint_{\overline{\Omega}_1} |v_m - v| \, dxdy = \iint_{\overline{\Omega}_1} |f(u_m) - f(u)| \, dxdy;$$

so that, by virtue of (6.108),

$$\iint_{\overline{\Omega}_1} |v_m - v| \; dxdy \leqslant \|f'\|_{L^\infty} \iint_{\overline{\Omega}_1} |u_m - u| \; dxdy, \qquad \text{for all } m.$$

Consequently, using (6.109),

$$\lim_{m\to\infty} \iint_{\overline{\Omega}_1} |v_m - v| \; dxdy = 0.$$

We have therefore shown that

$$v_m \to f(u) \quad \text{ in } L^1_{\text{loc}}(\Omega) \quad \text{ as } m \to \infty. \tag{6.114}$$

Next, use (6.113) to compute

$$\frac{\partial v_m}{\partial x} - f'(u)\frac{\partial u}{\partial x} = f'(u_m)\frac{\partial u_m}{\partial x} - f'(u)\frac{\partial u_m}{\partial x} + f'(u)\frac{\partial u_m}{\partial x} - f'(u)\frac{\partial u}{\partial x};$$

so that, applying the triangle inequality,

$$\left|\frac{\partial v_m}{\partial x} - f'(u)\frac{\partial u}{\partial x}\right| \leqslant |f'(u_m) - f'(u)| \left|\frac{\partial u_m}{\partial x}\right| + \|f'\|_{L^\infty} \left|\frac{\partial u_m}{\partial x} - \frac{\partial u}{\partial x}\right| \tag{6.115}$$

Observe that, by virtue of the continuity of f' and the statements in (6.110) and (6.111),

$$|f'(u_m(x,y)) - f'(u(x,y))| \left|\frac{\partial u_m}{\partial x}(x,y)\right| \to 0, \quad \text{as } m \to \infty, \quad \text{for a.e. } (x,y) \in \Omega. \tag{6.116}$$

Also, using (6.112),

$$|f'(u_m(x,y)) - f'(u(x,y))| \left|\frac{\partial u_m}{\partial x}(x,y)\right| \leqslant 2\|f'\|_{L^\infty} Z(x,y), \qquad \text{for a.e. } (x,y) \in \overline{\Omega}_1.$$

We can then apply the Lebesgue dominated convergence theorem to conclude from (6.116) that

$$\lim_{m\to\infty} \iint_{\overline{\Omega}_1} |f'(u_m(x)) - f'(u(x))| \left|\frac{\partial u_m}{\partial x}(x)\right| \; dxdy = 0 \tag{6.117}$$

It then follows from (6.109), (6.115) and (6.117) that

$$\lim_{m\to\infty} \iint_{\overline{\Omega}_1} \left|\frac{\partial v_m}{\partial x} - f'(u)\frac{\partial u}{\partial x}\right| \; dxdy = 0,$$

for any open subset Ω_1 of Ω such that $\overline{\Omega}_1 \subset \Omega$ and $\overline{\Omega}_1$ is compact; so that,

$$\frac{\partial v_m}{\partial x} \to f'(u)\frac{\partial u}{\partial x} \quad \text{ in } L^1_{\text{loc}}(\Omega) \quad \text{ as } m \to \infty. \tag{6.118}$$

Hence, it follows from (6.114), (6.118) and Proposition 6.4.6 that $f(u)$ has a weak partial derivative with respect to x given by (6.107).

Similar arguments and calculations show that, if u has a weak partial derivative with respect to y that is in $L^1_{\text{loc}}(\Omega)$, then $f(u)$ has a weak partial derivative with respect to y given by

$$\frac{\partial}{\partial y}[f \circ u] = f'(u)\frac{\partial u}{\partial y}.$$

■

Proposition 6.4.8. Let $u \in L^1_{\text{loc}}(\Omega)$ have a weak partial derivative with respect to x such that $\dfrac{\partial u}{\partial x} \in L^1_{\text{loc}}(\Omega)$. Then, the function $v = |u|$ has a weak partial derivative with respect to x given by

$$\frac{\partial |u|}{\partial x} = \begin{cases} \dfrac{\partial u}{\partial x}, & \text{if } u > 0; \\[2ex] 0, & \text{if } u = 0; \\[2ex] -\dfrac{\partial u}{\partial x}, & \text{if } u < 0. \end{cases} \tag{6.119}$$

An analogous result is obtained for the case in which u has a weak partial derivative with respect to y such that $\dfrac{\partial u}{\partial y} \in L^1_{\text{loc}}(\Omega)$.

Proof: For each $\varepsilon > 0$, define $f_\varepsilon(s) = \sqrt{s^2 + \varepsilon}$ for all $s \in \mathbb{R}$. Then, $f_\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ with derivative

$$f'_\varepsilon(s) = \frac{s}{\sqrt{s^2 + \varepsilon}}, \quad \text{for all } s \in \mathbb{R}. \tag{6.120}$$

From $s^2 + \varepsilon > s^2$, for all $s \in \mathbb{R}$, we obtain that

$$|s| < \sqrt{s^2 + \varepsilon}, \quad \text{for all } s \in \mathbb{R},$$

from which we get that

$$|f'_\varepsilon(s)| < 1, \quad \text{for all } s \in \mathbb{R}.$$

Thus, f'_ε is bounded for all $\varepsilon > 0$. Hence, Proposition 6.4.7 applies to f_ε. Thus, $f_\varepsilon(u)$ has a weak partial derivative with respect to x given by

$$\frac{\partial}{\partial x}[f_\varepsilon(u)] = f'_\varepsilon(u)\frac{\partial u}{\partial x};$$

so that, in view of (6.120),

$$\frac{\partial}{\partial x}[f_\varepsilon(u)] = \frac{u}{\sqrt{u^2 + \varepsilon}}\frac{\partial u}{\partial x}, \quad \text{for all } \varepsilon > 0. \tag{6.121}$$

Hence, for any $\varphi \in C_c^1(\Omega)$, according to (6.121) and the definition of weak derivatives,

$$\iint_\Omega f_\varepsilon(u)\frac{\partial \varphi}{\partial x}\, dxdy = -\iint_\Omega \frac{u}{\sqrt{u^2+\varepsilon}}\frac{\partial u}{\partial x}\varphi\, dxdy, \qquad \text{for all } \varepsilon > 0. \tag{6.122}$$

Now, from the fact that

$$\lim_{\varepsilon\to 0^+} f_\varepsilon(u) = |u|$$

and the Lebesgue dominated convergence theorem we obtain that

$$\lim_{\varepsilon\to 0^+}\iint_\Omega f_\varepsilon(u)\frac{\partial \varphi}{\partial x}\, dxdy = \iint_\Omega |u|\frac{\partial \varphi}{\partial x}\, dxdy, \qquad \text{for all } \varphi \in C_c^1(\Omega). \tag{6.123}$$

Next, define

$$\begin{array}{rcl} \Omega_- & = & \{(x,y)\in\Omega \mid u(x,y) < 0\}; \\ \Omega_o & = & \{(x,y)\in\Omega \mid u(x,y) = 0\}; \\ \Omega_+ & = & \{(x,y)\in\Omega \mid u(x,y) > 0\}. \end{array}$$

Then,

$$\iint_{\Omega_o} \frac{u}{\sqrt{u^2+\varepsilon}}\frac{\partial u}{\partial x_i}\varphi\, dxdy = 0, \qquad \text{for all } \varepsilon > 0,$$

and all $\varphi \in C_c^1(\Omega)$; so that,

$$\lim_{\varepsilon\to 0^+}\iint_{\Omega_o} \frac{u}{\sqrt{u^2+\varepsilon}}\frac{\partial u}{\partial x}\varphi\, dxdy = 0, \qquad \text{for all } \varphi \in C_c^1(\Omega). \tag{6.124}$$

Applying the Lebesgue dominated convergence theorem in Ω_- and Ω_+ we obtain

$$\lim_{\varepsilon\to 0^+}\iint_{\Omega_-} \frac{u}{\sqrt{u^2+\varepsilon}}\frac{\partial u}{\partial x}\varphi\, dxdy = \iint_{\Omega_-} -\frac{\partial u}{\partial x}\varphi\, dxdy, \qquad \text{for all } \varphi \in C_c^1(\Omega), \tag{6.125}$$

and

$$\lim_{\varepsilon\to 0^+}\iint_{\Omega_+} \frac{u}{\sqrt{u^2+\varepsilon}}\frac{\partial u}{\partial x}\varphi\, dxdy = \iint_{\Omega_+} \frac{\partial u}{\partial x}\varphi\, dxdy, \qquad \text{for all } \varphi \in C_c^1(\Omega). \tag{6.126}$$

Consequently, letting $\varepsilon \to 0^+$ on both sides of (6.122) and using (6.123), (6.124), (6.125) and (6.126), we obtain

$$\iint_\Omega |u|\frac{\partial \varphi}{\partial x}\, dxdy = -\left[\iint_{\Omega_-} -\frac{\partial u}{\partial x}\varphi\, dxdy + \iint_{\Omega_-} 0\varphi\, dxdy + \iint_{\Omega_+} \frac{\partial u}{\partial x}\varphi\, dxdy\right],$$

for all $\varphi \in C_c^1(\Omega)$, which shows that $|u|$ has a weak partial derivative with respect to x given by (6.120). ■

Definition 6.4.9 (Positive and Negative Parts of u)**.** Given $u\colon \Omega \to \mathbb{R}$, we define the positive of u to be the function $u^+\colon \Omega \to \mathbb{R}$ given by

$$u^+(x,y) = \max\{u(x,y), 0\}, \qquad \text{for all } (x,y) \in \Omega.$$

The negative part of u is the function $u^-\colon \Omega \to \mathbb{R}$ defined by

$$u^-(x,y) = \max\{-u(x,y), 0\}, \qquad \text{for all } (x,y) \in \Omega.$$

It follows from the definition of u^+ and u^- that

$$u = u^+ - u^-$$

and

$$|u| = u^+ + u^-.$$

Consequently,

$$u^+ = \frac{1}{2}|u| + \frac{1}{2}u \tag{6.127}$$

and

$$u^- = \frac{1}{2}|u| - \frac{1}{2}u \tag{6.128}$$

Suppose we are given that $u \in L^1_{\text{loc}}(\Omega)$ has a weak partial derivative with respect to x, $\dfrac{\partial u}{\partial x}$, that is in $L^1_{\text{loc}}(\Omega)$. It then follows from Proposition 6.4.8 that $|u|$ has a weak partial derivative with respect to x given in (6.119). Then, using (6.127) and Proposition 6.4.3, we obtain that u^+ has a weak partial derivative with respect to x given by

$$\frac{\partial u^+}{\partial x} = \frac{1}{2}\frac{\partial |u|}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial x},$$

or, in view of (6.119),

$$\frac{\partial u^+}{\partial x} = \begin{cases} \dfrac{\partial u}{\partial x}, & \text{if } u > 0; \\[2ex] 0, & \text{if } u \leqslant 0. \end{cases}$$

Similarly, using (6.119), (6.128), Propositions 6.4.3 and Proposition 6.4.8, u^- has a weak partial derivative with respect to x given by

$$\frac{\partial u^-}{\partial x} = \begin{cases} 0, & \text{if } u \geqslant 0; \\[2ex] -\dfrac{\partial u}{\partial x}, & \text{if } u < 0. \end{cases}$$

Similar calculations hold true for the weak partial derivative with respect to y.

6.4.5 The Space $W^{1,2}(\Omega)$

The arguments presented in the previous section carry over to the case in which the functions are assumed to be in $L^2_{\text{loc}}(\Omega)$. In this case, we say that $u \in L^2_{\text{loc}}(\Omega)$ has a weak partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ in $L^2_{\text{loc}}(\Omega)$; or simply, L^2 weak partial derivatives. In this case we obtain the following approximation results.

Proposition 6.4.10. Let $u \in L^2_{\text{loc}}(\Omega)$ have weak partial derivatives that are in $L^2_{\text{loc}}(\Omega)$. Then, $u_\varepsilon \in C^1(\Omega_\varepsilon)$,

$$\frac{\partial u_\varepsilon}{\partial x} = \psi_\varepsilon * \frac{\partial u}{\partial x},$$

$$\frac{\partial u_\varepsilon}{\partial y} = \psi_\varepsilon * \frac{\partial u}{\partial y},$$

and

$$\frac{\partial u_\varepsilon}{\partial x} \to \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial y} \to \frac{\partial u}{\partial y} \quad \text{in} \quad L^2_{\text{loc}}(\Omega) \quad \text{as } \varepsilon \to 0^+.$$

Thus, weak derivatives can be approximated in $L^2_{\text{loc}}(\Omega)$ by C^1 functions.

Proposition 6.4.11. A function $u \in L^2_{\text{loc}}(\Omega)$ has a weak partial derivative $\dfrac{\partial u}{\partial x} \in L^2_{\text{loc}}(\Omega)$ if and only if there exists a sequence of functions (u_m) in $C^1(\Omega)$ and a function $v \in L^2_{\text{loc}}(\Omega)$ such that

$$u_m \to u \quad \text{and} \quad \frac{\partial u_m}{\partial x} \to v \quad \text{in } L^2_{\text{loc}}(\Omega) \quad \text{as } m \to \infty. \tag{6.129}$$

A similarly assertion holds true for the weak partial derivative with respect to y.

Define $W^{1,2}(\Omega)$ to be the space of functions $u \in L^2(\Omega)$ that have weak partial derivatives in $L^2(\Omega)$. We can endow $W^{1,2}(\Omega)$ with the norm

$$\|u\|_{1,2} = \sqrt{\iint_\Omega u^2 \, dxdy + \iint_\Omega |\nabla u|^2 \, dxdy}, \quad \text{for } u \in W^{1,2}(\Omega). \tag{6.130}$$

We note that the norm given in (6.130) is induced by the real inner product

$$\langle u, v\rangle_{1,2} = \int_\Omega uv \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \text{for all } u, v \in W^{1,2}(\Omega). \tag{6.131}$$

It follows from the definitions of the norm and inner product in $W^{1,2}(\Omega)$ given in (6.130) and (6.131), respectively, that $W^{1,2}(\Omega)$ is a subspace of $L^2(\Omega)$, which has norm and inner product given by

$$\|u\|_{L^2} = \sqrt{\iint_\Omega u^2 \, dxdy}, \quad \text{for } u \in L^2(\Omega), \tag{6.132}$$

and

$$\langle u, v\rangle_{L^2} = \int_\Omega uv\ dx, \qquad \text{for all } u, v \in L^2(\Omega), \tag{6.133}$$

respectively.

If the integrals in the definitions in (6.130), (6.131), (6.132) and (6.133) are understood to be Lebesgue integrals, then $L^2(\Omega)$ and $W^{1,2}(\Omega)$ are real Hilbert spaces. That is, they are inner product spaces that are complete with respect to the metric induced by their respective norms.

6.4.6 The Space $H_o^1(\Omega)$

In section 6.3 we introduced the Sobolev space $H_o^1(\Omega)$ as the completion of the space V_o with respect to the norm $\|\cdot\|\colon V_o \to \mathbb{R}$ in V_o given in (6.79). Thus, the elements in $H_o^1(\Omega)$ are equivalence classes of Cauchy sequences in V_o determined by the relation

$$(\varphi_m) \sim (\psi_m) \text{ if and only if } \lim_{m\to\infty} \|\varphi_m - \psi_m\| = 0, \tag{6.134}$$

for Cauchy sequences (φ_m) and (ψ_m) in V_o.

In this section we show how to identify a Cauchy sequence (φ_m) in V_o (with respect to the norm in V_o given in (6.79)) with a function $u \in W^{1,2}(\Omega)$ with the property that

$$\lim_{m\to\infty} \|\varphi_m - u\|_{1,2} = 0, \tag{6.135}$$

where $\|\cdot\|_{1,2}$ denotes the norm in $W^{1,2}(\Omega)$ defined in (6.130).

Let (φ_m) be Cauchy sequence in V_o with respect to the norm in V_o given in (6.79). It then follows by virtue of the Poincaré inequality in (6.16) that the sequences

$$(\varphi_m), \qquad \left(\frac{\partial \varphi_m}{\partial x}\right) \text{ and } \left(\frac{\partial \varphi_m}{\partial y}\right) \tag{6.136}$$

are Cauchy sequences in $L^2(\Omega)$. Then, since $L^2(\Omega)$ is a complete normed vector space, each one of the sequences in (6.136) converges in $L^2(\Omega)$ as $m \to \infty$. Let u, v_x and v_y denote the $L^2(\Omega)$ limits of the sequences in (6.136), respectively, as $m \to \infty$; so that,

$$\varphi_m \to u \quad \text{ in } L^2(\Omega) \quad \text{ as } m \to \infty, \tag{6.137}$$

$$\frac{\partial \varphi_m}{\partial x} \to v_x \quad \text{ in } L^2(\Omega) \quad \text{ as } m \to \infty, \tag{6.138}$$

and

$$\frac{\partial \varphi_m}{\partial y} \to v_y \quad \text{ in } L^2(\Omega) \quad \text{ as } m \to \infty. \tag{6.139}$$

We may assume, passing to a subsequences if necessary, that

$$\varphi_m(x, y) \to u(x, y) \quad \text{ as } \quad m \to \infty \qquad \text{for a.e. } (x, y) \in \Omega, \tag{6.140}$$

$$\frac{\partial \varphi_m}{\partial x}(x,y) \to v_x(x,y) \quad \text{as } m\to\infty \quad \text{for a.e. } (x,y)\in\Omega, \tag{6.141}$$

and

$$\frac{\partial \varphi_m}{\partial y}(x,y) \to v_y(x,y) \quad \text{as } m\to\infty \quad \text{for a.e. } (x,y)\in\Omega, \tag{6.142}$$

We may also assume that there exists $Z \in L^2(\Omega)$ such that $Z \geqslant 0$ a.e. in Ω, and

$$|\varphi_m(x,y)| \leqslant Z(x,y), \quad \text{for a.e. } (x,y)\in\Omega \text{ and all } m\in\mathbb{N}, \tag{6.143}$$

Similarly, there exist $Z_x \in L^2(\Omega)$ and $Z_y \in L^2(\Omega)$ such that $Z_x \geqslant 0$ and $Z_y \geqslant 0$ a.e. in Ω, and

$$\left|\frac{\partial \varphi_m}{\partial x}(x,y)\right| \leqslant Z_x(x,y), \quad \text{for a.e. } (x,y)\in\Omega, \tag{6.144}$$

and

$$\left|\frac{\partial \varphi_m}{\partial y}(x,y)\right| \leqslant Z_y(x,y), \quad \text{for a.e. } (x,y)\in\Omega. \tag{6.145}$$

Next, let $\psi \in C_c^1(\Omega)$ and use integration by parts to compute

$$\iint_\Omega \varphi_m \frac{\partial \psi}{\partial x}\, dxdy = -\iint_\Omega \frac{\partial \varphi_m}{\partial x}\psi\, dxdy, \quad \text{for all } m\in\mathbb{N}. \tag{6.146}$$

Thus, in view of (6.145), (6.143), (6.141), (6.140), and by virtue of the Cauchy–Schwarz inequality and the Lebesgue dominated convergence theorem, we get from (6.146), after letting $m\to\infty$, that

$$\iint_\Omega u \frac{\partial \psi}{\partial x}\, dxdy = -\iint_\Omega v_x\psi\, dxdy, \quad \text{for all } \psi\in C_c^1(\Omega). \tag{6.147}$$

Similarly, using (6.145), (6.143), (6.142) and (6.140), we can show that

$$\iint_\Omega u \frac{\partial \psi}{\partial y}\, dxdy = -\iint_\Omega v_y\psi\, dxdy, \quad \text{for all } \psi\in C_c^1(\Omega). \tag{6.148}$$

It follows from (6.147) and (6.148) that v_x and v_y are the weak partial derivatives of u. Since these are in $L^2(\Omega)$, we have shown that $u \in W^{1,2}(\Omega)$. Furthermore, by virtue of (6.137), (6.138) and (6.139),

$$\lim_{m\to\infty} \|\varphi_m - u\|_{1,2} = 0,$$

which is (6.135).

We have therefore shown that to every Cauchy sequence (φ_m) in V_o there corresponds an element $u \in W^{1,2}(\Omega)$ such that (6.135) holds true. Furthermore, in view of (6.134), if (ψ_m) is any Cauchy sequence in V_o in the equivalence class of (φ_m), it follows from the triangle inequality that

$$\lim_{m\to\infty} \|\psi_m - u\|_{1,2} = 0.$$

Consequently, to every equivalence class of Cauchy sequences in V_o, we can associate an element $u \in W^{1,2}(\Omega)$ according to

$$\lim_{m\to\infty} \|\varphi_m - u\|_{1,2} = 0, \tag{6.149}$$

where (φ_m) is any representative of the equivalence class. Thus, $H_o^1(\Omega)$ can be identified with a subspace of $W^{1,2}(\Omega)$ consisting of functions $u \in W^{1,2}(\Omega)$ for which there is a sequence, (φ_m), of functions in $V_o = C_c^1(\Omega)$ for which (6.149) holds true; that is, $H_o^1(\Omega)$ is the topological closure of V_o in $W^{1,2}(\Omega)$.

Appendix A

Some Inequalities

A.1 The Cauchy–Schwarz Inequality

Theorem A.1.1 (The Cauchy–Schwarz Inequality for Functions)**.** Let f and g be continuous functions on $[a,b]$. Then

$$\left|\int_a^b f(x)g(x)\ dx\right| \leqslant \sqrt{\int_a^b |f(x)|^2\ dx}\sqrt{\int_a^b |g(x)|^2\ dx}. \tag{A.1}$$

Furthermore, equality in (A.1) occurs if and only if there exists a scalar λ such that

$$f(x) = \lambda g(x), \qquad \text{for all } x \in [a,b]. \tag{A.2}$$

In terms of the L^2 norm, $\|\cdot\|_2$, this inequality can be written as

$$\left|\int_a^b f(x)g(x)\ dx\right| \leqslant \|f\|_2\|g\|_2.$$

Proof: Define $P\colon \mathbb{R} \to \mathbb{R}$ by

$$P(t) = \int_a^b [f(x) - tg(x)]^2\ dx, \qquad \text{for all } t \in \mathbb{R}. \tag{A.3}$$

Expanding the integrand in (A.3) we obtain

$$P(t) = \int_a^b [f(x)]^2\ dx - 2t\int_a^b f(x)g(x)\ dx + t^2\int_a^b [g(x)]^2\ dx, \qquad \text{for } t \in \mathbb{R}. \tag{A.4}$$

We see in (A.4) that the function $P(t)$ given in (A.3) is a quadratic polynomial in t.

It follows from (A.3) that $P(t) \geqslant 0$ for all $t \in \mathbb{R}$. Consequently, $P(t)$ can have at most one real zero, or root. Hence, the discriminant,

$$4\left(\int_a^b f(x)g(x)\ dx\right)^2 - 4\int_a^b [g(x)]^2\ dx \cdot \int_a^b [f(x)]^2\ dx,$$

cannot be positive. Thus,

$$\left(\int_a^b f(x)g(x)\ dx\right)^2 - \int_a^b [g(x)]^2\ dx \cdot \int_a^b [f(x)]^2\ dx \leqslant 0,$$

or

$$\left(\int_a^b f(x)g(x)\ dx\right)^2 \leqslant \int_a^b [g(x)]^2\ dx \cdot \int_a^b [f(x)]^2\ dx,$$

from which the inequality in (A.1) follows.

Furthermore, equality in (A.1) occurs if and only if there exists $t_o \in \mathbb{R}$ such that $P(t_o) = 0$. Thus, using the definition of $P(t)$ in (A.3), we obtain that

$$\int_a^b [f(x) - t_o g(x)]^2\ dx = 0. \tag{A.5}$$

Thus, since we are assuming that f and g are continuous on $[a, b]$, it follows from (A.5) that

$$f(x) - t_o g(x) = 0, \quad \text{for all } x \in [a, b], \tag{A.6}$$

as a consequence of the Basic Lemma I (see Lemma 3.2.5 on page 29 in these notes). We can now see that (A.2) follows from (A.6) with $\lambda = t_o$. ■

The argument in the proof of Theorem A.1.1 can be used to prove the inequality in the general setting of a vector space V with a real inner product $\langle \cdot, \cdot \rangle$, also known as an inner–product space.

Theorem A.1.2 (The Cauchy–Schwarz Inequality)**.** Let V denote an inner–product space with inner–product $\langle \cdot, \cdot \rangle$. Let $\|v\| = \sqrt{\langle v, v \rangle}$, for all $v \in V$ denote the norm in V associated with the inner product. Then,

$$|\langle v, w \rangle| \leqslant \|v\|\|w\|, \quad \text{for all } v, w \in V. \tag{A.7}$$

Furthermore, equality in (A.7) holds true if and only if

$$v = \lambda w, \tag{A.8}$$

for some scalar λ.

A.2 Poincaré Inequality

We first state and prove the one–dimensional version of the Poincaré Inequality.

Theorem A.2.1 (One–dimensional Poincaré Inequality)**.** There is exists a constant C that depends on the length of the interval $[a, b]$ such that

$$\int_a^b |f(x)|^2\ dx \leqslant C \int_a^b |f'(x)|^2\ dx, \quad \text{for all } f \in C_o^1([a, b]). \tag{A.9}$$

Proof: Let $f \in C_o^1([a,b])$. Then, $f(a) = 0$ and, using the Fundamental Theorem of Calculus,

$$f(x) = \int_a^x f'(t)\ dt, \qquad \text{for all } x \in [a,b]. \tag{A.10}$$

Taking the absolute value on both sides of (A.10) we get the estimate

$$|f(x)| \leqslant \int_a^x |f'(t)|\ dt, \qquad \text{for all } x \in [a,b]. \tag{A.11}$$

Next, apply the Cauchy–Schwarz inequality (A.1) on the right–hand side of the estimate in (A.11) to get

$$|f(x)| \leqslant \sqrt{x-a} \cdot \left(\int_a^x |f'(t)|^2\ dt \right)^{1/2}, \qquad \text{for all } x \in [a,b], \tag{A.12}$$

where we have taken $g = 1$ in (A.1).

Squaring on both sides of (A.12) then yields

$$|f(x)|^2 \leqslant (x-a) \cdot \int_a^x |f'(t)|^2\ dt, \qquad \text{for all } x \in [a,b],$$

from which we get that

$$|f(x)|^2 \leqslant (x-a) \cdot \int_a^b |f'(t)|^2\ dt, \qquad \text{for all } x \in [a,b]. \tag{A.13}$$

Finally, integrate on both sides of the estimate in (A.13) to get

$$\int_a^b |f(x)|^2\ dx \leqslant \frac{(b-a)^2}{2} \cdot \int_a^b |f'(t)|^2\ dt. \tag{A.14}$$

The inequality in (A.9) now follows from (A.14) with $C = \dfrac{(b-a)^2}{2}$. ■

Next, we state and prove the Poincaré inequality in two dimensions.

Theorem A.2.2 (Two–dimensional Poincaré Inequality)**.** Let $\Omega \subset \mathbb{R}^2$ be open and bounded. There is exists a constant C that depends on the diameter of Ω such that

$$\iint_\Omega |u(x,y)|^2\ dxdy \leqslant C \iint_\Omega |\nabla u(x,y)|^2\ dxdy, \qquad \text{for all } u \in C_c^1(\Omega,\mathbb{R}), \tag{A.15}$$

where $C_c^1(\Omega,\mathbb{R})$ denotes the space of C^1, real–valued functions that have compact support in Ω.

Proof. Let Ω be a bounded, open subset of $\mathbb{R}^2$ with piece–wise smooth boundary, $\partial\Omega$. Given $u \in C_c^1(\Omega,\mathbb{R})$, we can extend to be 0 outside of Ω; so that, $u \in C_c^1(\mathbb{R}^2,\mathbb{R})$.

Since Ω is bounded, there exists a rectangle $R = [x_o, x_1] \times [y_o, y_1]$ such that $\overline{\Omega} \subseteq R$. Thus,

$$u(x,y) = 0, \qquad \text{for all } (x,y) \in \partial R. \tag{A.16}$$

Furthermore, we may assume that

$$x_1 - x_o \leqslant d \quad \text{ and } \quad y_1 - y_o \leqslant d, \tag{A.17}$$

where d is the diameter of Ω; that is,

$$d = \sup_{u,v\in\Omega} |u - v|. \tag{A.18}$$

For fixed $y \in \mathbb{R}$, define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = u(t,y), \qquad \text{for } t \in \mathbb{R}.$$

Then,

$$f'(t) = u_x(t,y), \qquad \text{for } t \in \mathbb{R}.$$

Thus, by the fundamental theorem of Calculus, for $(x,y) \in \Omega$,

$$f(x) - f(x_o) = \int_{x_o}^{x} u_x(t,y)\ dt,$$

or

$$u(x,y) - u(x_o,u) = \int_{x_o}^{x} u_x(t,y)\ dt;$$

so that, in view of (A.16),

$$u(x,y) = \int_{x_o}^{x} u_x(t,y)\ dt, \qquad \text{for } (x,y) \in \Omega. \tag{A.19}$$

Taking absolute values on both sides of (A.19) and applying the Cauchy–Schwarz inequality on the right–hand side, we obtain that

$$|u(x,y)| \leqslant \sqrt{x - x_o}\sqrt{\int_{x_o}^{x} |u_x(t,y)|^2\ dt}, \qquad \text{for } (x,y) \in \Omega. \tag{A.20}$$

Squaring on both sides of (A.20) we obtain the estimate

$$|u(x,y)|^2 \leqslant (x - x_o)\int_{x_o}^{x_1} |u_x(t,y)|^2\ dt, \qquad \text{for } (x,y) \in \Omega, \tag{A.21}$$

since $[x_o, x] \subset [x_o, x_1]$, for $(x,y) \in \Omega$.

Integrate on both sides of (A.21) with respect to x from x_o to x_1 to get

$$\int_{x_o}^{x_1} |u(x,y)|^2\ dx \leqslant \frac{(x_1 - x_o)^2}{2}\int_{x_o}^{x_1} |u_x(t,y)|^2\ dt,$$

or, in view of (A.18),

$$\int_{x_o}^{x_1} |u(x,y)|^2 \, dx \leqslant \frac{d^2}{2} \int_{x_o}^{x_1} |u_x(x,y)|^2 \, dx, \tag{A.22}$$

where d is given in (A.18).

Next, integrate on both sides of (A.22) with respect to y from y_o to y_1 to get

$$\int_{y_o}^{y_1} \int_{x_o}^{x_1} |u(x,y)|^2 \, dxdy \leqslant \frac{d^2}{2} \int_{y_o}^{y_1} \int_{x_o}^{x_1} |u_x(x,y)|^2 \, dxdy,$$

or

$$\iint_R |u(x,y)|^2 \, dxdy \leqslant \frac{d^2}{2} \iint_R |u_x(x,y)|^2 \, dxdy. \tag{A.23}$$

Now, since $u = 0$ outside of Ω, it follows from (A.23) that

$$\iint_\Omega |u(x,y)|^2 \, dxdy \leqslant \frac{d^2}{2} \iint_\Omega |u_x(x,y)|^2 \, dxdy. \tag{A.24}$$

Finally, since

$$|u_x(x,y)|^2 \leqslant |u_x(x,y)|^2 + |u_y(x,y)|^2 = |\nabla u(x,y)|^2, \quad \text{for all } (x,y) \in \Omega,$$

the inequality in (A.15) follows from (A.24) with $C = \dfrac{d^2}{2}$. ■

A.3 Wirtinger's Inequality

We used the following inequality in the proof of the isoperimetric theorem in Section 5.4.

Theorem A.3.1 (Wirtinger's Inequality). Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a 2π–periodic function with mean 0; so that,

$$f(t + 2\pi) = f(t), \quad \text{for all } t \in \mathbb{R}; \tag{A.25}$$

$$f'(t + 2\pi) = f'(t), \quad \text{for all } t \in \mathbb{R};$$

and

$$\int_0^{2\pi} f(t) \, dt = 0. \tag{A.26}$$

Then,

$$\int_0^{2\pi} (f(t))^2 \, dt \leqslant \int_0^{2\pi} (f'(t))^2 \, dt. \tag{A.27}$$

Equality in (A.27) occurs if and only if

$$f(t) = a\cos t + b\sin t, \quad \text{for all } t \in \mathbb{R}, \tag{A.28}$$

for some real constants a and b.

We will prove Theorem A.3.1 by applying the argument in the proof of the following result, which can be viewed as another version of the one–dimensional Poincaré inequality:

Theorem A.3.2. Let $f \in C_o^1([0,\pi]$. Then,

$$\int_0^\pi (f(t))^2\ dt \leqslant \int_0^\pi (f'(t))^2\ dt. \tag{A.29}$$

Equality in (A.29) occurs if and only if

$$f(t) = c\sin t, \qquad \text{for all } t \in [0,\pi], \tag{A.30}$$

for some real constant c.

Proof: In this proof we follow the arguments in Section 7.6 of [HLP34].

Implicit in the definition of $C_o^1([0,\pi]$ is the fact that

$$\lim_{t\to 0^+} f'(t) \text{ and } \lim_{t\to\pi^-} f'(t) \text{ exist.} \tag{A.31}$$

The idea of the proof is to find a function $u \in C^1((0,\pi),\mathbb{R})$ with the property that

$$\int_0^\pi (f'(t))^2\ dt - \int_0^\pi (f(t))^2\ dt = \int_0^\pi (f'(t) - u(t)f(t))^2\ dt. \tag{A.32}$$

We show how to find the function u.

Assuming that we have found $u \in C^1((0,\pi),\mathbb{R})$ for which (A.32) holds true, compute

$$\begin{aligned}\int_0^\pi (f'(t) - u(t)f(t))^2\ dt \quad &= \quad \int_0^\pi (f'(t))^2\ dt - 2\int_0^\pi u(t)f(t)f'(t)\ dt \\ &\qquad + \int_0^\pi (u(t))^2(f(t))^2\ dt\end{aligned} \tag{A.33}$$

Combining (A.32) and (A.33) we then get that

$$2\int_0^\pi u(t)f(t)f'(t)\ dt - \int_0^\pi (f(t))^2((u(t))^2+1)\ dt = 0. \tag{A.34}$$

Next, write

$$2\int_0^\pi u(t)f(t)f'(t)\ dt = \int_0^\pi u(t)\frac{d}{dt}[(f(t))^2]\ dt$$

and integrate by parts to compute

$$2\int_0^\pi u(t)f(t)f'(t)\ dt = (f(t))^2u(t)\Big|_0^\pi - \int_0^\pi (f(t))^2u'(t)\ dt;$$

so that, since we are assuming that $f(0) = f(\pi) = 0$,

$$2\int_0^\pi u(t)f(t)f'(t)\ dt = -\int_0^\pi (f(t))^2 u'(t)\ dt. \tag{A.35}$$

Combining (A.35) and (A.34) then yields

$$\int_0^\pi (f(t))^2(u'(t) + (u(t))^2 + 1)\ dt = 0,$$

from which we obtain that

$$u'(t) + (u(t))^2 + 1 = 0, \qquad \text{for all } t \in (0,\pi), \tag{A.36}$$

where we have used the Basic Lemma 1 on page 29 of these notes.

It follows from (A.36) that if $u \in C^1((0,\pi),\mathbb{R})$ is such that (A.32) holds true, then u must solve the first order differential equation

$$\frac{du}{dt} = -(1+u^2),$$

which we can solve by separation of variables to yield

$$u(t) = -\tan(t-c), \tag{A.37}$$

for some constant of integration c.

For the function u in (A.37) to be defined in the interval $(0,\pi)$, we take c in (A.37) to be $\dfrac{\pi}{2}$; so that,

$$u(t) = -\tan\left(t - \frac{\pi}{2}\right), \qquad \text{for } 0 < t < \pi,$$

which we can write as

$$u(t) = \frac{\cos t}{\sin t}, \qquad \text{for } 0 < t < \pi. \tag{A.38}$$

Substituting u in (A.38) into the integrand on the right–hand side of (A.32), we obtain that

$$\int_0^\pi (f'(t))^2\ dt - \int_0^\pi (f(t))^2\ dt = \int_0^\pi \left(f'(t) - \frac{(\cos t)f(t)}{\sin t}\right)^2\ dt. \tag{A.39}$$

Thus, we need to make sure that the integral on the right–hand side of (A.39) is defined. This is where we need to use the assumption in (A.31). Indeed, using the assumption that $f(0) = 0$ and L'Hospital's Rule, we compute

$$\lim_{t\to 0^+} \frac{(\cos t)f(t)}{\sin t} = \lim_{t\to 0^+} \frac{(\cos t)f'(t) - (\sin t)f(t)}{\cos t} = \lim_{t\to 0^+} f'(t),$$

which exists, according to (A.31). Similarly,

$$\lim_{t\to\pi^-}\frac{(\cos t)f(t)}{\sin t}=\lim_{t\to\pi^-}\frac{(\cos t)f'(t)-(\sin t)f(t)}{\cos t}=\lim_{t\to\pi^-}f'(t)$$

also exists.

We can now see that the inequality in (A.29) follows from (A.39). Also, equality in (A.29)) occurs, according to (A.39), if and only if

$$\int_0^\pi\left(f'(t)-\frac{(\cos t)f(t)}{\sin t}\right)^2\,dt=0. \tag{A.40}$$

Thus, using the Basic Lemma I on page 29 in these notes, we obtain from (A.40) that

$$f'(t)-\frac{(\cos t)f(t)}{\sin t}=0,\qquad \text{for all } t\in(0,\pi). \tag{A.41}$$

Consequently, f must solve the differential equation

$$\frac{f'(t)}{f(t)}=\frac{\cos t}{\sin t},\qquad \text{for all } t\in(0,\pi), \tag{A.42}$$

which we can solve by integrating on both sides of (A.42) to get

$$\ln|f(t)|=\ln|\sin t|+c_1,\qquad \text{for all } t\in(0,\pi). \tag{A.43}$$

Thus, exponentiating on both sides of (A.43), and using the continuity of f and sin,

$$f(t)=c\sin t,\qquad \text{for all } t\in[0,\pi],$$

for some real constant c, which is the assertion in (A.30). ■

Proof of Theorem A.3.1: Let $f\in C^1(\mathbb{R},\mathbb{R})$ be a 2π–periodic function with mean 0; so that, (A.25) and (A.26) hold true.

Consider the function $h\colon\mathbb{R}\to\mathbb{R}$ given by

$$h(t)=f(t)-f(t+\pi),\qquad \text{for all } t\in\mathbb{R}. \tag{A.44}$$

Note that

$$h(0)=f(0)-f(\pi), \tag{A.45}$$

and

$$h(\pi)=f(\pi)-f(2\pi);$$

so that,in view of (A.25),

$$h(\pi)=f(\pi)-f(0). \tag{A.46}$$

Thus, for the case in which $f(0)-f(\pi)\neq 0$, $h(0)$ and $h(\pi)$ have opposite signs, in view of (A.45) and (A.46). Consequently, we can invoke the Intermediate

Value Theorem to conclude that there exists $t_o \in [0, \pi)$ such that $h(t_o) = 0$. It then follows from (A.44) that

$$f(t_o) = f(t_o + \pi).$$

Set

$$c = f(t_o) = f(t_o + \pi), \tag{A.47}$$

and define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(t) = f(t) - c, \quad \text{ for all } t \in \mathbb{R}. \tag{A.48}$$

It follows from (A.47) and (A.48) that

$$g(t_o) = 0 \quad \text{ and } \quad g(t_o + \pi) = 0. \tag{A.49}$$

We also have from (A.48) that $g \in C^1(\mathbb{R}, \mathbb{R})$, 2π–periodic,

$$g'(t) = f'(t), \quad \text{ for all } t \in \mathbb{R}, \tag{A.50}$$

and

$$\int_0^{2\pi} (g(t))^2\ dt = \int_0^{2\pi} (f(t))^2\ dt + 2\pi c^2, \tag{A.51}$$

where we have used (A.26).

Motivated by the definition of the function u in (A.38) in the proof of Theorem A.3.2, we define

$$u(t) = \frac{\cos(t - t_o)}{\sin(t - t_o)}, \quad \text{ for } t \neq t_o + k\pi, \text{ for } k \in \mathbb{Z}, \tag{A.52}$$

and consider the integral

$$\int_0^{2\pi} (g'(t) - u(t)g(t))^2\ dt = \int_0^{2\pi} (f'(t) - u(t)g(t))^2\ dt, \tag{A.53}$$

where we have used (A.50).

We first show that the integral on the right–hand side of (A.53) is defined. To see this observe that the function ug is continuous on $[0, 2\pi)$ except at the points t_o and $t_o + \pi$. At the points of discontinuity, however, we can use (A.49) and L'Hospital's Rule to compute

$$\lim_{t \to t_o} \frac{g(t)\cos(t - t_o)}{\sin(t - t_o)} = \lim_{t \to t_o} \frac{f'(t)\cos(t - t_o) - g(t)\sin(t - t_o)}{\cos(t - t_o)} = f'(t_o),$$

where we have used (A.50); so that,

$$\lim_{t \to t_o} [u(t)g(t)] = f'(t_o). \tag{A.54}$$

Similarly,

$$\lim_{t \to t_o + \pi} [u(t)g(t)] = f'(t_o + \pi). \tag{A.55}$$

Thus, the integrand on the right–hand side can be redefined to be a continuous function on $[0, 2\pi]$; thus, the integral in (A.53) is finite.

The calculations in the proof of Theorem A.52 also reveal that the function u given in (A.52) solves the differential equation

$$u'(t) = -((u(t))^2 + 1), \quad \text{for } t \neq t_o + k\pi, \text{ for } k \in \mathbb{Z}, \tag{A.56}$$

Next, we compute

$$\begin{aligned}\int_0^{2\pi} (g'(t) - u(t)g(t))^2 \, dt &= \int_0^{2\pi} (g'(t))^2 \, dt - \int_0^{2\pi} 2g(t)g'(t)u(t) \, dt \\ &\quad + \int_0^{2\pi} (u(t))^2 (g(t))^2 \, dt,\end{aligned} \tag{A.57}$$

where

$$\int_0^{2\pi} 2g(t)g'(t)u(t) \, dt = \int_0^{2\pi} u(t) \frac{d}{dt}[(g(t))^2] \, dt;$$

so that, integrating by parts,

$$\int_0^{2\pi} 2g(t)g'(t)u(t) \, dt = u(t)(g(t))^2 \Big|_0^{2\pi} - \int_0^{2\pi} (g(t))^2 u'(t) \, dt;$$

Thus, since u and g are 2π periodic functions, and in view of (A.54) and (A.55),

$$\int_0^{2\pi} 2g(t)g'(t)u(t) \, dt = -\int_0^{2\pi} (g(t))^2 u'(t) \, dt,$$

or, using (A.56),

$$\int_0^{2\pi} 2g(t)g'(t)u(t) \, dt = \int_0^{2\pi} (g(t))^2 (u(t))^2 \, dt + \int_0^{2\pi} (g(t))^2 \, dt. \tag{A.58}$$

Combining (A.57) and (A.58) yields

$$\int_0^{2\pi} (g'(t) - u(t)g(t))^2 \, dt = \int_0^{2\pi} (g'(t))^2 \, dt - \int_0^{2\pi} (g(t))^2 \, dt,$$

or, using (A.50) and (A.51),

$$\int_0^{2\pi} (f'(t))^2 \, dt - \int_0^{2\pi} (f(t))^2 \, dt = 2\pi c^2 + \int_0^{2\pi} (g'(t) - u(t)g(t))^2 \, dt. \tag{A.59}$$

We can now see that the inequality in (A.27) follows from (A.59). Furthermore, if equality in (A.27) holds true, we obtain from (A.59) that

$$c = 0 \tag{A.60}$$

and g must solve the differential equation

$$g' - ug = 0. \tag{A.61}$$

The differential equation in (A.61) can be solved in the same way that we solved (A.41) in the proof of Theorem A.3.2 to obtain that

$$g(t) = A\sin(t - t_o), \quad \text{for all } t \in \mathbb{R}, \tag{A.62}$$

for some constant A.

Combining (A.48), (A.60) and (A.62), we then obtain

$$f(t) = A\sin(t - t_o), \quad \text{for all } t \in \mathbb{R},$$

which can be written as

$$f(t) = -A\sin(t_o)\cos(t) + A\cos(t_o)\sin(t), \quad \text{for all } t \in \mathbb{R}. \tag{A.63}$$

Setting $a = -A\sin(t_o)$ and $b = A\cos(t_o)$ in (A.63), we obtain the assertion in (A.28), and the proof of Wirtinger's Inequality is now complete. ■

Appendix B

Theorems About Integration

B.1 Differentiating Under the Integral Sign

Solutions of problems in the Calculus of Variations often require the differentiation of functions defined in terms of integrals of other functions. In many instance this involves differentiation under the integral sign. In this appendix we preset a few results that specify conditions under which differentiation under the integral sign is valid.

Proposition B.1.1 (Differentiation Under the Integral Sign)**.** Suppose that $h\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is a function whose partial derivative with respect to y exists for almost all $(x,y)\in[a,b]\times\mathbb{R}$. Define $H\colon\mathbb{R}\to\mathbb{R}$ by

$$H(y)=\int_a^b h(x,y)\;dx,\qquad \text{for all}\quad y\in\mathbb{R}. \tag{B.1}$$

Assume that the functions h and $\dfrac{\partial h}{\partial y}$ are absolutely integrable over $[a,b]$. Then, H is differentiable and its derivative is given by

$$H'(y)=\int_a^b \frac{\partial}{\partial y}[h(x,y)]\;dx. \tag{B.2}$$

Remark B.1.2. A function $f\colon [a,b]\to\mathbb{R}$ is said to be absolutely integrable if

$$\int_a^b |f(x)|\;dx<\infty, \tag{B.3}$$

where the integral on the left–hand side of (B.3) is taken to be the Lebesgue integral of $|f|$; thus, in the statement of Proposition B.1.1 we are assuming that

$|h|$ and $\left|\dfrac{\partial h}{\partial y}\right|$ are Lebesgue integrable. In the proof of Proposition B.1.1 that we will present in this appendix, we will assume that $|h|$ and $\left|\dfrac{\partial h}{\partial y}\right|$ are bounded and Riemann integrable; thus, we will be able to use the theory of Riemann integration. However, the statement in the proposition is true in the generality in which it was stated, and can be proved using the full power of the theory of Lebesgue integration.

Before proving special case of Proposition B.1.1, we present a few results about bounded, Riemann integrable functions. We note that continuous functions satisfy these two properties on $[a, b]$.

Lemma B.1.3. Let $f\colon [a,b] \to \mathbb{R}$ be bounded and Riemann integrable over $[a, b]$. Define $F\colon [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\ dt, \qquad \text{for all } x \in [a,b]. \tag{B.4}$$

Then, F is continuous on $[a, b]$.

Proof: Since we are assume that f is bounded on $[a, b]$, there exists a constant $M > 0$ such that

$$|f(t)| \leqslant M, \qquad \text{for all } t \in [a,b]. \tag{B.5}$$

For $x, y \in [a, b]$, use the definition of F in (B.4) to compute

$$F(y) - F(x) = \int_x^y f(t)\ dt. \tag{B.6}$$

Then, taking absolute value on both sides of (B.6),

$$|F(y) - F(x)| \leqslant \int_x^y |f(t)|\ dt,$$

for the case in which $x \leqslant y$. Thus, in view of (B.5),

$$|F(y) - F(x)| \leqslant M|y - x|,$$

from which the continuity of F follows. ■

Lemma B.1.4. Let $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying $x \mapsto f(x, y)$ is bounded and Riemann integrable for all $y \in \mathbb{R}$, and $y \mapsto f(x, y)$ is continuous for almost all $x \in [a, b]$. Define $F\colon [a,b] \to \mathbb{R}$ by

$$F(y) = \int_a^b f(x,y)\ dx, \qquad \text{for all } y \in \mathbb{R}. \tag{B.7}$$

Then, F is continuous in $\mathbb{R}$.

Proof: Fix $y \in \mathbb{R}$. Given $\varepsilon > 0$, for each $x \in [a, b]$ there exists $\delta(x) > 0$ such that

$$|z - y| < \delta(x) \implies |f(x, z) - f(x, y)| < \varepsilon.$$

Now, since $[a, b]$ is compact, we can find $\delta > 0$ such that

$$|z - y| < \delta \implies |f(x, z) - f(x, y)| < \varepsilon, \quad \text{for all } x \in [a, b].$$

Consequently, using the definition of F in (B.7),

$$|z - y| < \delta \implies |F(z) - F(y)| < (b - a)\varepsilon,$$

from which the continuity of F at y follows. ■

Proof of special case of Proposition B.1.1. We assume that $|h|$ and $\left|\dfrac{\partial h}{\partial y}\right|$ are bounded and Riemann integrable.

Fix $y \in \mathbb{R}$ and let $z \in \mathbb{R}$. Then,

$$h(x, z) = h(x, y) + \frac{\partial h}{\partial y}(x, y)(z - y) + o(|z - y|), \tag{B.8}$$

where the expression $o(|z - y|)$ is understood to mean

$$\lim_{z \to y} \frac{o(|z - y|)}{|z - y|} = 0. \tag{B.9}$$

By virtue of the compactness of $[a, b]$, we may assume that (B.8) and (B.9) hold uniformly for all $x \in [a, b]$.

Thus, using the definition of H in (B.1) we get that

$$H(z) = H(y) + \int_a^b \frac{\partial h}{\partial y}(x, y)(z - y)\ dx + o(|z - y|)(b - a), \tag{B.10}$$

where we have substituted the expression for $h(x, z)$ into (B.1) to obtain $H(z)$.

Rearranging the expression in (B.10)we get

$$H(z) - H(y) = (z - y)\int_a^b \frac{\partial h}{\partial y}(x, y)\ dx + (b - a)o(|z - y|), \tag{B.11}$$

Next, divide on both sides of (B.11) by $z - y$, assuming that $z \neq y$, to get

$$\frac{H(z) - H(y)}{z - y} = \int_a^b \frac{\partial h}{\partial y}(x, y)\ dx + (b - a)\frac{o(|z - y|}{z - y}), \quad \text{for } z \neq y. \tag{B.12}$$

Finally, let $z \to y$ on both sides of (B.12) and use (B.9) to get

$$\lim_{z \to y} \frac{H(z) - H(y)}{z - y} = \int_a^b \frac{\partial h}{\partial y}(x, y)\ dx,$$

which is the assertion in (B.2). ■

Proposition B.1.5 (Differentiation Under the Integral Sign and Fundamental Theorem of Calculus)**.** Suppose that $h\colon [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Define

$$H(y,t)=\int_a^t h(x,y,t)\ dx, \quad \text{for all } y\in\mathbb{R},\ t\in\mathbb{R}. \tag{B.13}$$

Assume that h has partial derivatives $\dfrac{\partial}{\partial y}[h(x,y,t)]$ and $\dfrac{\partial}{\partial t}[h(x,y,t)]$ that are absolutely integrable over $[a,b]$. Then, H has partial derivatives are given by

$$\frac{\partial}{\partial y}[H(y,t)]=\int_a^t \frac{\partial}{\partial y}[h(x,y,t)]\ dx \tag{B.14}$$

and

$$\frac{\partial}{\partial t}[H(y,t)]=h(t,y,t)+\int_a^t \frac{\partial}{\partial t}[h(x,y,t)]\ dx. \tag{B.15}$$

Proposition B.1.5 can be viewed as a generalization of the Fundamental Theorem of Calculus and is a special case of Leibnitz Rule.

Proof of special case of Proposition B.1.5. As in the proof of Proposition B.1.1, we prove this proposition for the special case in which the partial derivatives $\dfrac{\partial}{\partial y}[h(x,y,t)]$ and $\dfrac{\partial}{\partial t}[h(x,y,t)]$ are bounded and Riemann integrable.

The proof of the assertion in (B.14) follows from Proposition B.1.1 because t is fixed when computing the partial derivative with respect to y.

To establish the assertion in (B.15), fix t and y in $\mathbb{R}$, let $\tau\in\mathbb{R}$ and use (B.13) to compute

$$H(y,\tau)=\int_a^\tau h(x,y,\tau)\ dx. \tag{B.16}$$

Next, subtract the expression in (B.13) from that in (B.16)to get

$$H(y,\tau)-H(y,t)=\int_a^\tau h(x,y,\tau)\ dx-\int_a^t h(x,y,t)\ dx,$$

which we can rewrite as

$$\begin{aligned} H(y,\tau)-H(y,t) &= \int_a^\tau h(x,y,\tau)\ dx-\int_a^t h(x,y,\tau)\ dx \\ &\quad +\int_a^t h(x,y,\tau)\ dx-\int_a^t h(x,y,t)\ dx, \end{aligned}$$

or

$$\begin{aligned} H(y,\tau)-H(y,t) &= \int_t^\tau h(x,y,\tau)\ dx \\ &\quad +\int_a^t [h(x,y,\tau)-h(x,y,t)]\ dx. \end{aligned} \tag{B.17}$$

We shall first see that the continuity of h implies that

$$\lim_{\tau\to t}\frac{1}{\tau-t}\int_t^\tau h(x,y,\tau)\;dx = h(t,y,t). \tag{B.18}$$

To establish (B.18), first observe that

$$h(t,y,t) = \frac{1}{\tau-t}\int_t^\tau h(t,y,t)\;dx;$$

so that,

$$\frac{1}{\tau-t}\int_t^\tau h(x,y,\tau)\;dx - h(t,y,t) = \frac{1}{\tau-t}\int_t^\tau [h(x,y,\tau)-h(t,y,t)]\;dx. \tag{B.19}$$

Let $\varepsilon > 0$ be given. Since we are assuming that h is continuous, there exists $\delta > 0$ such that

$$|x-t|<\delta \text{ and } |\tau-t|<\delta \implies |h(x,y,\tau)-h(t,y,t)|<\varepsilon. \tag{B.20}$$

Thus, assuming that $t < \tau$, we obtain from (B.20) and (B.19) that

$$|\tau-t|<\delta \implies \left|\frac{1}{\tau-t}\int_t^\tau h(x,y,\tau)\;dx - h(t,y,t)\right| < \varepsilon,$$

which shows (B.18) for the case $t < \tau$. Similar calculations yield the result for $t > \tau$.

Next, use the assumption that the partial derivative of h with respect to t exists to get

$$h(x,y,\tau) = h(x,y,t) + \frac{\partial h}{\partial t}(x,y,t)(\tau-t) + o(|\tau-t|).$$

Then,

$$h(x,y,\tau) - h(x,y,t) = (\tau-t)\frac{\partial h}{\partial t}(x,y,t) + o(|\tau-t|);$$

so that, integrating on both sides with respect to x from a to t

$$\int_a^t [h(x,y,\tau)-h(x,y,t)]\;dx = (\tau-t)\int_a^t \frac{\partial h}{\partial t}(x,y,t)\;dx + o(|\tau-t|).$$

Consequently,

$$\lim_{\tau\to t}\frac{1}{\tau-t}\int_a^t [h(x,y,\tau)-h(x,y,t)]\;dx = \int_a^t \frac{\partial h}{\partial t}(x,y,t)\;dx. \tag{B.21}$$

Next, divide both sides of the expression in (B.17) by $\tau - t$, for $\tau \neq t$, to get

$$\begin{aligned}\frac{H(y,\tau)-H(y,t)}{\tau-t} &= \frac{1}{\tau-t}\int_t^\tau h(x,y,\tau)\;dx \\ &\quad + \frac{1}{\tau-t}\int_a^t [h(x,y,\tau)-h(x,y,t)]\;dx,\end{aligned} \tag{B.22}$$

for $\tau \neq t$.

Finally, let $\tau \to t$ in (B.22), and use the results in (B.18) and (B.21), to get from (B.22) that

$$\lim_{\tau \to t} \frac{H(y,\tau) - H(y,t)}{\tau - t} = h(t,y,t) + \int_a^t \frac{\partial h}{\partial t}(x,y,t)\ dx,$$

which is the assertion in (B.15). ■

B.2 The Divergence Theorem

We begin by stating the two–dimensional version of the divergence theorem. We then present some consequences of the result.

Let U denote an open subset of $\mathbb{R}^2$ and Ω a subset of U such that $\overline{\Omega} \subset U$. We assume that Ω is bounded with boundary, $\partial\Omega$, that can be parmetrized by $\sigma\colon [0,1] \to \mathbb{R}$, where $\sigma(t) = (x(t), y(t))$, for $t \in [0,1]$, with $x, y \in C^1([0,1], \mathbb{R})$ satisfying

$$(\dot{x}(t))^2 + (\dot{y}(t))^2 \neq 0, \quad \text{for all } t \in [0,1], \tag{B.23}$$

(where the dot on top of the variable indicates derivative with respect to t), and $\sigma(0) = \sigma(1)$. Implicit in the definition of a parametrization is the assumption that the map $\sigma\colon [0,1) \to \mathbb{R}^2$ is one–to–one on $[0,1)$. Thus, $\partial\Omega$ is a simple closed curve in U. Observe that the assumption in (B.23) implies that at every point $\sigma(t) \in \partial\Omega$, a tangent vector

$$\sigma'(t) = (\dot{x}(t), \dot{y}(t)), \quad \text{for } t \in [0,1]. \tag{B.24}$$

Let $F\colon U \to \mathbb{R}^2$ denote a C^1 vector field in U; so that,

$$F(x,y) = (P(x,y), Q(x,y)), \quad \text{for } (x,y) \in U, \tag{B.25}$$

where $P\colon U \to \mathbb{R}$ and $Q\colon U \to \mathbb{R}$ are C^1, real–valued functions defined on U.

The **divergence** of the vector field $F \in C^1(U, \mathbb{R}^2)$ given in (B.25) is a scalar field $\text{div}F\colon U \to \mathbb{R}$ defined by

$$\text{div}F(x,y) = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y) \quad \text{for } (x,y) \in U. \tag{B.26}$$

Example B.2.1. Imagine a two–dimensional fluid moving through a region U in the xy–plane. Suppose the velocity of the fluid at a point $(x,y) \in \mathbb{R}^2$ is given by a C^1 vector field $V : U \to \mathbb{R}^2$ in units of distance per time. Suppose that we also know the density of the fluid, $\rho(x,y)$ at any point $(x,y) \in U$ (in units of mass per area), and that $\rho\colon U \to \mathbb{R}$ is a C^1 scalar field. Define

$$F(x,y) = \rho(x,y)V(x,y), \quad \text{for } (x,y) \in U. \tag{B.27}$$

Then F has units of mass per unit length, per unit time. The vector field F in (B.27) is called the **flow field** and it measures the amount of fluid per unit time that goes through a cross section of unit length perpendicular to the direction of V. Thus, to get a measure of the amount of fluid per unit time that crosses the boundary $\partial\Omega$ in direction away from the region Ω, we compute the line integral

$$\oint_{\partial\Omega} F \cdot \widehat{n}\ ds, \tag{B.28}$$

where ds is the element of arc–length along $\partial\Omega$, and $\widehat{n}$ is unit vector that is perpendicular to the curve $\partial\Omega$ and points away from Ω. The expression in (B.28) is called the **flux** of the flow field F across $\partial\Omega$ and it measures the amount of fluid per unit time that crosses the boundary $\partial\Omega$.

On the other hand, the divergence, $\text{div}F$, of the flow field F in (B.27) has units of mass/time $\times$ length^2, and it measures the amount of fluid that diverges from a point per unit time per unit area. Thus, the integral

$$\iint_{\Omega} \text{div}F\ dxdy \tag{B.29}$$

measures the total amount of fluid leaving the region Ω per unit time. In the case where there are not sinks or sources of fluid inside the region Ω, the integrals in (B.28) and (B.29) must be equal; so that,

$$\iint_{\Omega} \text{div}F\ dxdy = \oint_{\partial\Omega} F \cdot \widehat{n}\ ds. \tag{B.30}$$

This is the statement of the **conservation of mass** principle in two dimensions. The expression in (B.30) is the two–dimensional **Divergence Theorem**.

Theorem B.2.2 (The Divergence Theorem in $\mathbb{R}^2$)**.** Let U be an open subset of $\mathbb{R}^2$ and Ω an open subset of U such that $\overline{\Omega} \subset U$. Suppose that Ω is bounded with boundary $\partial\Omega$. Assume that $\partial\Omega$ is a piece–wise C^1, simple, closed curve. Let $F \in C^1(U, \mathbb{R}^2)$. Then,

$$\iint_{\Omega} \text{div}F\ dxdy = \oint_{\partial\Omega} F \cdot \widehat{n}\ ds, \tag{B.31}$$

where $\widehat{n}$ is the outward, unit, normal vector to $\partial\Omega$ that exists everywhere on $\partial\Omega$, except possibly at finitely many points.

For the special case in which $\partial\Omega$ is parmatrized by $\sigma \in C^1([0,1], \mathbb{R}^2)$ satisfying (B.24), $\sigma(0) = \sigma(1)$, the map $\sigma\colon [0,1) \to \mathbb{R}^2$ is one–to–one, and σ is oriented in the counterclockwise sense, the outward unit normal to $\partial\Omega$ is given by

$$\widehat{n}(\sigma(t)) = \frac{1}{|\sigma'(t)|}(\dot{y}(t), -\dot{x}(t)), \qquad \text{for } t \in [0,1]. \tag{B.32}$$

Note that the vector $\widehat{n}$ in (B.32) is a unit vector that is perpendicular to the vector $\sigma'(t)$ in (B.24) that is tangent to the curve at $\sigma(t)$. In follows from

(B.32) that, for the C^1 vector field F given in (B.25), the line integral on the right–hand side of (B.31) can be written as

$$\oint_{\partial\Omega} F\cdot\widehat{n}\; ds = \int_0^1 (P(\sigma(t)), Q(\sigma(t)))\cdot\frac{1}{|\sigma'(t)|}(\dot{y}(t), -\dot{x}(t))\; |\sigma'(t)|\; dt,$$

or

$$\oint_{\partial\Omega} F\cdot\widehat{n}\; ds = \int_0^1 [P(\sigma(t))\dot{y}(t) - Q(\sigma(t))\dot{x}(t)]\; dt,$$

which we can write, using differentials, as

$$\oint_{\partial\Omega} F\cdot\widehat{n}\; ds = \oint_{\partial\Omega}(Pdy - Qdx). \tag{B.33}$$

Thus, using the definition of the divergence of F in (B.26) and (B.33), we can rewrite (B.31) as

$$\iint_{\Omega}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\; dxdy = \oint_{\partial\Omega}(Pdy - Qdx), \tag{B.34}$$

which is another form of the Divergence Theorem in (B.31).

Applying the Divergence Theorem (B.31) to the vector field $F = (Q, -P)$, where $P, Q \in C^1(U, \mathbb{R})$, yields from (B.34) that

$$\iint_{\Omega}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\; dxdy = \oint_{\partial\Omega}(Pdx + Qdy),$$

which is **Green's Theorem**.

As an application of the Divergence Theorem as stated in (B.34), consider the case of the vector field $(P, Q) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$. In this case (B.34) yields

$$\iint_{\Omega} 2\; dxdy = \oint_{\partial\Omega}(xdy - ydx),$$

or

$$2\;\text{area}(\Omega) = \oint_{\partial\Omega}(xdy - ydx),$$

from which we get the formula

$$\text{area}(\Omega) = \frac{1}{2}\oint_{\partial\Omega}(xdy - ydx), \tag{B.35}$$

for the area of the region Ω enclosed by a simple closed curve $\partial\Omega$.

B.3 Integration by Parts

Integration by parts is, perhaps, the most important tool in the calculations in the Calculus of Variations and variational methods. We present here two–dimensional versions of the integration by parts formulas. These can be derived by using the Divergence Theorem, which was presented in the previous section.

Let U denote an open subset of $\mathbb{R}^2$ and Ω be an open, bounded subset of U such that $\overline{\Omega} \subset U$. Assume that the boundary of Ω, $\partial\Omega$, is a piece–wise C^1, simple, closed curve.

Let u and v denote two real–valued functions defined in U and assume that $u, v \in C^1(U, \mathbb{R})$.

Example B.3.1. Define the vector field $F \colon U \to \mathbb{R}^2$ by

$$F(x,y) = u(x,y)v(x,y)\,\widehat{i}, \qquad \text{for all } (x,y) \in U.$$

Then,

$$\operatorname{div}(F) = \frac{\partial}{\partial x}[uv] = \frac{\partial u}{\partial x}\,v + u\,\frac{\partial v}{\partial x}.$$

Then, applying the Divergence Theorem in (B.30),

$$\iint_\Omega \left(\frac{\partial u}{\partial x}\,v + u\,\frac{\partial v}{\partial x}\right)\,dxdy = \oint_{\partial\Omega} uv\widehat{i}\cdot\widehat{n}\,ds,$$

which we can rewrite as

$$\iint_\Omega \frac{\partial u}{\partial x}\,v\,dxdy + \iint_\Omega u\,\frac{\partial v}{\partial x}\,dxdy = \oint_{\partial\Omega} uvn_1\,ds, \tag{B.36}$$

where n_1 is the first component of the outward unit normal vector $\widehat{n}$ on the boundary $\partial\Omega$ or Ω.

From (B.36) we obtain the integration by parts formula

$$\iint_\Omega u\,\frac{\partial v}{\partial x}\,dxdy = \oint_{\partial\Omega} uvn_1\,ds - \iint_\Omega \frac{\partial u}{\partial x}\,v\,dxdy. \tag{B.37}$$

Similar calculations can be used to obtain a second integration by parts formula

$$\iint_\Omega u\,\frac{\partial v}{\partial y}\,dxdy = \oint_{\partial\Omega} uvn_2\,ds - \iint_\Omega \frac{\partial u}{\partial y}\,v\,dxdy, \tag{B.38}$$

where n_2 is the second component of the outward unit normal vector $\widehat{n}$ on the boundary $\partial\Omega$ or Ω.

B.4 Green's Identities

In this section we use the divergence theorem in two dimensions to derive a very useful set of formulas known as green identities. The setup is similar to the one in the previous two sections.

Let U denote an open subset of $\mathbb{R}^2$ and Ω be an open, bounded subset of U such that $\overline{\Omega} \subset U$. Assume that the boundary of Ω, $\partial\Omega$, is a piece–wise C^1, simple, closed curve.

Let u and v denote two real–valued functions defined in U and assume that $u, v \in C^2(U, \mathbb{R})$; that is, the second partial derivatives of u and v are continuous in U. Define the vector field $F \colon U \to \mathbb{R}^2$ by

$$F(x,y) = v(x,y)\nabla u(x,y), \qquad \text{for all } (x,y) \in \mathbb{R}^2,$$

where ∇u denotes the gradient of u; that is,

$$\nabla u(x,y) = \frac{\partial u}{\partial x}(x,y)\widehat{i} + \frac{\partial u}{\partial y}(x,y)\widehat{j}, \quad \text{for all } (x,y) \in U.$$

More succinctly, we can write

$$F = v\nabla u, \tag{B.39}$$

and

$$\nabla u = \frac{\partial u}{\partial x}\widehat{i} + \frac{\partial u}{\partial y}\widehat{j},$$

with the understanding that u, v and ∇u are functions defined in U.

Then,

$$\begin{aligned}
\text{div}F &= \text{div}(v\nabla u) \\
&= \text{div}\left(v\frac{\partial u}{\partial x}\widehat{i} + v\frac{\partial u}{\partial y}\widehat{j}\right) \\
&= \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right);
\end{aligned}$$

so that,

$$\text{div}F = \nabla v \cdot \nabla u + v\Delta u, \tag{B.40}$$

where $\nabla v \cdot \nabla u$ is a dot–product of ∇u and ∇v, and Δu is the two–dimensional Laplacian of u:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Applying the Divergence Theorem in (B.31) to F given in (B.39) and using (B.40), we get

$$\iint_\Omega (\nabla v \cdot \nabla u + v\Delta u)\ dxdy = \oint_{\partial\Omega} v\nabla u \cdot \widehat{n}\ ds, \tag{B.41}$$

where $\nabla u \cdot \widehat{n}$ is the directional derivative of u in the direction of the outward unit normal to the boundary $\partial\Omega$, denoted by $\dfrac{\partial u}{\partial n}$; so that,

$$\frac{\partial u}{\partial n} = \nabla u \cdot \widehat{n}, \quad \text{on } \partial\Omega.$$

We therefore obtain from (B.41) that

$$\iint_\Omega v\Delta u\ dxdy = \oint_{\partial\Omega} v\frac{\partial u}{\partial n}\ ds - \iint_\Omega \nabla v \cdot \nabla u\ dxdy. \tag{B.42}$$

The expression in (B.42) is usually referred to a **Green's Identity I**.

Interchanging the roles of u and v in (B.42) we obtain

$$\iint_{\Omega} u\Delta v \ dxdy = \oint_{\partial\Omega} u\frac{\partial v}{\partial n} \ ds - \iint_{\Omega} \nabla u \cdot \nabla v \ dxdy. \tag{B.43}$$

Subtracting the expression in (B.43) from the expression in (B.42) we get

$$\iint_{\Omega} (v\Delta u - u\Delta v) \ dxdy = \oint_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) \ ds. \tag{B.44}$$

The expression in (B.44) is called a **Green's Identity II**.

Appendix C

Continuity of Functionals

In many of the examples presented in these notes, we consider functionals defined on the vector space $V = C^1([a,b],\mathbb{R})$ of the form

$$J(y) = \int_a^b F(x, y(x), y'(x))\, dx, \qquad \text{for } y \in V,$$

where $F\colon [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function. In this appendix we discuss continuity properties of this type of functionals with respect to the norm some norm defined in V.

C.1 Definition of Continuity

In general, let V denote a normed linear space with norm $\|\cdot\|$. We say that a functional $J\colon V \to \mathbb{R}$ is continuous at $u_o \in V$ if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|u - u_o\| < \delta \Rightarrow |J(u) - J(u_o)| < \varepsilon.$$

If J is continuous at every $u_o \in V$, we say that J is continuous in V.

Example C.1.1. Let $V = C([a,b],\mathbb{R})$ be endowed with the norm

$$\|y\| = \max_{a\leqslant x\leqslant b} |y(x)|, \qquad \text{for every } y \in V. \tag{C.1}$$

Let $g\colon [a,b]\times\mathbb{R}$ be a continuous function and define $J\colon V \to \mathbb{R}$ by

$$J(y) = \int_a^b g(x, y(x))\, dx, \qquad \text{for all } y \in V. \tag{C.2}$$

We will show that J is continuous in V.

First, we consider the spacial case in which

$$g(x, 0) = 0, \qquad \text{for all } x \in [a,b], \tag{C.3}$$

and show that the functional J defined in (C.2) is continuous at y_o, where $y_o(x) = 0$ for all $x \in [a, b]$; that is, in view of (C.3), (C.2) and the definition of continuity given at the start of this section, we show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$v \in V \text{ and } \|v\| < \delta \Rightarrow |J(v)| < \varepsilon. \tag{C.4}$$

We first show that, if $g\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and (C.3) holds true, then for every $\eta > 0$ there exists $\delta > 0$ such that

$$|s| < \delta \Rightarrow |g(x, s)| < \eta. \tag{C.5}$$

To establish this claim, we argue by contradiction. Assume to the contrary that there exists $\eta_o > 0$ such that, for every $n \in \mathbb{N}$, there exists $s_n \in \mathbb{R}$ and $x_n \in [a, b]$ such that

$$|s_n| < \frac{1}{n} \quad \text{and} \quad |g(x_n, y_n)| \geqslant \eta_o, \quad \text{for all } n. \tag{C.6}$$

Now, since $[a, b]$ is compact, we may assume (after passing to a subsequence, if necessary) that there exists $x_o \in [a, b]$ such that

$$x_n \to x_o \quad \text{as} \quad n \to \infty. \tag{C.7}$$

It follows from (C.6), (C.7) and the assumption that g is continuous that

$$|g(x_o, 0)| \geqslant \eta_o. \tag{C.8}$$

However, (C.8) is in direct contradiction with (C.3), since $\eta_o > 0$. We have therefore shown that, if $g\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and (C.3) holds true, then, for every $\eta > 0$, there exists $\delta > 0$ such that (C.5) is true.

Let $\varepsilon > 0$ be given and put

$$\eta = \frac{\varepsilon}{b - a}. \tag{C.9}$$

By what we have just proved, there exists $\delta > 0$ for which (C.5) holds true. Let $v \in V$ be such that $\|v\| < \delta$; then, by the definition of the norm $\|\cdot\|$ in (C.1),

$$|v(x)| < \delta, \quad \text{for all } x \in [a, b]. \tag{C.10}$$

We then get from (C.5) and (C.10) that

$$|g(x, v(x))| < \eta, \quad \text{for all } x \in [a, b]. \tag{C.11}$$

Consequently, integrating on both sides of the estimate in (C.11) from a to b,

$$\int_a^b |g(x, v(x))| \, dx < \eta(b - a). \tag{C.12}$$

In view of (C.12) and (C.9), we see that we have shown that

$$v \in V \text{ and } \|v\| < \delta \Rightarrow \int_a^b |g(x, v(x))| \, dx < \varepsilon,$$

from which (C.4) follows.

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