

Assignment #11

Due on Wednesday, November 13, 2019

Read Section 4.4 on *Differentiable Paths* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

Read Section 4.5.1 on *Differentiability of Paths* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

Read Section 5.1.1 on *Arc Length* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

Background and Definitions

- (*Reparametrizations*) Let $\sigma: [a, b] \rightarrow \mathbb{R}^n$ be a differentiable, one-to-one path. Suppose also that $\sigma'(t)$, is never the zero vector. Let $h: [c, d] \rightarrow [a, b]$ be a differentiable, one-to-one and onto map such that $h'(t) \neq 0$ for all $t \in [c, d]$.

Define $\gamma(t) = \sigma(h(t))$ for all $t \in [c, d]$.

The path $\gamma: [c, d] \rightarrow \mathbb{R}^n$ is called a *reparametrization* of σ

- (*Arc Length Parameter*) Let I denote an open interval in \mathbb{R} , and $\sigma: I \rightarrow \mathbb{R}^n$ be a parametrization of a curve C . For fixed $a \in I$, define

$$s(t) = \int_a^t \|\sigma'(\tau)\| d\tau \quad \text{for all } t \in I. \quad (1)$$

The parameter $s = s(t)$ measures the length along the curve C from the point $\sigma(a)$ to the point $\sigma(t)$.

Do the following problems

1. Show that the arclength parameter defined in (1) is differentiable on I and compute $s'(t)$ for all $t \in I$. Deduce that $s(t)$ is a strictly increasing function of t in I .
2. Let $\gamma: [c, d] \rightarrow \mathbb{R}^n$ be a reparametrization of $\sigma: [a, b] \rightarrow \mathbb{R}^n$.
 - (a) Show that γ is a differentiable, one-to-one path.
 - (b) Compute $\gamma'(t)$ and show that it is never the zero vector.
 - (c) Show that σ and γ have the same image in \mathbb{R}^n .

3. Let C be a curve parametrized by

$$\sigma(t) = \sigma(t) = (e^{kt} \cos t, e^{kt} \sin t), \quad \text{for } t \in [0, 2\pi],$$

where $k \neq 0$. Compute the arc length of C .

4. A particle is following a path in three-dimensional space given by

$$\sigma(t) = (e^t, e^{-t}, 1 - t) \quad \text{for } t \in \mathbb{R}.$$

At time $t_0 = 1$, the particle flies off on a tangent.

- (a) Where will the particle be at time $t_1 = 2$?
 - (b) Will the particle ever hit the xy -plane? If so, find the location on the xy plane where the particle hits.
5. Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geq 0\}$; i.e., C is the upper unit semi-circle. C can be parametrized by

$$\sigma(\tau) = (\tau, \sqrt{1 - \tau^2}), \quad \text{for } -1 \leq \tau \leq 1.$$

- (a) Compute $s(t)$, the arclength along C from $(-1, 0)$ to the point $\sigma(t)$, for $-1 \leq t \leq 1$.
- (b) Compute $s'(t)$, for $-1 < t < 1$, and sketch the graph of s as function of t .
- (c) Show that $\cos(\pi - s(t)) = t$ for all $-1 \leq t \leq 1$, and deduce that

$$\sin(s(t)) = \sqrt{1 - t^2}, \quad \text{for all } -1 \leq t \leq 1.$$