Assignment #6

Due on Wednesday, October 9, 2019

Read Section 3.3 on *Continuous Functions* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

- *(Continuous Function)* Let U denote an open subset of $\mathbb{R}^n$. A function $F\colon U \to \mathbb{R}^m$ is said to be continuous at $v \in U$ if and only if $\displaystyle\lim_{\|w-v\|\to 0} \|F(w) - F(v)\| = 0$.
- *(Image)* If $A \subseteq U$, the *image of* A under the map $F\colon U \to \mathbb{R}^m$, denoted by $F(A)$, is defined as the set $F(A) = \{w \in \mathbb{R}^m \mid w = F(v) \text{ for some } v \in A\}$.
- *(Pre–image)* If $B \subseteq \mathbb{R}^m$, the *pre–image of* B under the map $F\colon U \to \mathbb{R}^m$, denoted by $F^{-1}(B)$, is defined as the set $F^{-1}(B) = \{v \in U \mid F(v) \in B\}$.

  Note that $F^{-1}(B)$ is always defined even if F does not have an inverse map.

Do the following problems

1. Use the triangle inequality to prove that, for any v and w in $\mathbb{R}^n$,
$$\Big|\|v\| - \|w\|\Big| \leqslant \|v - w\|.$$
Use this inequality to deduce that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by
$$f(v) = \|v\| \quad \text{for all } \ v \in \mathbb{R}^n$$
is continuous on $\mathbb{R}^n$.

2. Let f and g denote two real–valued functions defined on an open region, D, in $\mathbb{R}^2$. Prove that the vector field $F\colon D \to \mathbb{R}^2$, defined by
$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in D,$$
is continuous on D if and only f and g are both continuous on D.

3. Let U denote an open subset of $\mathbb{R}^n$ and let $F\colon U \to \mathbb{R}^m$ and $G\colon U \to \mathbb{R}^m$ be two given functions.

   (a) Explain how the sum $F+G$ is defined.

   (b) Prove that if both F and G are continuous on U, then their sum is also continuous.

   (*Suggestion:* Use the triangle inequality.)

4. In each of the following, given the function $F\colon U \to \mathbb{R}^m$ and the set B, compute the pre–image $F^{-1}(B)$.

   (a) $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2+y^2 \\ x^2-y^2 \end{pmatrix}$, and $B = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

   (b) Let $D = \mathbb{R}^2 \backslash \{(0,0)\} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 > 0\}$ (the punctured plane), and define $f\colon D \to \mathbb{R}$ by

   $$f(x,y) = \frac{1}{\sqrt{x^2+y^2}}, \quad \text{for} \quad (x,y) \in D.$$

   Put $B = \{2\}$.

   (c) $f\colon D \to \mathbb{R}$ is as in part (b), and $B = \{0\}$.

5. Compute the image of the given sets under the following maps:

   (a) $\sigma\colon \mathbb{R} \to \mathbb{R}^2$, $\sigma(t) = (\cos t, \sin t)$, for all $t \in \mathbb{R}$. Compute $\sigma(\mathbb{R})$.

   (b) $f\colon D \to \mathbb{R}$ and D are as given in part (b) of the previous problem. Compute $f(D)$.