Solutions to Assignment #17

1. Prove that if $ad - bc \neq 0$, then the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible and compute A^{-1}.

***Solution**:* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume that $ad - bc \neq 0$.
First consider the case $a \neq 0$. Perform the elementary row operations: $(1/a)R_1 \to R_1$ and $-cR_1 + R_2 \to R_2$, successively, on the augmented matrix

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right), \tag{1}$$

we obtain the augmented matrix

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & -(cb/a) + d & -c/a & 1 \end{array}\right),$$

or

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \Delta/a & -c/a & 1 \end{array}\right), \tag{2}$$

where we have set $\Delta = ad - bc$; thus $\Delta \neq 0$.
Next, perform the elementary row operation $\dfrac{a}{\Delta}R_2 \to R_2$ on the augmented matrix in (2) to get

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -c/\Delta & a/\Delta \end{array}\right). \tag{3}$$

Finally, perform the elementary row operation $-\dfrac{b}{a}R_2 + R_1 \to R_1$ on the augmented matrix in (3) to get

$$\left(\begin{array}{cc|cc} 1 & 0 & d/\Delta & -b/\Delta \\ 0 & 1 & -c/\Delta & a/\Delta \end{array}\right). \tag{4}$$

From (4) we then see that, if $\Delta = ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix},$$

or

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{5}$$

Observe that the formula for A^{-1} in (5) also works for the case $a = 0$. In this case, $\Delta = -bc$; so that $b \neq 0$ and $c \neq 0$, and

$$A^{-1} = \frac{1}{-bc}\begin{pmatrix} d & -b \\ -c & 0 \end{pmatrix},$$

or

$$A^{-1} = \begin{pmatrix} -d/bc & 1/c \\ 1/b & 0 \end{pmatrix},$$

and we can check that

$$\begin{pmatrix} -d/bc & 1/c \\ 1/b & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and therefore A is invertible in this case as well. □

2. Let A, B and C denote matrices in $\mathbb{M}(m, n)$. Prove the following statements regarding row equivalence.

   (a) A is row equivalent to itself.

   Solution: Observe that $IA = A$, and I is an elementary matrix since it is obtained from I by performing, for instance, the elementary row operation: $(1)R_1 \to R_1$. □

   (b) If A is row equivalent to B, then B is row equivalent to A.

   Solution: If A is row equivalent to B, then there exist elementary matrices, $E_1, E_2, \ldots, E_k \in \mathbb{M}(m, m)$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = B. \tag{6}$$

   Since all elementary matrices are invertible, $E_1^{-1}, E_2^{-1}, \ldots E_k^{-1}$ exist. These are also elementary matrices. Multiplying on the left of both sides of (6) by E_k^{-1} yields

$$E_k^{-1}(E_k E_{k-1} \cdots E_2 E_1 A) = E_k^{-1} B.$$

   Applying the associative property we then get

$$E_{k-1} \cdots E_2 E_1 A = E_k^{-1} B. \tag{7}$$

   We can continue in this fashion multiplying successively by

$$E_{k-1}^{-1}, \ldots, E_2^{-1}, E_1^{-1}$$

on the left of both sides of (7) then yields

$$A = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} B,$$

which shows that B is row equivalent to A, since inverses of elementary matrices as elementary matrices. □

(c) If A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

Solution: Assume that A is row equivalent to B and B is row equivalent to C. Then, there exist elementary matrices,

$$E_1, E_2, \ldots, E_k, F_1, F_2, \ldots, E_\ell \in \mathbb{M}(m, m)$$

such that

$$E_k E_{k-1} \cdots E_2 E_1 A = B, \tag{8}$$

and

$$F_\ell F_{\ell-1} \cdots F_2 F_1 B = C. \tag{9}$$

Multiplying by $F_1, F_2, \ldots, F_\ell$ on the left in both sides of (8) successively then yields

$$F_\ell F_{\ell-1} \cdots F_2 F_1 E_k E_{k-1} \cdots E_2 E_1 A = F_\ell F_{\ell-1} \cdots F_2 F_1 B,$$

which in view of (9) then yields

$$F_\ell F_{\ell-1} \cdots F_2 F_1 E_k E_{k-1} \cdots E_2 E_1 A = C.$$

Hence, A is row equivalent to C. □

Note: these properties are usually known as *reflexivity, symmetry* and *transitivity,* respectively, and they define an *equivalence relation.*

3. Use Gaussian elimination to determine whether the matrix

$$A = \begin{pmatrix} 1 & -4 & 1 \\ 0 & 3 & -1 \\ -3 & 0 & 1 \end{pmatrix}$$

is invertible or not. If A is invertible, compute its inverse.

Solution: Begin with the augmented matrix

$$\begin{pmatrix} 1 & -4 & 1 & | & 1 & 0 & 0 \\ 0 & 3 & -1 & | & 0 & 1 & 0 \\ -3 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Next, apply the elementary row operation $3R_1 + R_3 \to R_3$ on the augmented matrix (10) to get

$$\begin{pmatrix} 1 & -4 & 1 & | & 1 & 0 & 0 \\ 0 & 3 & -1 & | & 0 & 1 & 0 \\ 0 & -12 & 4 & | & 3 & 0 & 1 \end{pmatrix}. \tag{11}$$

Finally, apply the elementary row operation $4R_2 + R_3 \to R_3$ to the augmented matrix in (11) to get

$$\begin{pmatrix} 1 & -4 & 1 & | & 1 & 0 & 0 \\ 0 & 3 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 3 & 4 & 1 \end{pmatrix}, \tag{12}$$

and observe that the matrix on the left–hand side of (12) cannot be the turned into the identity in $\mathbb{M}(3,3)$ by elementary row operations. We therefore conclude that A is not invertible. □

4. Let A denote an $m \times n$ matrix.

   (a) Show that if $m < n$, then A is singular.

   Proof: Assume that $A \in \mathbb{M}(m,n)$ where $m < n$. Then,

   $$Ax = \mathbf{0}$$

   is a homogeneous system of m linear equations in n unknowns. Since there are more equations than unknowns, it follows from the Fundamental Theorem of Homogeneous Systems that $Ax = \mathbf{0}$ has nontrivial solutions. Hence, A is singular. □

   (b) Prove that A is singular if and only if the columns of A are linearly dependent in $\mathbb{R}^m$.

Proof: Assume that $A \in \mathbb{M}(m, n)$ is singular and write $A = [v_1 \quad v_2 \quad \cdots \quad v_n]$, where $v_1, v_2, \ldots, v_n$ are the columns of A. Consider the equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = \mathbf{0}, \tag{13}$$

which can be written in matrix form as

$$Ax = \mathbf{0}. \tag{14}$$

Since A is singular, the matrix equation in (14) has a nontrivial solution and this implies that the vector equation in (13) has a nontrivial solution. Consequently, the columns of A are linearly dependent.

Conversely, if the columns of A are linearly dependent, then the vector equation in (13) has a nontrivial solution, which implies that the matrix equation $Ax = \mathbf{0}$ has a nontrivial solution and therefore A is singular. □

5. Let A denote an $n \times n$ matrix. Prove that A is invertible if and only if A is nonsingular.

   Proof: Assume that A is invertible. Then, A has a left–inverse, B. It then follows that the equation

   $$Ax = \mathbf{0}$$

   has only the trivial solution and therefore A is nonsingular.

   Conversely, suppose that A is nonsingular; then, the equation

   $$Ax = 0$$

   has only the trivial solution. Consequently the columns of A are linearly independent and therefore they form a basis for $\mathbb{R}^n$. Denote the columns of A by $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. Then, any vector in $\mathbb{R}^n$ is a linear combination of the vectors in $\{v_1, v_2, \ldots, v_n\}$. In particular, there exist c_{ij}, for $1 \leqslant i, j \leqslant n$, such that

   $$\begin{array}{rcl} c_{11} v_1 + c_{21} v_2 + \cdots + c_{n1} v_n & = & e_1 \\ c_{12} v_1 + c_{22} v_2 + \cdots + c_{n2} v_n & = & e_2 \\ \vdots & \vdots & \vdots \\ c_{1n} v_1 + c_{2n} v_2 + \cdots + c_{nn} v_n & = & e_1, \end{array}$$

   where $\{e_1, e_2, \cdots, e_n\}$ is the standard basis is $\mathbb{R}^n$. We then get that

   $$A \begin{pmatrix} c_{1,j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{pmatrix} = e_j$$

for $j = 1, 2, \ldots, n$. Consequently, if we set $C = [c_{ij}]$ for $1 \leqslant i, j \leqslant n$, we see that

$$AC_j = e_j,$$

where C_j is the j^{th} column of C; in other words

$$AC = [AC_1 \quad AC_2 \quad \cdots \quad AC_n] = [e_1 \quad e_2 \quad \cdots \quad e_n] = I.$$

We have therefore shown that A has right–inverse, C.

Next, transpose the equation $AC = I$ to obtain

$$(AC)^T = I^T,$$

or

$$C^T A^T = I,$$

which shows that A^T has a left–inverse. It then follows that A^T is invertible. Consequently, its transpose, $(A^T)^T = A$ is also invertible. $\square$