Multivariable Calculus with Applications to the Life Sciences

Lecture Notes

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Chapter 1

Preface

All questions of interest in the sciences involve more than one variable and functions of more than one variable. Thus, the single variable Calculus that we have learned up to this point is very limited in its applicability to the analysis of problems arising in the sciences. Even in the case in which the functions of interest in some application can be assumed to be functions of a single variable (as illustrated in the example from epidemiology to be discussed in the next section), the fact that a problem requires more than one of those functions puts us in the realm of multiple variables. It is for that reason that we need to learn the concepts and methods of Multivariable Calculus.

In this course we will learn Multivariable Calculus in the context of problems in the life sciences. Throughout these notes, as well as in the lectures and homework assignments, we will present several examples from Epidemiology, Population Biology, Ecology and Genetics that require the methods of Calculus in several variables.

In addition to applications of Multivariable Calculus, we will also look at problems in the life sciences that require applications of probability. In particular, the use of probability distributions to study problems in which randomness, or chance, is involved, as is the case in the study of genetic mutations.

Chapter 2

Introductory Examples

In this chapter we present two examples that will help motivate the mathematical topics that will be covered in this course. The first example is a system of equations from Epidemiology that provides a simple model for the spread of a contagious disease. The second example is from Population Ecology and prescribes the interactions between predator and prey species in simple model.

2.1 Modeling the Spread of a Disease

Example 2.1.1 (A simple SIR Model)**.** In a simple mathematical model for a disease that is spread through infections transmitted between individuals in a population, the population is divided into three compartments pictured in Figure 2.1.1. The first compartment, $S(t)$, denotes the set of individuals in the

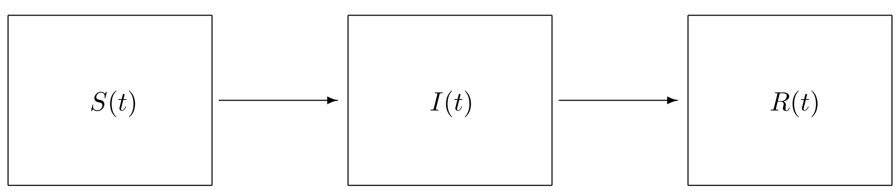

Figure 2.1.1: SIR Compartments

population that are susceptible to acquiring the disease at time t; the second compartment, $I(t)$, denotes the set of infected individual who can also infect others, also at time t; and the third compartment, $R(t)$, denotes the set of individuals who had the disease and who have recovered from the disease at time t.

We assume that the functions S, I and R; are differentiable functions of time. Thus, the techniques that we learned in single variable Calculus can be applied to these functions. We also assume that the total number of individuals in the population,

$$N = S(t) + I(t) + R(t),$$

is constant.

Susceptible individuals can get infected through contact with infectious individuals and move to the infected class. This is indicated by the arrow going from the $S(t)$ compartment to the $I(t)$ compartment in Figure 2.1.1. In this simple model, we assume that the individuals in compartment $R(t)$ can no longer get infected.

In addition to the assumptions that we have made so far, we also assume the following:

- The rate at which susceptible individuals get infected is proportional to product of number of susceptible individuals and the number of infected individuals with constant of proportionality $\beta > 0$. We can write this in symbols as
$$\text{Rate of Infection} = \beta S I.$$
- The rate at which infected individuals recover is proportional to the number of infected individuals with constant of proportionality $\gamma > 0$. We can write this in symbols as
$$\text{Rate of Recovery} = \gamma I$$

We would like to understand the flow of individuals from one compartment to another according to the flow arrows pictured in Figure 2.1.1 and the assumptions that we have stated so far. One way to understand the flows is to look at the rates of change of the numbers of individuals in each compartment. For instance, the rate of change of the number of individuals in the infected compartment,

$$I'(t) \quad \text{or} \quad \frac{dI}{dt},$$

has to be accounted for by the rate at which individuals enter the compartment from the susceptible class by way of infections, and the number of individuals that leave the class by way of recovery. We can express this mathematically by means of the equation

$$\frac{dI}{dt} = \beta S I - \gamma I \tag{2.1}$$

The equation in (2.1) is an example of what is known as a *conservation principle*; it expresses the fact that, since the total number of individuals in the population is to remain constant, the rates of change of the number of individuals in a given compartment have to be accounted for by the rates at which individuals enter or leave a given class, or compartment.

The expression in (2.1) is also an example of a *differential equation*. Similar considerations lead to two additional differential equations

$$\frac{dS}{dt} = -\beta S I \tag{2.2}$$

and

$$\frac{dR}{dt} = \gamma I. \tag{2.3}$$

Putting the differential equations in (2.1), (2.2) and (2.3) together leads to the following system of differential equations:

$$\begin{cases} \dfrac{dS}{dt} &= -\beta SI; \\[2ex] \dfrac{dI}{dt} &= \beta SI - \gamma I; \\[2ex] \dfrac{dR}{dt} &= \gamma I. \end{cases} \tag{2.4}$$

The system in (2.4) is known in the literature as the Kermack–McKendrick SIR model. It first appeared in the scientific literature in 1927.

One of the goals of this course is to develop some of the concepts from Multivariable Calculus that will help us in the analysis of systems like the one in (2.4). An examination of the right–hand side of the equations in (2.4) reveals that the quantities $S(t)$, $I(t)$ and $R(t)$ have to be studied simultaneously, since their rates of change are intertwined. Thus, it makes sense to consider the triple

$$(S(t), I(t), R(t)), \qquad \text{for } t \text{ in some interval of time.} \tag{2.5}$$

The expression in (2.6) defines a **vector–valued function** of a single variable, t. As t varies, the image of the function defined in (2.6) traces a curve in three dimensional space, as pictured in Figure 2.1.2. This curve is an example of a **parametrized curve**, an this is where we begin our study of the topics from Multivariable Calculus in this course.

2.2 Preliminary Analysis of a Simple SIR Model

In many cases, analysis of two dimensional systems suffices in many applications. We illustrate this in the following example in which perform a preliminary analysis of the SIR model developed in Example 2.1.1.

Example 2.2.1 (Preliminary Analysis of a Simple SIR Model)**.** We begin with the observation that the system in (2.4) that the total size, N, of the population is constant. Thus, from the equation

$$S(t) + I(t) + R(t) = N, \qquad \text{for all } t,$$

we can solve for $R(t)$ in terms of $S(t)$ and $I(t)$ to get

$$R(t) = N - S(t) - I(t), \qquad \text{for all } t.$$

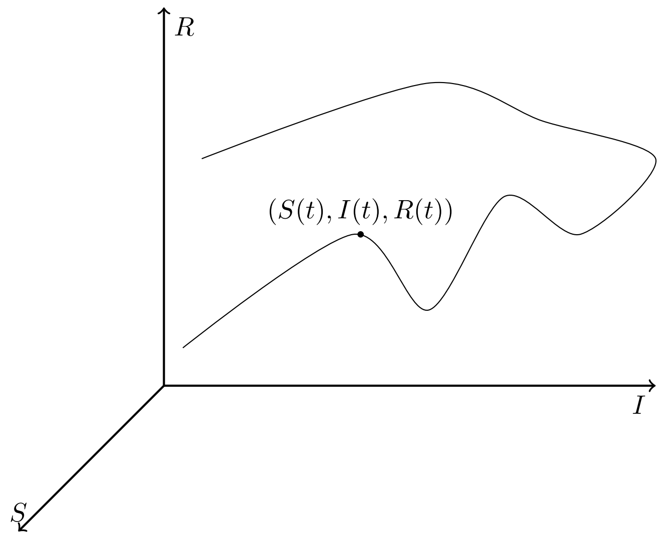

Figure 2.1.2: Curve in SIR–Space

Thus, if we can determine the number of susceptible and infectious individuals at any time t, we'll be able to determine the number of recovered individuals at any time t. Hence, it suffices to study the two–dimensional system

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI; \\ \\ \dfrac{dI}{dt} = \beta SI - \gamma I. \end{cases} \tag{2.6}$$

We would like to determine the pairs $(S(t), I(t))$, which can be pictured as points in the SI–plane, whose components satisfy the equations in (2.6).

Suppose that initially (at time $t = 0$) there are I_o infectious individuals and S_o susceptible individuals. We would like to determine $S(t)$ and $I(t)$ for $t > 0$.

The initial point (S_o, I_o) is shown in Figure 2.2.3, as well as a possible solution curve. In the rest of this example we will see how to justify the shape of the curve drawn in Figure 2.2.3.

The system of equations in (2.6) gives information about the derivatives

$$S'(t) = -\beta S(t)I(t) \tag{2.7}$$

and

$$I'(t) = \beta S(t)I(t) - \gamma I(t), \tag{2.8}$$

of the quantities S and I, respectively. It follows from (2.7) that, in the case in which, S and I are positive, $S'(t) < 0$; so that, the number of susceptible

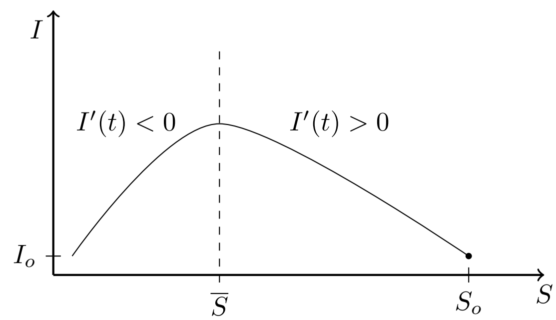

Figure 2.2.3: Curve in the SI–Plane

individuals in the population can only decrease. On the other, rewriting (2.8) as

$$I'(t) = \beta I(t)\left[S(t) - \frac{\gamma}{\beta}\right], \tag{2.9}$$

we see that the answer to the question of whether the number of infected individuals will increase or decrease will depend on whether or not S is bigger than the value

$$\overline{S} = \frac{\gamma}{\beta}. \tag{2.10}$$

The value of $\overline{S}$ is shown in Figure 2.2.3 for the case in which

$$S_o > \overline{S}. \tag{2.11}$$

In this case, for values of t for which $\overline{S} < S(t) \leqslant S_o$, $I'(t) > 0$, as indicated in Figure 2.2.3. Thus, the number of infected individuals will increase, and therefore the disease will spread in this case. Note also that, in the case in which (2.11), the number of infected individuals will increase to a largest value at a time $\bar{t}$ for which $I'(\bar{t}) = 0$ (see Figure 2.2.3). The number of infectious individuals reaching a maximum value indicates an epidemic. After reaching the maximum value, the number of infectious individuals begins to decrease because, according to (2.9) and (2.10), $S(t) < \overline{S}$ implies that $I'(t) < 0$, as shown in Figure 2.2.3.

On the hand other hand, in the case in which

$$S_o < \overline{S}, \tag{2.12}$$

$S(t) < \overline{S}$ for all $t \geqslant 0$; so that, according to (2.9) and (2.10), $I'(t) < 0$ for all $t \geqslant 0$ and, therefore, the number of infected individuals will decrease from I_o and the disease will not spread.

Finally, observe that, in view of (2.10), the inequality in (2.11) can be rewritten as

$$S_o > \frac{\gamma}{\beta},$$

from which we get that

$$\frac{\beta S_o}{\gamma} > 1. \tag{2.13}$$

The expression on the left–hand side of the inequality in (2.13) is usually denoted by R_o, and is called the **reproduction number**. It is a very important number in epidemiology. When it can be computed, or estimated, R_o provides important information that can be used to determine whether a given disease will spread or not. In particular, since the inequality in (2.13) is equivalent to (2.11), we see that, if $R_o > 1$, the disease will spread. On the other hand, if $R_o < 1$, it follows from (2.12) that the disease will not spread.

2.3 A Predator–Prey System

Examples of applications that are amenable to the two–dimensional analysis illustrated in the previous section are provided by systems that model the interaction of two species that live in the same ecosystem. The simplest of those types of systems is the following predator–prey system known as the Lotka–Volterra system.

Example 2.3.1 (Lotka–Volterra System)**.** Let $x(t)$ and $y(t)$ denote the population densities of two species living in the same ecosystem at time t. We assume that the x and y are differentiable functions of t. Assume also that the population of density y depends solely on the density of the species of density x. We may quantify this by prescribing that, in the absence of the species of density x, the per–capita growth rate of species of density y is a negative constant:

$$\frac{y'(t)}{y(t)} = -\gamma, \quad \text{for all } t \text{ with } x(t) = 0, \tag{2.14}$$

for some positive constant γ. We will see later in this course that (2.14) implies that the population of density y will eventually go extinct in the absence of the species of density x.

On the other hand, in the absence of the species of density y, the species of density x will experience unlimited growth according to

$$\frac{x'(t)}{x(t)} = \alpha, \quad \text{for all } t \text{ with } y(t) = 0, \tag{2.15}$$

where α is a positive constant.

When both species are present, the per–capita growth rate of the species of population density x is given by

$$\frac{x'(t)}{x(t)} = \alpha - \beta y, \quad \text{for all } t, \tag{2.16}$$

where β is a positive constant, and the species of density y has a per–capita growth rate given by

$$\frac{y'(t)}{y(t)} = -\gamma + \delta x, \quad \text{for all } t, \tag{2.17}$$

for some positive constant δ.

The equations in (2.16) and (2.17) describe a predator–prey interaction in which predator species, y, relies solely on the prey species, x, for sustenance, while the only factor that can hinder the growth of the prey species, x, is the presence of the predator species y.

The equations in (2.16) and (2.17) form a system of differential equations,

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy; \\ \\ \dfrac{dy}{dt} = \delta xy - \gamma y, \end{cases} \tag{2.18}$$

known as the Lotka–Volterra system. We will analyze this system later on in these notes.

Chapter 3

Parametrized Curves

The curve pictured in Figure 2.1.2 is an example of a **parametrized curve** in three–dimensional space. It is the image of a vector–valued function of a single variable. In the example discussed in the previous Section 2.1, this function is given by

$$(S(t), I(t), R(t)), \tag{3.1}$$

for t in some interval of time, J. In this case, we call t a parameter, and the curve traced by the points (3.1) is a parametrized curve in three dimensions.

In many applications, phenomena can be described by parametrized curves in two dimensions. We saw instances of this in Example 2.2 and in the Lotka–Volterra system derived in Section 2.3. For that reason, we begin this chapter by studying parametrized curves in the plane.

3.1 Parametrized Curves in the Plane

The set of points $(S(t), I(t))$ discussed in Example 2.2.1 trace out a curve in the SI–place, pictured in Figure 2.2.3, as the parameter t varies. The solutions $(x(t), y(t))$ of the Lotka–Volterra system in (2.18) trace out curves in the xy–plane as t varies. In both of these instances we obtain a parametrized curves in the plane.

Definition 3.1.1 (Parametrized Curves in the xy–plane)**.** Let J denote an interval of real numbers and $x\colon J \to \mathbb{R}$ and $y\colon J \to \mathbb{R}$ denote functions that are differentiable[1] in J. The set of points, C, in the xy–place with coordinates $(x(t), y(t))$ for $t \in J$ is called a parametrized curve.

[1] The function $x\colon J \to \mathbb{R}$ is differentiable at $t \in J$ means that $\lim\limits_{h\to 0} \dfrac{x(t+h) - x(t)}{h}$ exists. This limit is usually denoted by $x'(t)$ and, geometrically, it gives the slope of the tangent line to the graph of $x = x(t)$ at the point $(t, x(t))$. We say that x is differentiable in J if x is differentiable at every t in J.

Example 3.1.2. Let

$$x(t) = e^t, \qquad \text{for all } t \in \mathbb{R},$$

and

$$y(t) = e^{-t}, \qquad \text{for all } t \in \mathbb{R},$$

and

$$C = \{(x(t), y(t)) \mid t \in \mathbb{R}\}.$$

Then, C is a parametrized curve.

In this example we see how to sketch C.

Observe that $x(t) > 0$ and $y(t) > 0$ for all t because the exponential function is always positive (i.e. $e^a > 0$ for all $a \in \mathbb{R}$); thus, the curve C must lie in the first quadrant. Observe also that

$$x(t) \cdot y(t) = 1, \qquad \text{for all } t,$$

so that C must be the portion of the parabola

$$xy = 1,$$

that lies in the first quadrant (see Figure 3.1.1).

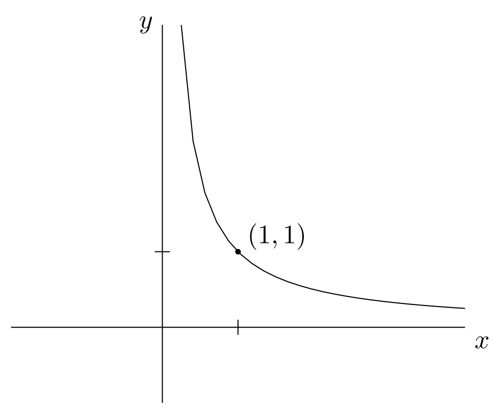

Figure 3.1.1: Sketch of C in Example 3.1.2

Note the $(x(0), y(0)) = (1, 1)$; so, the point $(1, 1)$ is on the curve C; this point is shown in the picture in Figure 3.1.1.

Definition 3.1.3 (Parametrizations)**.** Give a curve, C, in the plane, we say that C is parametrizable if we can find a parametrization, $(x(t), y(t))$ for $t \in J$, where J is an interval in $\mathbb{R}$, such that

$$C = \{(x(t), y(t)) \mid t \in J\}.$$

In what follows we provide several examples of curves in the plane and their parametrizations.

Example 3.1.4 (Points)**.** Given a point, (x_o, y_o), in the plane, the constant functions

$$x(t) = x_o, \quad \text{for all } t \in \mathbb{R},$$

and

$$y(t) = y_o, \quad \text{for all } t \in \mathbb{R},$$

form a parametrization of the point (x_o, y_o).

Example 3.1.5 (Lines)**.** Given a point, $P(x_o, y_o)$, in the plane, and a direction vector $\overrightarrow{v} = a\widehat{i} + b\widehat{j}$, we would like to construct a parametrization of the line, L, through the point P in the direction of the vector $\overrightarrow{v}$. An arbitrary point

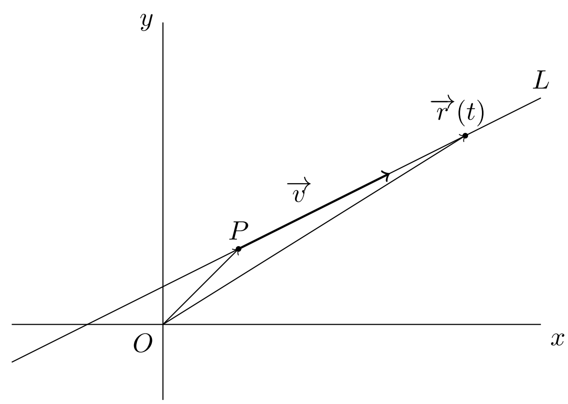

Figure 3.1.2: Sketch of Line L in Example 3.1.5

$\overrightarrow{r}(t) = (x(t), y(t))$ on the line L can be reached from the origin, O, by first going from the origin to the point P along the vector $\overrightarrow{OP}$ shown in Figure 3.1.5, and then going from the point P along the direction of $\overrightarrow{v}$ through a scalar multiple, $t\overrightarrow{v}$, of the vector $\overrightarrow{v}$. This is expressed as the vector equation

$$\overrightarrow{r}(t) = \overrightarrow{OP} + t\overrightarrow{v}. \tag{3.2}$$

The expression in (3.2) is the **vector–parametric** equation of the line L. As t varies over all real values, $\overrightarrow{r}(t)$ traces every point on L. For instance, when $t = 0$, $\overrightarrow{r}(0) = \overrightarrow{OP}$ determines the point $P(x_o, y_o)$; when $t = 1$, $\overrightarrow{r}(1) = \overrightarrow{OP} + \overrightarrow{v}$ is the point at the tip of the vector $\overrightarrow{v}$, which lies on the line L. when its tail is at the point P.

The vector $\overrightarrow{OP}$ can be written as $x_o\widehat{i} + y_o\widehat{j}$, or as

$$\overrightarrow{OP} = (x_o, y_o). \tag{3.3}$$

Similarly, the vector $\overrightarrow{v} = a\widehat{i} + b\widehat{j}$ can also be written as $\overrightarrow{v} = (a, b)$. The scalar multiple, $t\overrightarrow{v}$, of $\overrightarrow{v}$ is then

$$t\overrightarrow{v} = t(a, b) = (ta, tb),$$

or

$$t\overrightarrow{v} = (at, bt). \tag{3.4}$$

Combining the expressions in (3.2), (3.3) and and (3.4) yields

$$(x(t), y(t)) = (x_o, y_o) + (at, bt),$$

or

$$(x(t), y(t)) = (x_o + at, y_o + bt), \tag{3.5}$$

where we have used vector addition in the right–hand side of (3.5).

Equating corresponding components of the vectors in (3.5) yields the **parametric equations** of the line L:

$$\begin{cases} x = x_o + at; \\ y = y_o + bt, \end{cases}$$

for $t \in \mathbb{R}$.

Example 3.1.6 (Line Segments)**.** Consider a pair of distinct points $P(x_o, y_o)$ and $Q(x_1, y_1)$ in the xy–plane. We would like to construct a parametrization of

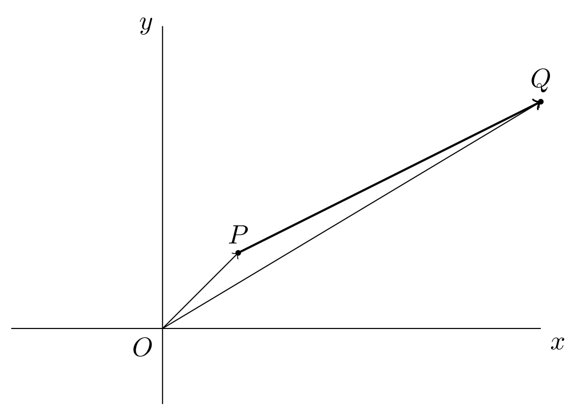

Figure 3.1.3: Sketch of Line Segment in Example 3.1.6

the directed line segment connecting the point P to the point Q.

We first construct the vector $\overrightarrow{v} = \overrightarrow{PQ}$ as follows

$$\overrightarrow{PQ} = (x_1 - x_o, y_1 - y_o),$$

or

$$\overrightarrow{PQ} = (x_1 - x_o)\widehat{i} + (y_1 - y_o)\widehat{j}$$

We first parametrize the line segment from P to Q as we parametrized the line L in Example 3.1.5 by means of the vector–parametric equation in equation (3.2),

$$\overrightarrow{r}(t) = \overrightarrow{OP} + t\overrightarrow{PQ}, \quad \text{for } 0 \leqslant t \leqslant 1, \tag{3.6}$$

except that this time t has to be restricted to $0 \leqslant t \leqslant 1$. A value of $t = 0$ in (3.6) corresponds to the point P and a value of $t = 1$ corresponds to the vector

$$\overrightarrow{r}(t) = \overrightarrow{OP} + \overrightarrow{PQ},$$

which locates the point Q (see the sketch in Figure 3.1.3).

From the vector–parametric equation in (3.6) we obtain the parametric equations

$$\left\{ \begin{array}{rcl} x & = & x_o + (x_1 - x_o)t; \\ y & = & y_o + (y_1 - y_o)t, \end{array} \right. \qquad \text{for } 0 \leqslant t \leqslant 1,$$

for the segment connecting P to Q.

Example 3.1.7. Find a parametrization for the directed line segment from $P(5, 3)$ to $Q(1, 1)$.

***Solution**:* First, compute the direction vector $\overrightarrow{v} = \overrightarrow{PQ}$ to get

$$\overrightarrow{v} = (1 - 5, 1 - 3) = (-4, -2).$$

Then, the parametric equations of the line segment are

$$\left\{ \begin{array}{rcl} x & = & 5 - 4t; \\ y & = & 3 - 2t, \end{array} \right. \qquad \text{for } 0 \leqslant t \leqslant 1,$$

A sketch of the segment is shown in Figure □

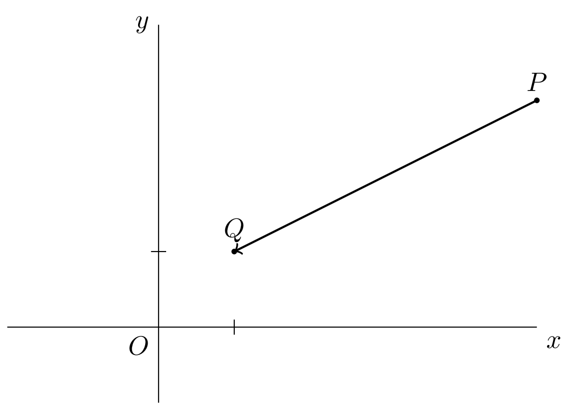

Figure 3.1.4: Sketch of Line Segment from $(5, 3)$ to $(1, 1)$

Example 3.1.8 (Circles)**.** Let C denote the circle in the xy–plane of radius $r > 0$ and centered at the point (x_o, y_o). Then, every point (x, y) in C is at a distance of r from the point (x_o, y_o). Thus,

$$(x - x_o)^2 + (y - y_o)^2 = r^2. \tag{3.7}$$

Divide both sides of the equation in (3.7) by r^2 to obtain

$$\frac{(x-x_o)^2}{r^2} + \frac{(y-y_o)^2}{r^2} = 1,$$

or

$$\left(\frac{x-x_o}{r}\right)^2 + \left(\frac{y-y_o}{r}\right)^2 = 1. \tag{3.8}$$

Recalling the trigonometric identity

$$\cos^2 t + \sin^2 t = 1, \qquad \text{for all } t \in \mathbb{R},$$

we can set

$$\frac{x-x_o}{r} = \cos t \quad \text{and} \quad \frac{y-y_o}{r} = \sin t,$$

or

$$x - x_o = r\cos t \quad \text{and} \quad y - y_o = r\sin t;$$

so that, the equations

$$\left\{\begin{array}{rcl} x & = & x_o + r\cos t; \\ y & = & y_o + r\sin t, \end{array}\right. \tag{3.9}$$

give a parametrization of C provided the values of the parameter t are confined to an interval of real numbers of length 2π; for example, $0 \leqslant t < 2\pi$, or $-\pi < t \leqslant \pi$. Observe that the direction given by the parametrization in (3.9) is in the counterclockwise sense (see Figure 3.1.5). To see why this assertion is true, note

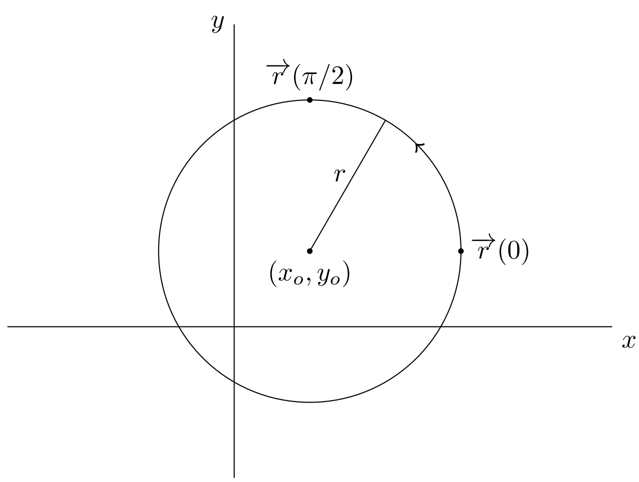

Figure 3.1.5: Sketch of Circle of Radius r and Center (x_o, y_o)

that $\overrightarrow{r}(0) = (x_o + r, y_o)$ and, a quarter of the time later, $\overrightarrow{r}(\pi/2) = (x_o, y_o + r)$.

Observe that the choice

$$\frac{x-x_o}{r} = \sin t \quad \text{and} \quad \frac{y-y_o}{r} = \cos t,$$

will also satisfy the equation of the circle of radius r around (x_o, y_o) in (3.8). Thus, the set of equations

$$\left\{\begin{array}{lcl} x & = & x_o + r\sin t; \\ y & = & y_o + r\cos t, \end{array}\right. \tag{3.10}$$

for $0 \leqslant t < 2\pi$ is also a parametrization of the circle C given by the equation (3.7). However this parametrization is oriented in the clockwise sense (see Figure 3.1.6).

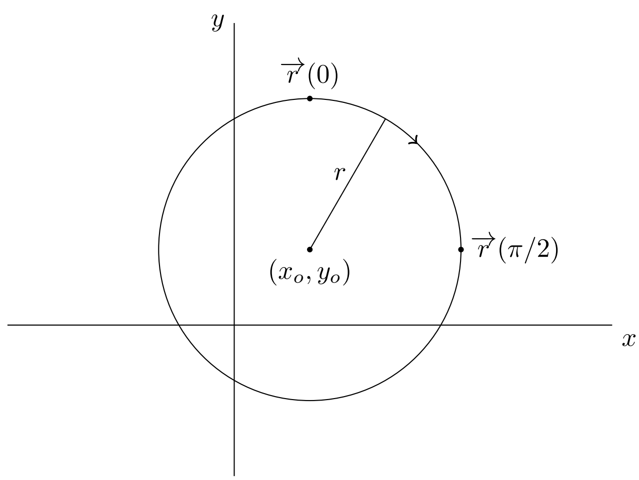

Figure 3.1.6: Sketch of Circle Parametrized by (3.10)

Example 3.1.9. Give a parametrization for the semicircle, C, from the point $P(0, 2)$ to the point $Q(0, 0)$.

Solution: Figure 3.1.7 shows a sketch of C. We use the parametrization in (3.10) with $x_o = 0$, $y_o = 1$ and $r = 1$, with t restricted to $0 \leqslant t \leqslant \pi$, to get

$$\left\{\begin{array}{lcl} x & = & \sin t; \\ y & = & 1 + \cos t, \end{array}\right. \qquad \text{for } 0 \leqslant t \leqslant \pi.$$

□

Example 3.1.10 (Ellipses)**.** The graph of the equation

$$\frac{(x - x_o)^2}{a^2} + \frac{(y - y_o)^2}{b^2} = 1 \tag{3.11}$$

is an ellipse with center (x_o, y_o) and vertices at $(x_o - a, y_o)$, $(x_o + a, y_o)$, $(x_o, y_o - b)$ and $(x_o, y_o + b)$. A possible sketch is shown in Figure 3.1.8 As we did when we

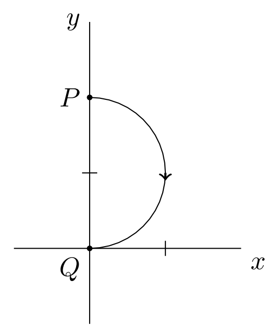

Figure 3.1.7: Sketch of Semicircle C in Example 3.1.9

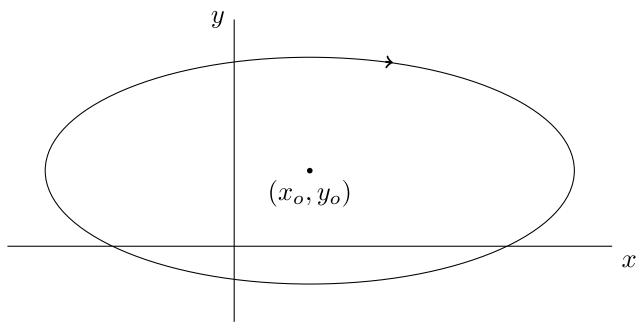

Figure 3.1.8: Sketch of Ellipse in Example 3.1.10

constructed parameterizations for circles, we can use the trigonometric identity

$$\cos^2 t + \sin^2 t = 1, \qquad \text{for all } t \in \mathbb{R}$$

and set

$$\frac{x - x_o}{a} = \cos t \qquad \text{and} \qquad \frac{y - y_o}{b} = \sin t,$$

to get

$$x - x_o = a\cos t \qquad \text{and} \qquad y - y_o = b\sin t;$$

so that, the equations

$$\left\{ \begin{array}{rcl} x & = & x_o + a\cos t; \\ y & = & y_o + b\sin t, \end{array} \right. \qquad \text{for } 0 \leqslant t < 2\pi, \tag{3.12}$$

parametrize the ellipse given by (3.11). As was the case for the circle, the parametrization in (3.12) is oriented in the counterclockwise sense as shown in Figure 3.1.8. Similarly, a parametrization in the clockwise sense is given by the equations

$$\left\{ \begin{array}{rcl} x & = & x_o + a\sin t; \\ y & = & y_o + b\cos t, \end{array} \right. \qquad \text{for } 0 \leqslant t < 2\pi, \tag{3.13}$$

has a clockwise orientation.

Example 3.1.11. Let C denote the portion of the ellipse given by the graph of the equation

$$4x^2 + y^2 = 4, \tag{3.14}$$

in the first quadrant of the xy–plane, from the point $P(0,2)$ to the point $Q(1,0)$. Give a parametrization for C.

Solution: Figure 3.1.9 shows a sketch of C.

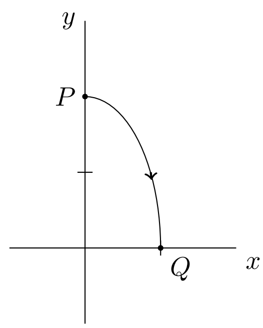

Figure 3.1.9: Sketch of Curve C in Example 3.1.11

Divide both sides of the equation in (3.14) by 4 to get

$$x^2 + \frac{y^2}{4} = 1. \tag{3.15}$$

We see from (3.15) that the ellipse is centered at the origin (so that, $x_o = y_o = 0$) with $a = 1$ and $b = 2$. Since the orientation is in the clockwise sense (see sketch in Figure 3.1.9), we use the parametric equations in (3.13) with t restricted to go from 0 to $\pi/2$:

$$\left\{\begin{array}{rcl} x & = & \sin t; \\ y & = & 2\cos t, \end{array}\right. \qquad \text{for } 0 \leqslant t < \pi/2. \tag{3.16}$$

□

Example 3.1.12 (Graphs of Functions)**.** Let f denote a differentiable function of a single variable defined over some open interval containing a and b, where $a < b$. We let C denote the portion of the graph of $y = f(x)$ for $a \leqslant x \leqslant b$; that is,

$$C = \{(x, f(x)) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b\}.$$

Figure 3.1.10 illustrates what may happen in a general situation. The curve C is the portion of the graph of f that lies between the points P and Q in Figure 3.1.10.

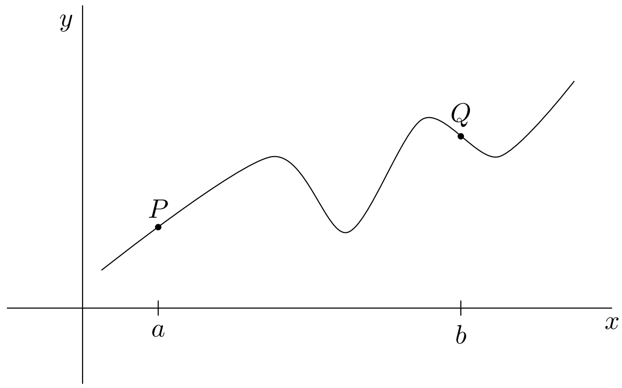

Figure 3.1.10: Sketch of Graph of $y = f(x)$

In order to paremetrize C, we can consider x as a parameter and set

$$x = t;$$

so that

$$y = f(t).$$

Hence, the equations

$$\left\{\begin{array}{rcl} x & = & t; \\ y & = & f(t), \end{array}\right. \qquad \text{for } a \leqslant t \leqslant b, \tag{3.17}$$

will parametrize C.

Example 3.1.13. Let C denote the portion of the parabola given by the equation

$$y = x^2$$

from the point $P(-1, 1)$ to the point $(2, 4)$.

Give a parametrization for C.

Solution: A sketch of C is shown in Figure 3.1.11.

Use the equations in (3.17) with $f(x) = x^2$, $a = -1$, and $b = 2$ to get

$$\left\{\begin{array}{rcl} x & = & t; \\ y & = & t^2, \end{array}\right. \qquad \text{for } -1 \leqslant t \leqslant 2.$$

$\square$

In the next example with construct another parametrization of the curve in Example 3.1.11.

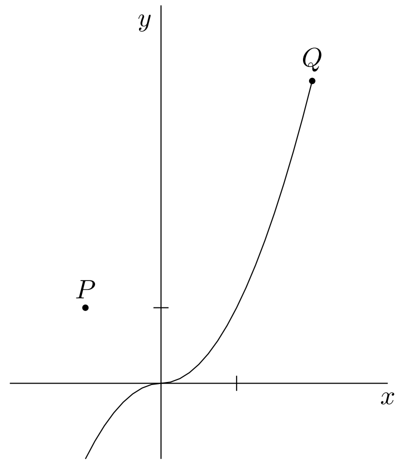

Figure 3.1.11: Sketch of Graph of $y = x^2$ from $x = -1$ to $x = 2$

Example 3.1.14. Let C denote the portion of the ellipse given by the graph of the equation

$$4x^2 + y^2 = 4, \tag{3.18}$$

in the first quadrant of the xy–plane, from the point $P(0, 2)$ to the point $Q(1, 0)$. Give a parametrization for C.

Solution: A sketch of the graph of C is shown in Figure 3.1.9.

Observe that C can also be realized as the graph of a function f that can be obtained by solving the equation in (3.18) for y. We obtain

$$f(x) = 2\sqrt{1 - x^2}, \quad \text{for } -1 \leqslant x \leqslant 1.$$

Thus, the equations

$$\left\{\begin{array}{rcl} x & = & t; \\ y & = & 2\sqrt{1 - t^2}, \end{array}\right. \qquad \text{for } 0 \leqslant t \leqslant 1,$$

also constitute a parametrization of the curve C. □

3.2 Differentiable Paths

Given functions x and y defined on an open interval J, the function $\overrightarrow{r} : J \to \mathbb{R}^2$ given by

$$\overrightarrow{r}(t) = (x(t), y(t)), \quad \text{for all } t \in J,$$

is called a **path**. If both x and y are differentiable, then $\overrightarrow{r}$ is said to be a **differentiable path** and its derivative is given by

$$\overrightarrow{r}'(t) = (x'(t), y'(t)), \quad \text{for all } t \in J.$$

Chapter 4

Vector Fields

In this course, we will focus on two–dimensional vector fields. These are function, $\overrightarrow{F}$, from a domain in $\mathbb{R}^2$ to $\mathbb{R}^2$; we write,

$$\overrightarrow{F} : D \to \mathbb{R}^2,$$

where D is the domain of the vector field.

In the first section of this chapter, we present a few examples of two–dimensional vector fields and their geometric representation and interpretations, and in the following section we present the concept of the flow of a field, which relates vector fields to that paths and parametrized curves that we studied in previous sections.

4.1 Examples of Vector Fields

We begin with examples in which the domain, D, of a vector field, $\overrightarrow{F}$, is the entire xy–plane. We then have that

$$\overrightarrow{F} : \mathbb{R}^2 \to \mathbb{R}^2.$$

4.2 The Flow of a Vector Field

Chapter 5

Real Valued Functions of Two Variables

5.1 Graph of functions of two variables

5.1.1 Sections and lever sets

5.1.2 Contour plots

5.1.3 Surfaces in three dimensions

5.2 Linear Functions

5.2.1 Definition of a linear function

5.2.2 Graphs of linear functions: planes in space

5.3 Vectors

5.3.1 The dot product

5.3.2 Norm of vectors

5.4 Differentiability

5.4.1 Partial derivatives

5.4.2 The Chain Rule

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5.4.4 The gradient of a function of two variables

5.4.5 Tangent plane to a surface

5.4.6 Linear approximations to a function of two variables

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Chapter 6

Linear Vector Fields in Two Dimensions

6.1 Definition of a Linear Vector Fields

A function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is said to be a linear vector field if it is given by an expression of the form

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cd + dy \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \tag{6.1}$$

where a, b, c and d are real numbers (constants).

We note that, starting in this chapter, we adopt the convention of using columns to denote vectors in in $\mathbb{R}^2$. We are also dropping the arrow above the symbol to denote names of vectors. The context will make it clear when we are talking about vectors and not numbers. We shall also refer to numbers as “scalars” to distinguish them from vectors.

Examples of linear vector fields are

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ -y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

and

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x - y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

The vector field associated with the Lotka–Volterra system, namley,

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x - \beta xy \\ \beta xy - \gamma y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

where α, β and γ are real constants, is not linear (Why?).

Another example of a two–dimensional vector field that is not linear is provided by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

6.2 Matrices and Matrix Algebra

The general form of linear field given in (6.1) can be written in a more compact way by using matrix notation. In this section we discuss definitions of matrices and matrix products and present some of their properties.

A matrix is an array of numbers organized in rows of and columns. An $m \times n$ matrix consists m rows and n columns. In this course we will deal only with the cases in which m and n are 1 or 2.

A 2×1 matrix is a column vector

$$\begin{pmatrix} x \\ y \end{pmatrix}, \tag{6.2}$$

2 rows and 1 column. We will use column vectors of the form in (6.2) to represent vectors in $\mathbb{R}^2$.

A 1×2 matrix is a row vector of the form

$$\begin{pmatrix} a & b \end{pmatrix}. \tag{6.3}$$

Denote the column vector in (6.2) by v and the row vector in (6.3) by R.

Definition 6.2.1 (Row–Column Product)**.** The row-column product, Rv, is the scalar obtained by

$$R\mathrm{v} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax + by. \tag{6.4}$$

We note two things about the product in (6.4):

(i) it is the first entry in the definition of the linear vector field in (6.1);

(ii) It is the dot product of the vectors $\overrightarrow{w} = a\widehat{i} + b\widehat{j}$ and $\overrightarrow{v} = x\widehat{i} + y\widehat{j}$.

Writing

$$\mathrm{w} = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{6.5}$$

the **transpose** of the column vector w, denoted by w^T, is the column vector obtained from w in (6.5) as follows

$$\mathrm{w}^T = \begin{pmatrix} a \\ b \end{pmatrix}^T = \begin{pmatrix} a & b \end{pmatrix}$$

Thus, given two column vectors

$$\mathrm{w} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \mathrm{v} = \begin{pmatrix} x \\ y \end{pmatrix},$$

we can from the row–column product of w^T and v to get

$$\mathrm{w}^T \mathrm{v} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax + by,$$

which is the dot product of the vectors w and v in $\mathbb{R}^2$. We then have that

$$\mathrm{w} \cdot \mathrm{v} = \mathrm{w}^T \mathrm{v}.$$

In general, a 2×2 matrix is an array, A, of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{6.6}$$

where a, b, c and d are real numbers.

The matrix A in (6.6) is made up of two rows

$$R_1 = \begin{pmatrix} a & b \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} c & d \end{pmatrix},$$

or two columns

$$\mathrm{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} b \\ d \end{pmatrix}.$$

We can therefore write

$$A = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \tag{6.7}$$

or

$$A = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix}.$$

We can use the row–column product defined in Definition 6.2.1 to define the product of a matrix and a vector.

Definition 6.2.2 (Product of Matrix and a Vector)**.** Given a 2×2 matrix A and a (column) vector v, the product, $A\mathrm{v}$, is the column vector obtained as follows: Write the matrix A in terms of its rows as in (6.7); then, compute

$$A\mathrm{v} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \mathrm{v} = \begin{pmatrix} R_1 \mathrm{v} \\ R_2 \mathrm{v} \end{pmatrix} \tag{6.8}$$

Thus, if A is as given in (6.6) and v is the column vector given in (6.2), then

$$A\mathrm{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cd + dy \end{pmatrix}. \tag{6.9}$$

Comparing (6.1) and (6.9) we see that the linear vector field in (6.1) can be written as multiplication by the 2×2 matrix A given in (6.6). We then have that

$$F\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{6.10}$$

According to (6.10), every linear vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ has a 2×2 matrix, A, associated with it.

Example 6.2.3. The matrix associated with the linear field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -4x \end{pmatrix}, \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

is

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}.$$

Definition 6.2.4 (Matrix Multiplication)**.** Given 2×2 matrices A and B, write the matrix A in terms of its rows as in (6.7), and write B in terms of its columns,

$$B = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix}.$$

The matrix product AB is the 2×2 matrix obtained as follows

$$AB = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 \end{bmatrix} = \begin{pmatrix} R_1\mathrm{v}_1 & R_1\mathrm{v}_2 \\ R_2\mathrm{v}_1 & R_2\mathrm{v}_2 \end{pmatrix}. \tag{6.11}$$

Example 6.2.5. Let A denote the 2×2 matrix

$$A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}, \tag{6.12}$$

and B the 2×2 matrix

$$B = \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix}. \tag{6.13}$$

Then, using the formula in (6.11), we compute

$$AB = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -10 & 2 \end{pmatrix}. \tag{6.14}$$

Example 6.2.6. Let A and B be as given in (6.12) and (6.13), respectively. We can also form the matrix product BA:

$$BA = \begin{pmatrix} 1 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 8 & -4 \\ 20 & -8 \end{pmatrix}. \tag{6.15}$$

Comparing the results in (6.14) and (6.15) we see that

$$AB \neq BA; \tag{6.16}$$

thus, matrix multiplication is not commutative.

In order to obtain (6.16), we have used the notion of matrix equality.

Definition 6.2.7 (Matrix Equality)**.** Two matrices are said to be equal if and only if corresponding entries in the matrix are the same.

Example 6.2.8. Let A and B denote the 2×2 matrices given by

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}.$$

Then,

$$AB = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.17}$$

Definition 6.2.9 (Zero Matrix)**.** The 2×2 matrix whose entries all all 0 is called the zero 2×2 matrix. We will denote it by the symbol O; so that,

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 6.2.10. Let A and B denote the 2×2 matrices given by

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -3/2 & 1 \\ -1/2 & 0 \end{pmatrix}.$$

We compute

$$AB = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} -3/2 & 1 \\ -1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We also compute

$$BA = \begin{pmatrix} -3/2 & 1 \\ -1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition 6.2.11 (Identity Matrix)**.** The 2×2 matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is called the 2×2 identity matrix. We will denote it by the symbol I; so that,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the matrices A and B in Example 6.2.10, we saw that

$$AB = BA = I.$$

When this happens we say that the matrix A is invertible.

Definition 6.2.12 (Invertible Matrix)**.** A 2×2 matrix A is said to be invertible if there exists a 2×2 matrix B such that

$$AB = BA = I. \tag{6.18}$$

If (6.18) holds true, we also say that B is the inverse of A and denote it by A^{-1}; so that,

$$AA^{-1} = A^{-1}A = I.$$

Definition 6.2.13 (Matrix Addition)**.** Given two matrices, A and B, of the same size, the matrix $A + B$ is obtained by adding corresponding entries. We have three cases two consider.

(i) Adding two 2×2 matrices.

Let A and B be 2×2 matrices given by

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

respectively. The sum $A + B$ is defined by

$$A + B = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}.$$

(ii) Adding two column–vectors in $\mathbb{R}^2$.

Let v_1 and v_2 be vectors in $\mathbb{R}^2$ given by

$$\mathrm{v}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},$$

respectively. The vector sum $\mathrm{v}_1 + \mathrm{v}_2$ is defined by

$$\mathrm{v}_1 + \mathrm{v}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}.$$

(iii) Adding two row–vectors in $\mathbb{R}^2$.

Let R_1 and R_2 be row–vectors in $\mathbb{R}^2$ given by

$$R_1 = \begin{pmatrix} a_1 & b_1 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} a_2 & b_2 \end{pmatrix},$$

respectively. The sum $R_1 + R_2$ is the row–vector defined by

$$R_1 + R_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \end{pmatrix}.$$

Remark 6.2.14. The addition of a column–vector and a row vector is not defined, neither is the addition of a 2×2 matrix and a vector.

Example 6.2.15. Let A and B denote 2×2 matrices given by

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix},$$

respectively. Then

$$A + B = O,$$

where O is the 2×2 zero matrix. We say that B is the additive inverse of A and write

$$B = -A.$$

Definition 6.2.16 (Scalar Multiplication)**.** Given a matrix A and a real number t, the matrix tA is obtained by multiplying every entry in the matrix by t. We have three cases two consider.

(i) Scalar multiple of a 2×2 matrix.

Let A be the 2×2 matrix given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The matrix tA is defined by

$$tA = \begin{pmatrix} ta & tb \\ tc & td \end{pmatrix}.$$

(ii) Scalar multiple of a column–vector in $\mathbb{R}^2$.

Let v be a vector in $\mathbb{R}^2$ given by

$$\mathrm{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The vector $t\mathrm{v}$ is defined by

$$t\mathrm{v} = \begin{pmatrix} tx \\ ty \end{pmatrix}.$$

(iii) Scalar multiple of a row–vector in $\mathbb{R}^2$.
Let R be a row–vector in $\mathbb{R}^2$ given by

$$R = \begin{pmatrix} a & b \end{pmatrix}.$$

The row–vector tR is defined by

$$tR = \begin{pmatrix} ta & tb \end{pmatrix}.$$

Example 6.2.17. Let A denote the 2×2 matrix given by

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}.$$

Compute $A^2 + 3A + 2I$, where I is the 2×2 identity matrix.

Solution: First, we compute

$$A^2 = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 6 \\ -3 & 7 \end{pmatrix}.$$

Then

$$\begin{aligned} A^2 + 3A + 2I &= \begin{pmatrix} -2 & 6 \\ -3 & 7 \end{pmatrix} + 3\begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 6 \\ -3 & 7 \end{pmatrix} + \begin{pmatrix} 0 & -6 \\ 3 & -9 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

so that

$$A^2 + 3A + 2I = O,$$

the 2×2 zero matrix. □

6.2.1 Properties of Matrix Products

In this section we list a few of the properties of the matrix and vector operations that have been defined so far. These operations will be used in various matrix calculations in these notes.

Proposition 6.2.18 (Distributive Properties)**.**

(i) For 2×2 matrices A, B and C,

$$A(B + C) = AB + AC.$$

(ii) For 2×2 matrices A, B and C,

$$(A + B)C = AC + BC.$$

(iii) For a 2×2 matrix A and column–vectors v_1 and v_2 in $\mathbb{R}^2$,

$$A(\mathrm{v}_1 + \mathrm{v}_2) = A\mathrm{v}_1 + A\mathrm{v}_2.$$

(iv) For 2×2 matrices A and B, and a column–vector v in $\mathbb{R}^2$,

$$(A + B)\mathrm{v} = A\mathrm{v} + B\mathrm{v}.$$

(v) For a scalar t and column–vectors v_1 and v_2 in $\mathbb{R}^2$,

$$t(\mathrm{v}_1 + \mathrm{v}_2) = t\mathrm{v}_1 + t\mathrm{v}_2.$$

(vi) For a scalars t and r, and a column–vector v in $\mathbb{R}^2$,

$$(t + r)\mathrm{v} = t\mathrm{v} + r\mathrm{v}.$$

Proposition 6.2.19 (Associative Properties)**.**

(i) For 2×2 matrices A, B and C,

$$A(BC) = (AB)C.$$

(ii) For a 2×2 matrix A, a column–vector v, and a scalar t,

$$A(t\mathrm{v}) = tA\mathrm{v}$$

Remark 6.2.20. The properties in Proposition 6.2.18 and 6.2.19 can be derived by using the definition of the operations in Definitions 6.2.4, 6.2.13 and 6.2.16.

Example 6.2.21. In Example 6.2.17 we saw that the matrix

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}. \tag{6.19}$$

satisfies the equation

$$A^2 + 3A + 2I = O. \tag{6.20}$$

We can rewrite (6.20) as

$$A^2 + 3A = -2I. \tag{6.21}$$

We can then use the distributive properties to rewrite the left–hand side in (6.21) to get

$$A(A + 3I) = -2I, \tag{6.22}$$

where we have also used the fact that $A = AI$.

Next, multiply on both sides of (6.22) by the scalar $-\dfrac{1}{2}$, and use the distributive and associative properties, to get

$$A\left[-\frac{1}{2}(A+3I)\right] = I. \tag{6.23}$$

It follows from (6.23) and Definition 6.2.12 that the matrix A given in (6.19) is invertible, and its inverse is given by

$$A^{-1} = -\frac{1}{2}(A+3I),$$

or

$$\begin{aligned} A^{-1} &= -\frac{1}{2}A - \frac{3}{2}I \\ &= -\frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1/2 & 3/2 \end{pmatrix} + \begin{pmatrix} -3/2 & 0 \\ 0 & -3/2 \end{pmatrix}; \end{aligned}$$

so that

$$A^{-1} = \begin{pmatrix} -3/2 & 1 \\ -1/2 & 0 \end{pmatrix}. \tag{6.24}$$

In the next section we will see another way to obtain the result in (6.24).

6.2.2 Invertible Matrices

In this section we will see that the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.25}$$

has an inverse provided that $ad - bc \neq 0$.

The expression $ad - bc$ is called the determinant of the matrix A in (6.25) is called the determinant of the matrix A and will denoted by $\det(A)$. We then have that

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc. \tag{6.26}$$

Assume that $\det(A) \neq 0$ and look for a 2×2 matrix B, given by

$$B = \begin{pmatrix} x & z \\ y & w \end{pmatrix}, \tag{6.27}$$

where x, y, z and w are unknowns to be determine shortly, such that

$$AB = I, \tag{6.28}$$

the 2×2 identity matrix, or

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or

$$\begin{pmatrix} ax + by & az + bw \\ cx + dy & cz + dw \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.29}$$

It follows from (6.29) that (6.28) is equivalent to the system of equations

$$\begin{cases} ax + by &= 1 \\ cx + dy &= 0 \\ az + bw &= 0 \\ cz + dw &= 1 \end{cases} \tag{6.30}$$

We first consider the case in which

$$a \neq 0 \quad \text{and} \quad c \neq 0. \tag{6.31}$$

In this case, we can solve the second equation in (6.30) for x to get

$$x = -\frac{d}{c} y, \tag{6.32}$$

and substitute into the first equation in (6.30) to get

$$-\frac{ad}{c} y + by = 1,$$

which can be solved for y to yield

$$y = -\frac{c}{ad - bc},$$

or

$$y = -\frac{c}{\det(A)}. \tag{6.33}$$

Combining (6.34) and (6.32) we get that

$$x = \frac{d}{\det(A)}. \tag{6.34}$$

Similarly, if (6.31) is true, then we can solve the third equation in (6.30) for z to get

$$z = -\frac{b}{a} w. \tag{6.35}$$

Substituting (6.35) into the last equation in (6.30) then yields

$$-\frac{bc}{a} w + dw = 1,$$

which can be solved for w to yield

$$w = \frac{a}{\det(A)}. \tag{6.36}$$

Combining (6.35) and (6.36) then yields

$$z = -\frac{b}{\det(A)}. \tag{6.37}$$

It follows from (6.27), (6.28), (6.34), (6.34), (6.37) and (6.36) that the matrix

$$B = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{6.38}$$

for the case in which $\det(A) \neq 0$ and (6.31) holds true, is such that

$$AB = I.$$

It can also be verified that $BA = I$ (see Problem 1 in Assignment #16). Thus the matrix in (6.38) is the inverse of A for the case in which $\det(A) \neq 0$ and (6.31) holds true.

Next, assume that $\det(A) \neq 0$ and

$$a = 0 \quad \text{or} \quad c = 0. \tag{6.39}$$

Suppose that $a = 0$; then,

$$\det(A) = -bc \neq 0; \tag{6.40}$$

so that

$$b \neq 0 \quad \text{and} \quad c \neq 0. \tag{6.41}$$

It then follows from the first equation in (6.30) that

$$by = 1,$$

from which we get that

$$y = \frac{1}{b}, \tag{6.42}$$

since $b \neq 0$ by the first condition in (6.41).

Next, use the second condition in (6.41) and (6.40) to rewrite (6.42) as

$$y = -\frac{c}{\det(A)}. \tag{6.43}$$

Next, use the second condition in (6.41) to solve the second equation in (6.30) to get

$$x = -\frac{d}{c} y. \tag{6.44}$$

Combining (6.44) and (6.43) then yields that

$$x = \frac{d}{\det(A)}. \tag{6.45}$$

Continuing with the assumption that $a = 0$, and using the first condition in (6.41), we obtain from the third equation in (6.30) that

$$w = 0, \tag{6.46}$$

we can rewrite as

$$w = \frac{a}{\det(A)}, \tag{6.47}$$

since $a = 0$.

Finally, substituting the result in (6.46) into the last equation in (6.30) we get

$$cz = 1,$$

which can be solved for z to yield

$$z = \frac{1}{c}, \tag{6.48}$$

in view of the second condition in (6.41). We can then use (6.39) to rewrite (6.48) as

$$z = -\frac{b}{\det(A)}. \tag{6.49}$$

We note that the results in (6.45), (6.43), (6.49) and (6.47) are precisely the results in (6.34), (6.34), (6.37) and (6.36), respectively, which is the result that we obtained in the previous case. Consequently, in this case as well we obtain that the inverse of A in (6.25) is given by B in (6.38).

The second option in (6.39) (namely, $c = 0$) yields the same result. We therefore conclude that, if A given in (6.25) is such that $\det(A) \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{6.50}$$

Example 6.2.22. Let A be the matrix in Example 6.2.21; namely,

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}. \tag{6.51}$$

Then, $\det(A) = 2$; so that, $\det(A) \neq 0$. Thus, we can use the formula in (6.50) to compute the inverse of A in (6.51) to get

$$A^{-1} = \frac{1}{2} \begin{pmatrix} -3 & 2 \\ -1 & 0 \end{pmatrix},$$

which yields the same matrix in (6.24) obtained in Example 6.2.21.

6.3 The Flow of Two–Dimensional Vector Fields

The goal of this section is to compute the flow of two–dimensional linear fields $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}, \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{6.52}$$

The flow of F in (6.52) is made up of curves parametrized by paths

$$\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$$

satisfying the differential equations

$$\begin{cases} \dfrac{dx}{dt} = ax + by; \\[2ex] \dfrac{dy}{dt} = cx + dy. \end{cases} \tag{6.53}$$

We can rewrite the system in (6.53) in vector form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \tag{6.54}$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.55}$$

In (6.54) the dot on top of a symbol for a variable indicates the derivative of that variable with respect to t; so that,

$$\dot{x} = \frac{dx}{dt} \quad \text{and} \quad \dot{y} = \frac{dy}{dt}.$$

Example 6.3.1. Consider system

$$\begin{cases} \dfrac{dx}{dt} = -9y; \\[2ex] \dfrac{dy}{dt} = x - 6y. \end{cases} \tag{6.56}$$

The matrix corresponding to the system in (6.56) is

$$A = \begin{pmatrix} 0 & -9 \\ 1 & -6 \end{pmatrix}. \tag{6.57}$$

Let v be the vector

$$\mathrm{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{6.58}$$

We note that
$$Av = \begin{pmatrix} 0 & -9 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ -3 \end{pmatrix};$$
so that,
$$Av = -3v \tag{6.59}$$
We will show that the path
$$\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$$
defined by
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c\, e^{-3t} v, \quad \text{for } t \in \mathbb{R}, \tag{6.60}$$
where c is a constant and v is the vector in (6.58), is a solution of the equation
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{6.61}$$
where A is the 2×2 matrix given in (6.57). Indeed, taking the derivative with respect to t on both sides of (6.60) yields
$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \frac{d}{dt}\left[c\, e^{-3t} v\right] \\ &= c\frac{d}{dt}\left[\, e^{-3t}\right] v \\ &= c(-3)e^{-3t} v \\ &= ce^{-3t}(-3v); \end{aligned}$$
so that, by virtue of (6.59)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = ce^{-3t} Av.$$
Consequently, using the properties of the matrix product,
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A\left[ce^{-3t} v\right]. \tag{6.62}$$
Comparing (6.60) and (6.62) we see that
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix},$$
which is (6.61).

We have therefore obtained solutions to the system in (6.56) which lie on a line determined by the vector v in (6.57). These solutions are sketched in

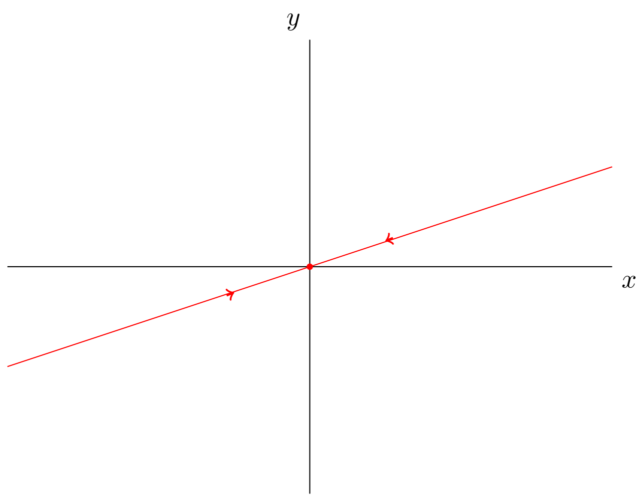

Figure 6.3.1: Sketch of (6.60)

Figure 6.3.1. The sketch shows the origin, corresponding to $c = 0$ in (6.60), and two half lines pointing towards the origin correspond to $c > 0$ (in the first quadrant), and to $c < 0$ (in the third quadrant). The lines point towards the origin because of the decreasing exponential in the definition of the solutions in (6.60).

Example 6.3.1 illustrates a special situation in the flow of linear fields. In some cases, trajectories along the flow will lie on lines through the origin. We shall refer to these special solutions as "line solutions."

Line solutions for a system of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \tag{6.63}$$

where A is a 2×2 matrix, occur when there exists a nonzero vector v in $\mathbb{R}^2$ such that

$$A\mathrm{v} = \lambda \mathrm{v}, \tag{6.64}$$

for some scalar λ. When this is the case, the paths

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c\, e^{\lambda t}\mathrm{v}, \quad \text{for } t \in \mathbb{R}, \tag{6.65}$$

for arbitrary constant c, yields solutions to the system in (6.63). The solutions in (6.65) yield the origin for the case $c = 0$, and two half-lines in the direction of the vector v pointing towards the origin if $\lambda < 0$, or away from the origin if $\lambda > 0$.

We are able to find line solutions of the system in (6.63) as long as we are able to find scalars λ for which the equation in (6.64). This is a very special situation; when it occurs, we call the scalar λ an eigenvalue of the matrix A; a corresponding nonzero vector v for which (6.64) holds true is called an eigenvector for λ.

6.3.1 Eigenvalues and Eigenvectors

Given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{6.66}$$

we say that a scalar λ is an eigenvalue of A if there exists a nonzero vector

$$\mathrm{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}. \tag{6.67}$$

Thus, in order to find eigenvalues for the matrix A in (6.66), we first need to find conditions on the entries of A that will guarantee that the equation in (6.67) has nonzero solutions. The equation in (6.67) can be written as system of two linear equations

$$\left\{ \begin{array}{rcl} ax + by & = & \lambda x; \\ cx + dy & = & \lambda y, \end{array} \right.$$

or

$$\left\{ \begin{array}{rcl} (a-\lambda)x + by & = & 0; \\ cx + (d-\lambda)y & = & 0. \end{array} \right. \tag{6.68}$$

By inspection we see that the system in (6.68) has the zero solution.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For this reason, this solution is usually referred to as the trivial solution. However, in some cases, the system in (6.68) might have infinitely many solutions (this would be the case in which the two equations in (6.68) represent the same line). This occurs, according to the result in Problems 3 and 4 in Assignment #15, when

$$(a-\lambda)(d-\lambda) - bc = 0,$$

or

$$\lambda^2 - (a+d)\lambda + ad - bc = 0. \tag{6.69}$$

The equation in (6.69) is called the characteristic equation of the matrix A in (6.66). Its solutions will be eigenvalues of A. The polynomial

$$p_{_A}(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc \tag{6.70}$$

is called the characteristic polynomial of A. The roots, or zeros, of $p_A(\lambda)$ are the eigenvalues of A. We have encountered the expression $ad-bc$ in the characteristic polynomial in (6.70); it is the determinant of A, denoted by $\det(A)$. The coefficient of expression $a+d$ is the sum of the entries along the main diagonal of A, and it is called the trace of A; we write,

$$\text{trace}(A) = a + d.$$

We can therefore write the characteristic polynomial of A as

$$p_A(\lambda) = \lambda^2 - \text{trace}(A)\lambda + \det(A). \tag{6.71}$$

Once we find an eigenvalue, λ, we can find a corresponding eigenvector by solving the system of equations in (6.68) for the specific value of λ. We illustrate this procedure in the following example.

Example 6.3.2. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}. \tag{6.72}$$

Solution: The trace of the matrix A in (6.72) is $\text{trace}(A) = 0$, its determinant is $\det(A) = -4$. Then, according to (6.71), the characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 - 4,$$

which factors into

$$p_A(\lambda) = (\lambda + 2)(\lambda - 2);$$

so that, the eigenvalues of A in (6.72) are

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 2. \tag{6.73}$$

Next, we compute eigenvectors corresponding to λ_1 and λ_2 in (6.73). We do this by solving the system of equations in (6.68) for $\lambda = \lambda_1$ and for $\lambda = \lambda_2$.

For $\lambda = -1$, the system in (6.68) yields

$$\begin{cases} 5x - y = 0; \\ 5x - y = 0, \end{cases}$$

which is the single equation

$$y = 5x. \tag{6.74}$$

Since we are looking for a nontrivial solution os the system, we can set $x = 1$ in (6.74) to get $y = 5$; so that,

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \tag{6.75}$$

is an eigenvector corresponding to $\lambda_1 = -2$.

Similarly, for $\lambda = \lambda_2 = 2$, we obtain the system

$$\left\{\begin{array}{rcl} x - y & = & 0; \\ x - y & = & 0, \end{array}\right.$$

which is equivalent to the equations

$$y = x,$$

from which we get that

$$\mathrm{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.76}$$

is an eigenvector corresponding to the eigenvalue $\lambda_2 = 2$. □

6.3.2 Line Solutions

An advantage of knowing eigenvalues and eigenvectors of a 2×2 matrix, A, is that they provide special kind of solutions to the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \tag{6.77}$$

For instance, if A has a real eigenvalue, λ, with a corresponding eigenvector, v, then the path $\begin{pmatrix} x \\ y \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2$ define by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c\, e^{\lambda t}\mathrm{v}, \qquad \text{for } t \in \mathbb{R},$$

where c is a constant, solves the system in (6.77). An instance of this fact was seen in Example 6.3.1.

For the case in which the matrix A has two real eigenvalues, λ_1 and λ_2, with $\lambda_1 \neq \lambda_2$, corresponding eigenvectors v_1 and v_2, respectively, do not lie on the same line (see Problem 3 in Assignment #18). Consequently, we obtain two distinct line solutions for the system in (6.77),

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = c_1\, e^{\lambda_1 t}\mathrm{v}_1 \quad \text{and} \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = c_2\, e^{\lambda_2 t}\mathrm{v}_2 \quad \text{for } t \in \mathbb{R},$$

where c_1 and c_2 are constants. Furthermore, it is shown in courses in differential equations, that the expression

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\, e^{\lambda_1 t}\mathrm{v}_1 + c_2\, e^{\lambda_2 t}\mathrm{v}_2 \quad \text{for } t \in \mathbb{R}, \tag{6.78}$$

where c_1 and c_2 are arbitrary constants, yields all solutions of the system in (6.77) (see also Problem 4 in Assignment #17).

The expression in (6.78) can be uses to aid in sketching the flow the vector field

$$A \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

We illustrate this in the next example.

Example 6.3.3. Sketch the flow of the vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - y \\ 5x - 3y \end{pmatrix}, \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2. \tag{6.79}$$

Solution: The flow of the field (6.79) is obtained by solving the pair of differential equations

$$\begin{cases} \dfrac{dx}{dt} &= 3x - y; \\ \\ \dfrac{dy}{dt} &= 5x - 3y. \end{cases} \tag{6.80}$$

The system (6.80) can in turn be written in vector form as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \tag{6.81}$$

where A is the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}. \tag{6.82}$$

We saw in Example 6.3.2 that the matrix A in (6.82) has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 2$, with corresponding eigenvectors given in (6.75) and (6.76), respectively; that is,

$$\mathrm{v}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathrm{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{6.83}$$

By the discussion preceding this example, all solutions to the system in (6.80) are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\, e^{-2t}\mathrm{v}_1 + c_2\, e^{2t}\mathrm{v}_2 \quad \text{for } t \in \mathbb{R}, \tag{6.84}$$

where c_1 and c_2 are arbitrary constants, and v_1 and v_2 are as given in (6.83).

The general form of the solutions given in (6.84) is very helpful in sketching the the flow of the field in (6.79). We first note that, if $c_1 = c_2 = 0$, (6.84) yields the origin, $(0, 0)$, as a solution. If $c_1 \neq 0$ and $c_2 = 0$, the flow curves will lie on the line through the origin parallel to the vector v_1; both solution curves will point towards the origin because the exponential e^{-2t} decreases with increasing t. On the other hand, if $c_1 = 0$ and $c_2 \neq 0$, the trajectories lie on the in the direction of the vector v_2 and point away from the origin because the exponential e^{2t} increases with increasing t. We have therefore obtained the origin and four line solutions. All of these are shown in Figure 6.3.2. Figure 6.3.2 shows other possible flow curves of the field in (6.79). In the next section we will see how to sketch those curves. □

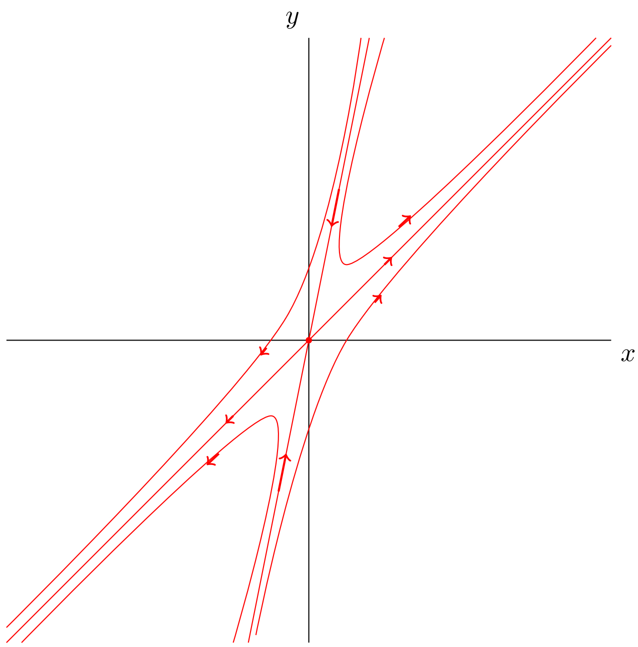

Figure 6.3.2: Sketch of Flow of the Field in (6.79)