Notes on Partial Differential Equations

Preliminary Lecture Notes

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Chapter 1

Preface

This course is an introduction to the theory and applications of partial differential equations (PDEs). PDEs are expressions involving functions of several variables and its partial derivatives in which we seek to find one of the functions, or a set of functions, subject to some initial conditions (if time is involved as one of the variables) or boundary conditions. They arise naturally when modeling physical or biological systems in which assumptions of continuity and differentiability are made about the quantities in question. In these notes we will discuss several modeling situations that give rise to PDEs.

In problems involving PDEs we are mainly interested in the question of existence of solutions. In some cases, answering this questions amounts to coming up with formulas for the solutions. In these notes we will discuss a few techniques for constructing solutions (e.g., separation of variables, series expansions and Green's function methods) for the special case of linear equations.

Linear partial differential equations are very important because they come up in many applications in the natural sciences. There are three major classes of linear PDEs: hyperbolic, parabolic and elliptic equations. Archetypal instances of these classes of PDEs are the classical equations of mathematical physics: the wave equation, the heat or diffusion equations, and Laplace's or Poisson's equations, respectively. In simple instances of these equations in one and two–dimensional space, we will show how to construct solutions subject to some initial conditions and some boundary conditions. These constructions will be based on the method of Fourier series expansions. We will also explore other methods for constructing solutions involving Green's functions and transform methods. All these methods rely on the linear structure of the equations.

In most cases, however, explicit constructions of solutions are not possible. In these cases, the only recourse we have is analytical proofs of existence, or nonexistence, and qualitative analysis to deduce properties of solutions. Once an existence theorem is obtained for a particular PDE problem, the next step in the analysis might involve approximation techniques to get information on the behavior and property of solutions. In these notes, we will discuss a few of those computational techniques.

Chapter 2

How Do PDEs Arise?

In general, a partial differential equation for a function, u, of several variables, $u(x_1, x_2, \ldots, x_n)$, is an expression of the form

$$F(x, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots, u_{x_n x_n}, \ldots) = 0, \tag{2.1}$$

where $x = (x_1, \ldots, x_n)$ and $u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots, u_{x_n x_n}, \ldots$ denote partial derivatives of u, for some function, F, of several variables. For example, in the simplest case in which u is a function of time, $t \in \mathbb{R}$, and a single space variable $x \in \mathbb{R}$, an instance of (2.1) is provided by

$$u_t - k u_{xx} = 0, \tag{2.2}$$

for some constant k.

While we are usually interested in knowing when equations like (2.1) and (2.2) have solutions subject to some initial and/or boundary conditions, in this chapter we will focus on the questions of how those equations arise in practice. For instance, the equation in (2.2) describes one–dimensional heat flow ($u(x,t)$ in this case denotes the temperature at time t and location x), or one–dimensional diffusion ($u(x,t)$ denotes the concentration of a substance at time t and location x). We begin by deriving a system of PDEs that describe the motion of fluids.

2.1 Modeling Fluid Flow

In this section we illustrate the use of a very important modeling principle, which we shall refer to as a **conservation principle**. This is a rather general principle that can be applied in situations in which the evolution in time of the quantity of a certain entity within a certain system is studied. For instance, suppose the quantity of a certain substance confined within a system is given by a continuous function of time, t, and is denoted by $Q(t)$ (the assumption of continuity is one that needs to be justified by the situation at hand). A conservation principle states that the rate at which a the quantity $Q(t)$ changes has to be accounted by how much of the substance goes into the system and how much of it goes out of the system. For the case in which Q is also assumed to be differentiable (again, this is a mathematical assumption that would need some justification), the conservation principle can be succinctly stated as

$$\frac{dQ}{dt} = \text{Rate of } Q \text{ in } - \text{ Rate of } Q \text{ out.} \tag{2.3}$$

In the cases to be considered in this section, the conservation principle in (2.3) might lead to a differential equation, or a system of differential equations, and so the theory of differential equations will be used to help in the analysis of the model.

In the derivation of the equations governing fluid motion, we will have the opportunity to apply the conservation principle in (2.3) several times.

Suppose we are following the motion of a fluid in some region R in three–dimensional space; see Figure 2.1.1. We assume that the fluid is a continuum

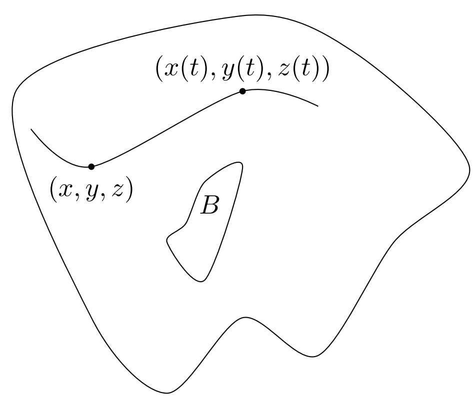

Figure 2.1.1: Region R

with density function $\rho(x, y, x, t)$, in units of mass per unit volume, so that the mass of a fluid element of volume $dV = dxdydz$ around a point (x, y, z) at time t is, approximately,

$$\rho(x, y, x, t)dV,$$

where dV denotes the volume of the fluid element. It then follows that the mass of fluid contained in a subregion B of R (see Figure 2.1.1) at time t is given by

$$M(B, t) = \iiint_B \rho(x, y, x, t)\ dV. \tag{2.4}$$

We assume throughout this discussion that ρ is a continuous function.

We also assume that each fluid element located at (x, y, z) at time t moves according to a velocity vector $\vec{u} = (u_1, u_2, u_3)$, where u_1, u_2 and u_3 are differentiable functions of (x, y, z, t). Thus, the path that a fluid element located at (x, y, z) at time $t = 0$ will follow is determined by the following system of ordinary differential equations

$$\begin{cases} \dfrac{dx}{dt} = u_1(x(t), y(t), z(t), t); \\ \\ \dfrac{dy}{dt} = u_2(x(t), y(t), z(t), t); \\ \\ \dfrac{dz}{dt} = u_3(x(t), y(t), z(t), t), \end{cases} \tag{2.5}$$

subject to the initial conditions

$$\begin{cases} x(0) = x; \\ y(0) = y; \\ z(0) = z. \end{cases} \tag{2.6}$$

If we assume that the components of the velocity field $\vec{u}$ are differentiable with continuous derivatives throughout the region R and for all times t (i.e., $\vec{u}$ is a C^1 vector field), then a solution to the system of ordinary differential equations in (2.5) subject to the initial conditions in (2.6) is guaranteed to exist over some maximal interval of time containing 0. The solution $(x(t), y(t), y(t))$ of the system in (2.5) subject to the initial conditions in (2.6) defines a path in space,

$$t \mapsto (x(t), y(t), y(t)),$$

for t in the maximal interval of existence, which describes the motion of a fluid element located at (x, y, z) at time $t = 0$. The path traced by the fluid element as it moves in time is called a **pathline**; Figure 2.1.1 shows what a pathline through (x, y, z) might look like. If we knew the velocity field at any point in space and at any time, we could compute the pathline through (x, y, z) by integrating the equations in (2.5) and imposing the initial conditions in (2.6):

$$\begin{aligned} x(t) &= x + \int_0^t u_1(x(\tau), y(\tau), z(\tau), \tau)\ d\tau; \\ y(t) &= y + \int_0^t u_2(x(\tau), y(\tau), z(\tau), \tau)\ d\tau; \\ z(t) &= z + \int_0^t u_3(x(\tau), y(\tau), z(\tau), \tau)\ d\tau. \end{aligned} \tag{2.7}$$

However, the velocity field is usually not known, and we need to do more modeling to find equations involving u_1, u_2 and u_3 that we hope we can solve.

2.1.1 The Continuity Equation

Consider a subregion, B, of R, with smooth boundary ∂B, as that pictured in Figure 2.1.1. The mass of the fluid contained at time t in that region, $M_B(t)$, is given by equation (2.4),

$$M_B(t) = \iiint_B \rho(x, y, x, t)\ dV. \tag{2.8}$$

By the principle of conservation of mass, the rate of change in the mass of fluid contained in B has to be accounted for by how much fluid is entering the region and how much is leaving per unit of time:

$$\frac{dM_B}{dt} = \text{ Rate of fluid into } B - \text{ Rate of fluid out of } B. \tag{2.9}$$

The equation in (2.9) is an instance of the conservation principle in (2.3).

If we assume that ρ is a C^1 function in R, we can compute the left–hand side of the equation by differentiating under the integral in (2.8):

$$\frac{dM_B}{dt} = \iiint_B \frac{\partial\rho}{\partial t}(x, y, x, t)\ dV. \tag{2.10}$$

Next, we compute the right–hand side of the expression in (2.9). Let $\vec{n}$ denote the unit vector normal to the boundary, ∂B, of the region B pointing outward. The outward unit normal, $\vec{n}(x, y, z)$, to the boundary of B is guaranteed to exist at every point $(x, y, z) \in \partial B$ if we assume that ∂B is a smooth surface. Then, the rate of fluid passing through an element of area, dA, on the surface ∂B can be expressed, approximately, as

$$\rho\ \vec{u} \cdot \vec{n}\ dA, \tag{2.11}$$

where $\vec{u} \cdot \vec{n}$ denotes the dot product of $\vec{u}$ and $\vec{n}$. Note that the expression in (2.11) is in units of mass per unit of time. Integrating the expression in (2.11) over the boundary of B yields the net **flux** of mass across the surface ∂B,

$$\iint_{\partial B} \rho\ \vec{u} \cdot \vec{n}\ dA. \tag{2.12}$$

Since the outward unit normal, $\vec{n}$, points away from the region B, the expression in (2.12) measures the flux of fluid away from the region B, if it is positive; if the expression in (2.12) is negative, it measures the net amount of fluid per unit time that enters B. We can therefore write the conservation principle in (2.9) as

$$\frac{dM_B}{dt} = -\iint_{\partial B} \rho\ \vec{u} \cdot \vec{n}\ dA. \tag{2.13}$$

To understand the reason for the minus sign on the right–hand side of the expression in (2.13), observe that a net increase in the amount of fluid in the region B, which yields a positive sign for the derivative in the left–hand side of (2.13), corresponds to a net amount of fluid flowing into the region B across the boundary ∂B.

Since we are assuming that the boundary of B is smooth, we can apply the Divergence Theorem to rewrite the integral in the right–hand side of (2.13) as follows:

$$\iint_{\partial B} \rho\, \vec{u}\cdot\vec{n}\; dA = \iiint_B \nabla\cdot(\rho\vec{u})\; dV, \tag{2.14}$$

where $\nabla\cdot(\rho\vec{u})$ denotes the divergence of the vector field $\rho\vec{u}$; that is,

$$\nabla\cdot(\rho\vec{u}) = \frac{\partial}{\partial x}(\rho u_1) + \frac{\partial}{\partial y}(\rho u_2) + \frac{\partial}{\partial z}(\rho u_3). \tag{2.15}$$

In view of (2.10) and (2.14), we see that we can rewrite the conservation equation in (2.13) as

$$\iiint_B \frac{\partial \rho}{\partial t}\; dV = -\iiint_B \nabla\cdot(\rho\vec{u})\; dV,$$

or

$$\iiint_B \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right]\; dV = 0. \tag{2.16}$$

If we assume that the vector field $\vec{u}$ and the scalar field ρ are C^1 functions over R and for all times t, then the fact that (2.16) holds true for any subregion B of R with smooth boundary implies that integrand on the left–hand side of (2.16) must be 0 over R and for all t; that is,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0, \quad \text{in } R \text{ and for all } t. \tag{2.17}$$

The equation in (2.17) is an example of a partial differential equation (PDE) involving the functions ρ, u_1, u_2 and u_3; in fact, using the definition of divergence (see (2.15)), the PDE in (2.17) can be rewritten as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_1) + \frac{\partial}{\partial y}(\rho u_2) + \frac{\partial}{\partial z}(\rho u_3) = 0. \tag{2.18}$$

The PDE in (2.17) is called the **continuity equation** and it expresses the conservation principle for a quantity of density ρ that flows according to a velocity field $\vec{u}$ in some region in space. For one–dimensional flow with linear density $\rho(x,t)$ and scalar velocity field $u(x,t)$, for $x \in \mathbb{R}$ and $t \in \mathbb{R}$, the continuity equation reads

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0; \tag{2.19}$$

see (2.18). The equation in (2.19) is an example of a first order PDE because the first derivatives of the functions ρ and u are involved. As it stands, the PDE in (2.19) involves two unknown functions, the density, ρ, and the velocity, u. Thus, we will need one more relation or equations in order for us to even begin to solve the problem posed by the modeling that led to the PDE in (2.19). An interesting example is provided by the following application to modeling traffic flow.

Example 2.1.1 (Modeling Traffic Flow)**.** Consider the unidirectional flow of traffic in a one–lane, straight road depicted in Figure 2.1.2. In this idealized road, vehicles are modeled by moving points. The location, x, of a point–vehicle is measured from some reference point along an axis parallel to the road. We

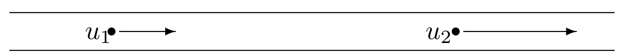

Figure 2.1.2: One–lane unidirectional flow

postulate a traffic density, $\rho(x,t)$, measured in units of number of cars per unit length of road at location x and time t. We interpret $\rho(x,t)$ as follows: Consider a section of the road from x to $x+\Delta x$ at time t. Let $\Delta N([x,x+\Delta x],t)$ denote the number of cars in the section $[x,x+\Delta x]$ at time t. We define $\rho(x,t)$ by the expression

$$\rho(x,t) = \lim_{\Delta x\to 0} \frac{\Delta N([x,x+\Delta x],t)}{\Delta x}, \tag{2.20}$$

provided that the limit on the right–hand side of (2.20) exists. It follows from (2.20) that, if a continuous traffic density, $\rho(x,t)$, is known for all x and t, then the number of cars in a section of the road from $x=a$ to $x=b$, where $a<b$, at time t is given by

$$\Delta N([a,b],t) = \int_a^b \rho(x,t)\ dx.$$

We assume that at each point x along the road and at each time t the velocity of vehicle at that location and time is dictated by a function $u(x,t)$, which we also assume to be a C^1 function. It follows from these assumptions and the derivations in this section that the one–dimensional equation of continuity in (2.19) applies to this situation.

Ideally, we would like to find a solution, ρ, to (2.19) subject to some initial condition

$$\rho(x,0) = \rho_o(x), \tag{2.21}$$

for some initial traffic density profile, ρ_o, along the road. In order to solve this problem, we postulate that u is a function of traffic density—the higher the density, the lower the traffic speed, for example. We may therefore write

$$u = f(\rho,\Lambda), \tag{2.22}$$

where f is a continuous function of ρ and a set of parameters, Λ. Some of the parameters might be a maximum density, $\rho_{\max}$, dictated by bumper to bumper traffic, and a maximum speed, $v_{\max}$; for instance, $v_{\max}$ is a speed limit. Given the parameters $\rho_{\max}$ and $v_{\max}$, the simplest model for the relationship between v and ρ is the **constitutive equation**

$$u = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{2.23}$$

We therefore arrive at the initial value problem (IVP):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + v_{\max}\dfrac{\partial}{\partial x}\left[\rho\left(1 - \dfrac{\rho}{\rho_{\max}}\right)\right] = 0 & \text{for } x \in \mathbb{R},\ t > 0; \\ \rho(x, 0) = \rho_o(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{2.24}$$

where we have incorporated the continuity equation in (2.19), the initial condition in (2.21), and the constitutive relation in (2.23), which is an instance of (2.22).

The partial differential equation model for traffic flow (2.24) presented in this section, based on the equation of continuity in (2.19) and a constitutive relation for the traffic velocity, u, and the traffic density ρ (of which (2.23) is just an example), was first introduced by Lighthill and Whitman in 1955 (see [LW55]); it was also treated by Richards in 1956, [Ric56]. In a subsequent section in these notes we will present an analysis of this model based on the **method of characteristics**.

We end this section with an alternate derivation of the conservation of mass equation in (2.17). In this approach we focus on the amount of fluid contained in a region B as the fluid in this region moves according to flow dictated by the velocity field $\vec{u}$. Suppose we begin to observe a portion of fluid in B at time $t = 0$. We assume that B is bounded and has smooth boundary ∂B. At some time $t > 0$, the portion of fluid in B has moved as a consequence of the fluid motion. We denote by B_t the portion of the fluid that we are following at time t (see Figure 2.1.3). To see how B_t comes about, consider a fluid element located at (x, y, z) at time $t = 0$. At time $t > 0$, the fluid element will be located at $(x(t), y(t), z(t))$, where the functions $x(t)$, $y(t)$ and $z(t)$ are solutions to the system of ordinary differential equations in (2.5) subject to the initial conditions in (2.6). We denote the point $(x(t), y(t), z(t))$ by $\varphi_t(x, y, z)$, and note that the map

$$(x, y, z) \mapsto \varphi_t(x, y, z), \qquad \text{for all } (x, y, z) \in R,$$

yields a C^1 map from R to R. Furthermore, φ_t is an invertible map for each t in the interval of existence for the initial value problem in (2.5) and (2.6). We shall refer to φ_t as the fluid flow map; it gives the location of a fluid element initially at (x, y, y) at time t as a result of fluid motion. It then follows that B_t is the image of B under the flow map φ_t; that is,

$$B_t = \varphi_t(B). \tag{2.25}$$

The total mass of the fluid in B_t is a function of time that we compute as follows

$$m(t) = \iiint_{B_t} \rho(\varphi_t(x, y, z), t)\ dV. \tag{2.26}$$

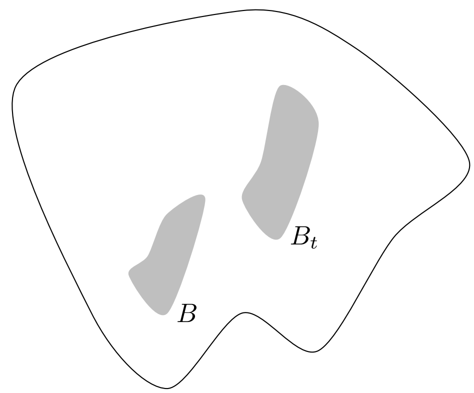

Figure 2.1.3: Balance of Forces

Note that

$$m(0) = \iiint_B \rho(x, y, y, 0)\ dV \equiv m_o, \tag{2.27}$$

which is the mass of the portion of fluid in the region B. As the flow of the fluid moves the region B, its shape might change. However, because of conservation of mass, the mass of fluid contained in B_t must the same as that contained in the region B at time $t = 0$; that is,

$$m(t) = m_o, \qquad \text{for all } t, \tag{2.28}$$

where m_o is the constant given in (2.27). It follows from (2.28) that

$$\frac{dm}{dt} = 0, \qquad \text{for all } t. \tag{2.29}$$

Before we compute $\frac{dm}{dt}$, we first rewrite the integral defining $m(t)$ in (2.26) by means of the change of variables provided by the flow map φ_t (see (2.25). We have

$$m(t) = \iiint_B \rho(x, y, z, t) J(x, y, z, t)\ dxdydz$$

where $J(x, y, z, t)$ the Jacobian of the map φ_t; that is, $J(x, y, z, t)$ is the determinant of the derivative map of φ_t. We then have that

$$\frac{dm}{dt} = \iiint_B \frac{\partial}{\partial t}[\rho J]\ dxdydz,$$

or

$$\frac{dm}{dt} = \iiint_B \left[\rho \frac{\partial J}{\partial t} + \frac{\partial \rho}{\partial t} J\right]\ dxdydz. \tag{2.30}$$

Making the change of variables provided by the flow map in the integral in (2.30) we obtain that

$$\begin{aligned}\frac{dm}{dt} &= \iiint_{B_t} \left[\rho(\varphi_t(x,y,z),t)\frac{1}{J(\varphi_t(x,y,z),t)}\frac{\partial}{\partial t}J(\varphi_t(x,y,z),t) \right. \\ &\qquad \left. +\frac{\partial}{\partial t}[\rho(\varphi_t(x,y,z),t)]\right] \, dV.\end{aligned} \tag{2.31}$$

It can be shown that

$$\frac{\partial}{\partial t}[J(\varphi_t(x,y,z),t) = J(\varphi_t(x,y,z),t) \ \nabla\cdot\vec{u}(\varphi_t(x,y,z),t), \tag{2.32}$$

see page 8 in [CM93]. Thus, substituting (2.32) into (2.31), we get

$$\frac{dm}{dt} = \iiint_{B_t} \left[(\nabla\cdot\vec{u}(\varphi_t(x,y,z),t))\rho(\varphi_t(x,y,z),t) + \frac{\partial}{\partial t}[\rho(\varphi_t(x,y,z),t)]\right] \, dV,$$

which we can write as

$$\frac{dm}{dt} = \iiint_{B_t} \left[(\nabla\cdot\vec{u})\rho + \frac{D\rho}{Dt}\right] \, dV, \tag{2.33}$$

where we have set

$$\begin{aligned}\frac{D\rho}{Dt} &= \frac{\partial}{\partial t}[\rho(\varphi_t(x,y,z),t)] \\ &= \frac{\partial}{\partial t}[\rho(x(t),y(t),z(t),t)] \\ &= \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt} + \frac{\partial\rho}{\partial t},\end{aligned} \tag{2.34}$$

where we have used the Chain Rule in the last step of the calculations in (2.34) and assumed that the density ρ is a C^1 field. We therefore have that

$$\frac{\partial}{\partial t}[\rho(x(t),y(t),z(t),t)] = \frac{\partial\rho}{\partial t} + u_1\frac{\partial\rho}{\partial x} + u_2\frac{\partial\rho}{\partial y} + u_3\frac{\partial\rho}{\partial z}, \tag{2.35}$$

where we have used the fact that $(x(t), y(t), z(t))$ solves the system of ordinary differential equations in (2.5). Writing (2.35) in vector notation we obtain

$$\frac{\partial}{\partial t}[\rho(x(t),y(t),z(t),t)] = \frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho, \tag{2.36}$$

where $\nabla\rho = \left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}\right)$ is the gradient of ρ. The expression in (2.36) is called the **material derivative** of the field ρ. It is also referred to as the **convective derivative** of ρ and is usually denoted by $\frac{D\rho}{Dt}$, so that

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho. \tag{2.37}$$

In general, given a C^1 scalar field, g, defined in a region R, the material derivative of g is given by

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \vec{u} \cdot \nabla g. \tag{2.38}$$

The material derivative of g in (2.38) expresses the rate of change of g along the pathlines as a result of the fact that the field g might change in time as well as a result of the motion of the fluid. The material derivative of a C^1 vector field, $\overrightarrow{G} = (g_1, g_2, g_3)$, is

$$\frac{D\overrightarrow{G}}{Dt} = \left(\frac{Dg_1}{Dt}, \frac{Dg_2}{Dt}, \frac{Dg_3}{Dt} \right),$$

which can be written as

$$\frac{D\overrightarrow{G}}{Dt} = \frac{\partial \overrightarrow{G}}{\partial t} + (\vec{u} \cdot \nabla)\overrightarrow{G}. \tag{2.39}$$

Combining (2.29) with (2.33) we get that

$$\iiint_{B_t} \left[(\nabla \cdot \vec{u})\rho + \frac{D\rho}{Dt} \right] dV = 0, \quad \text{for all } t. \tag{2.40}$$

It follows from (2.40) that

$$\frac{D\rho}{Dt} + (\nabla \cdot \vec{u})\rho = 0, \quad \text{in } R, \text{ for all } t, \tag{2.41}$$

where the material derivative, $\frac{D\rho}{Dt}$, of ρ is given in (2.37); that is,

$$\begin{aligned} \frac{D\rho}{Dt} &= \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho \\ &= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) - (\nabla \cdot \vec{u})\rho \end{aligned} \tag{2.42}$$

substituting the result of the calculations in (2.42) into (2.41) then yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \quad \text{in } R, \text{ for all } t,$$

which is the continuity equation in (2.17). We have also shown that the equation in (2.41) is an equivalent form of the continuity equation. We shall rewrite it here as

$$\frac{D\rho}{Dt} = -(\nabla \cdot \vec{u})\rho, \quad \text{in } R, \text{ for all } t. \tag{2.43}$$

2.1.2 Conservation of Momentum for an Ideal Fluid

The total momentum at time t of a the portion of fluid contained in a region B_t with smooth boundary, ∂B_t, is given by

$$\overrightarrow{\Pi}_B(t) = \iiint_{B_t} \rho(x(t), y(t), z(t), t)\vec{u}(x(t), y(t), z(t), t)\ dV,$$

or

$$\overrightarrow{\Pi}_B(t) = \iiint_{B_t} \rho(\varphi_t(x, y, x), t)\vec{u}(\varphi_t(x, y, x), t)\ dV,$$

and which we'll simply write as

$$\overrightarrow{\Pi}_B(t) = \iiint_{B_t} \rho\vec{u}\ dV, \tag{2.44}$$

(see Figure 2.1.3). The principle of conservation of momentum states that the rate of change of the total momentum of the fluid in B_t has to be accounted for by the balance of forces acting on B_t:

$$\frac{d\overrightarrow{\Pi}_B}{dt} = \text{ Balance of Forces on } B_t; \tag{2.45}$$

this is, in fact, Newton's second law of motion.

There are two types of forces acting on the portion of fluid in B_t that contribute to the balance of forces in the right–hand side of the equation in (2.45). There are forces of stress due to the fluid surrounding the region B_t, and there are external, or body forces, such as gravity or electromagnetic forces. We can then rewrite the conservation of momentum equation in (2.45) as

$$\frac{d\overrightarrow{\Pi}_B}{dt} = \overrightarrow{S}_B(t) + \overrightarrow{F}_B(t), \tag{2.46}$$

where $\overrightarrow{S}_B(t)$ denotes the total vector sum of the stress forces acting on B_t, and $\overrightarrow{F}_B(t)$ the total vector sum of body forces acting on B_t.

We assume that

$$\overrightarrow{F}_B(t) = \iiint_{B_t} \vec{f}\ dV, \tag{2.47}$$

where the vector field $\vec{f}(x, y, z, t)$ gives the total forces per unit volume acting on an element of fluid around the point (x, y, z) at time t.

In this section we shall make a special assumption when modeling the stress forces acting on the fluid. We assume that the fluid under consideration is an **ideal fluid**. This means that at any point, (x, y, y), on a surface in the fluid, the stress force per unit area exerted across the surface is given by

$$p(x, y, z, t)\vec{n}$$

where $\vec{n}$ is a unit vector perpendicular to the surface at (x, y, z) and time t, and $p(x, y, z, t)$ is a scalar field called the **pressure**. It then follows that

$$\overrightarrow{S}_B(t) = -\iint_{\partial B_t} p\vec{n}\ dA, \tag{2.48}$$

where $\vec{n}$ is the outward unit normal to ∂B_t.

Substituting the expressions in (2.48) and (2.47) into the conservation of momentum expression in (2.46) yields

$$\frac{d\overrightarrow{\Pi}_B}{dt} = -\iint_{\partial B_t} p\vec{n}\ dA + \iiint_{B_t} \vec{f}\ dV. \tag{2.49}$$

Writing the unit vector $\vec{n}$ in Cartesian coordinates, (n_1, n_2, n_3), we see that the stress forces term in (2.49) has components

$$-\iint_{\partial B_t} pn_1\ dA,\ -\iint_{\partial B_t} pn_2\ dA,\ \text{and}\ -\iint_{\partial B_t} pn_3\ dA.$$

Applying the divergence theorem to each of these components we get

$$-\iint_{\partial B_t} pn_1\ dA = -\iiint_{B_t} \frac{\partial p}{\partial x}\ dV$$

$$-\iint_{\partial B_t} pn_2\ dA = -\iiint_{B_t} \frac{\partial p}{\partial y}\ dV$$

and

$$-\iint_{\partial B_t} pn_3\ dA = -\iiint_{B_t} \frac{\partial p}{\partial z}\ dV.$$

Substituting these expressions into the definition of $\overrightarrow{S}_B(t)$ in (2.48) we obtain

$$\vec{S}_B(t) = -\iiint_{B_t} \nabla p\ dV, \tag{2.50}$$

where

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) \tag{2.51}$$

is the gradient of p. Combining (2.50), (2.48) and (2.46), we can rewrite the conservation of momentum equation in (2.49) as

$$\frac{d\overrightarrow{\Pi}_B}{dt} = -\iiint_{B_t} \nabla p\ dV + \iiint_{B_t} \vec{f}\ dV, \tag{2.52}$$

where ∇p is as given in (2.51).

Next, we see how to compute the left–hand side of the equation in (2.52),

$$\frac{d\overrightarrow{\Pi}_B}{dt} = \frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV, \tag{2.53}$$

according to the definition of momentum in (2.44).

Observe that, since B_t comes about as the result of the action of the flow map φ_t on B (see 2.25), we can rewrite the integral on the right–hand side of (2.53) as

$$\begin{aligned}\iiint_{B_t} \rho\vec{u}\ dV &= \iiint_{B_t} \rho(\varphi_t(x,y,z),t)\vec{u}(\varphi_t(x,y,z),t)\ dV \\ &= \iiint_{B} \rho(x,y,z,t)\vec{u}(x,y,z,t)J(x,y,z,t)\ dxdydz\end{aligned}$$

where $J(x,y,z,t)$ the Jacobian of the map φ_t; that is, $J(x,y,z,t)$ is the determinant of the derivative map of φ_t. We then have that

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B} \frac{\partial}{\partial t}[J\rho\vec{u}]\ dxdydz,$$

or

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B} \left[\frac{\partial J}{\partial t}\rho\vec{u} + \frac{\partial}{\partial t}[\rho\vec{u}]J\right]\ dxdydz. \tag{2.54}$$

Substituting (2.32) into (2.54) yields

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B} \left[(\nabla\cdot\vec{u})\rho\vec{u} + \frac{\partial}{\partial t}[\rho\vec{u}]\right] J\ dxdydz,$$

which can be written as

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B_t} \left[(\nabla\cdot\vec{u})\rho\vec{u} + \frac{\partial}{\partial t}[\rho\vec{u}]\right]\ dV. \tag{2.55}$$

Using the expression for the material derivative of a vector field in (2.39), we can rewrite (2.55) as

$$\frac{d}{dt}\iiint_{B_t} \rho\vec{u}\ dV = \iiint_{B_t} \left[\frac{D}{Dt}(\rho\vec{u}) + (\nabla\cdot\vec{u})\rho\vec{u}\right]\ dV. \tag{2.56}$$

Using the definition of the convective derivative for a vector field in (2.39) we have that

$$\frac{D}{Dt}(\rho\vec{u}) = \rho\frac{D\vec{u}}{Dt} + \frac{D\rho}{Dt}\vec{u}, \tag{2.57}$$

where

$$\begin{aligned}\frac{D\rho}{Dt} &= \frac{\partial\rho}{\partial t} + \vec{u}\cdot\nabla\rho \\ &= \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) - \rho\nabla\cdot\vec{u};\end{aligned}$$

it then follows from the conservation mass equation in (2.17) that

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{u}, \tag{2.58}$$

which is an alternate form of the conservation of mass principle.

Combining (2.57) and (2.58) then yields

$$\frac{D}{Dt}(\rho\vec{u}) = \rho\frac{D\vec{u}}{Dt} - (\nabla\cdot\vec{u})\rho\vec{u}. \tag{2.59}$$

Substituting the expression for $\frac{D}{Dt}(\rho\vec{u})$ in (2.59) into the expression for the rate of change of momentum in (2.56) yields

$$\frac{d}{dt}\iiint_{B_t}\rho\vec{u}\,dV = \iiint_{B_t}\rho\frac{D\vec{u}}{Dt}\,dV. \tag{2.60}$$

Substituting the expression for the rate of change of momentum in (2.60) into the left–hand side of (2.52) yields

$$\iiint_{B_t}\rho\frac{D\vec{u}}{Dt}\,dV = -\iiint_{B_t}\nabla p\,dV + \iiint_{B_t}\vec{f}\,dV,$$

or

$$\iiint_{B_t}\left[\rho\frac{D\vec{u}}{Dt} + \nabla p - \vec{f}\right]dV = 0, \quad \text{for all } t. \tag{2.61}$$

Assuming that the fields ρ, $\vec{u}$ and p are C^1 over R and for all times t, and that the field $\vec{f}$ is continuous over R and for all times t, we see that the integrand in the left–hand side of (2.61) is continuous over R and for all times t. Thus, since (2.61) holds true for all bounded subregions, B_t, of R with smooth boundary, we conclude that

$$\rho\frac{D\vec{u}}{Dt} + \nabla p - \vec{f} = 0, \quad \text{in } R, \text{ for all } t,$$

or

$$\rho\frac{D\vec{u}}{Dt} = -\nabla p + \vec{f}, \quad \text{in } R, \text{ for all } t, \tag{2.62}$$

which is the differential form of the conservation of momentum principle.

Observe that the PDE in (2.62) is a vector differential equation in three dimensions. As such, it is really a system of three first–order PDEs:

$$\left\{\begin{array}{rcl} \rho\dfrac{Du_1}{Dt} & = & -\dfrac{\partial p}{\partial x} + f_1; \\ \rho\dfrac{Du_2}{Dt} & = & -\dfrac{\partial p}{\partial y} + f_2; \\ \rho\dfrac{Du_3}{Dt} & = & -\dfrac{\partial p}{\partial z} + f_3. \end{array}\right. \tag{2.63}$$

The equations in (2.18) and (2.63) constitute a system of four first–order PDEs in the (possibly) unknown scalar fields u_1, u_2, u_3, ρ and p (the body forces field $\vec{f}$ can usually be determined from the outset). Thus, in order to have any hope for solving the system of conservation equations in (2.17) and (2.62), we need to have at least one more relation, or equation, involving the velocity field, $\vec{u}$, the density, ρ, and the pressure, p. Another relation will be provided by the principle of conservation of energy to be discussed in the next section.

The expression in (2.60) holds true for any C^1 vector field $\overrightarrow{G}$ in R,

$$\frac{d}{dt}\iiint_{B_t} \rho\overrightarrow{G}\ dV = \iiint_{B_t} \rho\frac{D\overrightarrow{G}}{Dt}\ dV,$$

or any C^1 scalar field g,

$$\frac{d}{dt}\iiint_{B_t} \rho g\ dV = \iiint_{B_t} \rho\frac{Dg}{Dt}\ dV, \tag{2.64}$$

where $\frac{Dg}{Dt}$ is the material derivative of g. This is known as the **Transport Theorem**. We will have opportunity to apply the transport theorem in (2.64) in the next section.

2.1.3 Conservation of Energy in Incompressible Flow

Consider the volume of the portion of the fluid in B_t at time t (see Figure 2.1.3),

$$v(t) = \iiint_{B_t} dV. \tag{2.65}$$

As the shape of the region B_t changes with the flow, the volume of B_t might change also. We compute the rate at which the volume changes by first rewriting the expression for $v(t)$ in (2.65) as

$$v(t) = \iiint_{B} J(\varphi_t(x,y,z),t)\ dV. \tag{2.66}$$

It follows from (2.66) that

$$\frac{dv}{dt} = \iiint_{B} \frac{\partial}{\partial t}[J(\varphi_t(x,y,z),t)]\ xdydz, \tag{2.67}$$

where

$$\frac{\partial}{\partial t}[J(\varphi_t(x,y,z),t) = (\nabla\cdot\vec{u}(\varphi_t(x,y,z),t))J(\varphi_t(x,y,z),t)\ ,$$

according to (2.32). We therefore obtain from (2.67) that

$$\frac{dv}{dt} = \iiint_{B}(\nabla\cdot\vec{u}(\varphi_t(x,y,z),t))J(\varphi_t(x,y,z),t)\ dxdydz,$$

which we can rewrite as

$$\frac{dv}{dt} = \iiint_{B_t} \nabla\cdot\vec{u}\ dV. \tag{2.68}$$

In an **incompressible flow** the volume of any portion of the fluid does not change with time. We therefore obtain from (2.68) that

$$\iiint_{B_t} \nabla \cdot \vec{u}\; dV = 0, \qquad \text{for all } t. \tag{2.69}$$

Since the expression in (2.69) holds true for any B_t in R, it follows that, for case in which the velocity field, $\vec{u}$, is C^1 in R, the condition for the flow to be incompressible is

$$\nabla \cdot \vec{u} = 0, \qquad \text{in } R \text{ for all } t. \tag{2.70}$$

We show in this section that, in an ideal incompressible fluid, the kinetic energy in the portion of the fluid in B_t is conserved.

The kinetic energy of the portion of the fluid B_t at time t is given by

$$E(t) = \frac{1}{2}\iiint_{B_t} \rho \|\vec{u}\|^2\; dV, \tag{2.71}$$

where $\|u\|^2 = \vec{u}\cdot\vec{u}$ is the square of the Euclidean norm of the velocity field $\vec{u}$.

The rate of change of E in (2.71) is given by the Transport Theorem in (2.64) to be

$$\frac{dE}{dt} = \frac{1}{2}\iiint_{B_t} \rho \frac{D}{Dt}[\|\vec{u}\|^2]\; dV, \tag{2.72}$$

where

$$\frac{D}{Dt}[\|\vec{u}\|^2] = 2\vec{u}\cdot\frac{D\vec{u}}{Dt},$$

so that, in view of (2.72),

$$\frac{dE}{dt} = \iiint_{B_t} \rho\vec{u}\cdot\frac{D\vec{u}}{Dt}\; dV,$$

or

$$\frac{dE}{dt} = \iiint_{B_t} \vec{u}\cdot\left(\rho\frac{D\vec{u}}{Dt}\right)\; dV. \tag{2.73}$$

Substituting the law of conservation of momentum expression for an ideal fluid in (2.62) into the right–hand side of (2.73) then yields

$$\frac{dE}{dt} = \iiint_{B_t} \vec{u}\cdot\left(-\nabla p + \vec{f}\right)\; dV,$$

which can be written as

$$\frac{dE}{dt} = -\iiint_{B_t} \nabla p\cdot\vec{u}\; dV + \iiint_{B_t} \vec{f}\cdot\vec{u}\; dV. \tag{2.74}$$

The right–most integral in (2.74) measures the rate at which body forces do work in the portion of fluid in B_t at time t. In order to understate the other integral in (2.74) we use the assumption that the fluid is incompressible, stated as the PDE in (2.70), to obtain

$$\nabla \cdot (p\vec{u}) = \nabla p \cdot \vec{u} + p\nabla \cdot \vec{u} = \nabla p \cdot \vec{u},$$

so that

$$\iiint_{B_t} \nabla p \cdot \vec{u}\; dV = \iiint_{B_t} \nabla \cdot (p\vec{u})\; dV. \tag{2.75}$$

Applying the divergence theorem to the integral on the right–hand side of (2.75) yields

$$\iiint_{B_t} \nabla p \cdot \vec{u}\; dV = \iint_{\partial B_t} p\vec{u} \cdot \vec{n}\; dA, \tag{2.76}$$

where $\vec{n}$ denotes the outward unit normal vector the boundary of B_t. Substituting the expression in (2.76) into the right–hand side of (2.74) then yields

$$\frac{dE}{dt} = -\iint_{\partial B_t} p\vec{u} \cdot \vec{n}\; dA + \iiint_{B_t} \vec{f} \cdot \vec{u}\; dV. \tag{2.77}$$

Observe that $-\displaystyle\iint_{\partial B_t} p\vec{u} \cdot \vec{n}\; dA$ gives the rate at which the stress forces are doing work on the portion of fluid in B_t. Hence, the equation in (2.77) is a statement of the conservation of kinetic energy.

2.1.4 Euler Equations for Incompressible, Ideal Fluids

Putting together the PDEs in (2.58), (2.62) and (2.70) we obtain the system of PDEs

$$\begin{cases} \dfrac{D\rho}{Dt} = 0; \\[2ex] \rho\dfrac{D\vec{u}}{Dt} = -\nabla p + \vec{f}; \\[2ex] \nabla \cdot \vec{u} = 0, \end{cases} \tag{2.78}$$

stating the principles of conservation of mass, conservation of momentum, and conservation of energy, respectively, for incompressible, ideal fluids. The equations in the system of PDEs in (2.78) are known as the Euler equations for incompressible, ideal fluids. Using the definition of the material derivative, $\dfrac{D}{Dt}$, in (2.38) and (2.39), the Euler equations in (2.78) can also be written as

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \vec{u} \cdot \nabla\rho = 0; \\[2ex] \rho\dfrac{\partial\vec{u}}{\partial t} + \rho(\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \vec{f}; \\[2ex] \nabla \cdot \vec{u} = 0, \end{cases} \tag{2.79}$$

The fields ρ, $\vec{u}$ and p in (2.79) are assumed to be C^1 functions defined in and open region R in $\mathbb{R}^3$, and for $t \geqslant 0$; the field $\vec{f}$ is assumed to be continuous in R and for all $t \geqslant 0$. The field $\vec{f}$ is usually know; but the functions ρ, $\vec{u}$ and p are unknown. We would like to obtain information about these functions for all times, t, and all points in R, given some initial conditions; for example,

$$\begin{cases} \rho(x,y,z,0) = \rho_o(x,y,z), & \text{for } (x,y,z) \in R; \\ \vec{u}(x,y,z,0) = \vec{u}_o(x,y,z), & \text{for } (x,y,z) \in R; \\ p(x,y,z,0) = p_o(x,y,z),, & \text{for } (x,y,z) \in R, \end{cases}$$

where ρ_o, $\vec{u}_o$ and p_o are given functions defined in R. Since, we want the flow to remain within the region R, we also impose the **boundary condition**

$$\vec{u} \cdot \vec{n} = 0, \quad \text{on } \partial R, \text{ for all } t, \tag{2.80}$$

where we are assuming that R has a smooth boundary ∂R. The condition in (2.80) forbids fluid to cross in or out of the boundary.

2.2 Modeling Diffusion

The random migration of small particles (e.g., pollen grains, large molecules, etc.) immersed in a stationary fluid is known as **diffusion**. This process, also known as Brownian motion, is caused by the random bombardment of the particles by the fluid molecules because of thermal excitation. Brownian motion can be modeled probabilistically by looking at motions of large ensemble of particles. This is a microscopic view. In this section we would like to provide a macroscopic model of diffusion based on a conservation principle.

Imagine that a certain number of Brownian particles moves within a region R in $\mathbb{R}^3$ pictured in Figure 2.2.4. Assume that there is a vector field $\overrightarrow{J}$ that

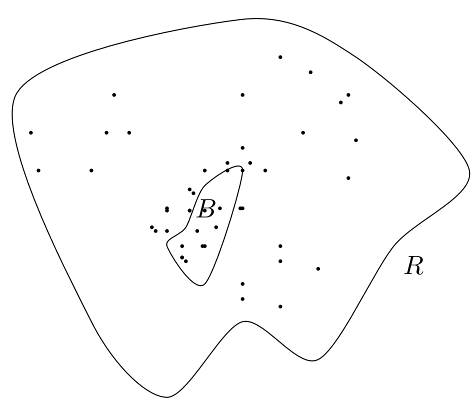

Figure 2.2.4: Brownian Particles in a Region R

gives a measure of the number of particles that cross a unit cross–sectional at point $(x, y, z) \in R$ and time t as follows

$$\overrightarrow{J}(x, y, z, t) \cdot \vec{n} dA$$

gives, approximately, the number of particles that cross a small section of the surface of area dA, per unit time, in a direction perpendicular to the surface at that point. It then follows that the number of particles per unit time crossing the smooth boundary of a region $B \subset R$ into that region (see Figure 2.2.4) is given by

$$-\iint_{\partial B} \overrightarrow{J}(x, y, z, t) \cdot \vec{n}\ dA, \tag{2.81}$$

where the minus sign in (2.81) takes into account that we are taking $\vec{n}$ to be the outward unit normal to ∂B. The expression in (2.81) is called the **flux** of particles across the boundary of B.

Assume that the concentration of particles in the region R at any time t is given by a C^1 scalar field, u, so that number of particles contained in the region B is given at time t is given by

$$N_B(t) = \iiint_B u(x, y, x, t)\ dxdydx, \qquad \text{for all } t. \tag{2.82}$$

Assuming that particles are not being created or destroyed, we get the conservation principle

$$\frac{dN_B}{dt} = -\iint_{\partial B} \overrightarrow{J}(x, y, z, t) \cdot \vec{n}\ dA \tag{2.83}$$

Since we are assuming that u is a C^1 field, we can differentiate under the integral sign in (2.82) to rewrite (2.83) as

$$\iiint_B \frac{\partial u}{\partial t}\ dxdydx = -\iint_{\partial B} \overrightarrow{J}(x, y, z, t) \cdot \vec{n}\ dA \tag{2.84}$$

If we also assume that the vector field $\overrightarrow{J}$ is a C^1 function, we can use the Divergence Theorem to rewrite the right–hand side of (2.84) to obtain

$$\iiint_B \frac{\partial u}{\partial t}\ dV = -\iiint_B \nabla \cdot \overrightarrow{J}\ dV$$

or

$$\iiint_B \left[\frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{J}\right]\ dV = 0. \tag{2.85}$$

Since (2.85) holds true for all bounded subsets, B, of R, and all times t, we obtain the PDE

$$\frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{J} = 0, \quad \text{in } R, \text{ for all } t. \tag{2.86}$$

The PDE in (2.86) has two unknown functions: the concentration u and the flux field $\overrightarrow{J}$. Thus, in order to complete the modeling, we need a constitutive equation relating u and $\overrightarrow{J}$. This is provided by Fick's First Law of Diffusion (see [Ber83, pg. 18]):

$$\overrightarrow{J} = -D\nabla \cdot u, \quad \text{in } R, \text{ for all } t, \tag{2.87}$$

where D is a proportionality constant known as the diffusion constant of the medium in which the particles are, or diffusivity. Observe that D in (2.87) has units of squared length per time. The expression in (2.87) postulates that the flux of Brownian particles is proportional to the negative gradient of the concentration. Thus, the diffusing particles will move from regions of high concentration to regions of low concentration.

Substituting the expression for $\overrightarrow{J}$ in (2.87) into the conservation equation in (2.86) we obtain

$$\frac{\partial u}{\partial t} - D\nabla \cdot (\nabla u) = 0, \quad \text{in } R, \text{ for all } t, \tag{2.88}$$

where we have used the assumption that D is constant.

Assuming that u is also a C^2 function, we can use the definitions of gradient and divergence to compute

$$\nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2.89}$$

The expression on the right–hand side of (2.89) is known as the **Laplacian** of u, and is usually denoted by the symbol Δu, so that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2.90}$$

Another notation for Δu found in various textbooks is $\nabla^2 u$.

In view of (2.89) and (2.90), we see that the PDE in (2.88) can be written as

$$\frac{\partial u}{\partial t} = D\Delta u, \quad \text{in } R, \text{ for all } t, \tag{2.91}$$

which is called the **diffusion equation**. The expression in (2.91) is also known as Fick's second equation (see [Ber83, pg. 20]), or Fick's Second Law of Diffusion.

For the case of in which the diffusing substance is constrained to move in one space direction (say, parallel to the x–axis), the diffusion equation in (2.91) becomes

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \tag{2.92}$$

The equation in (2.92) applies to the situation in which medium containing Brownian particles is in a cylindrical region of constant cross sectional area and axis parallel to the x–axis. In later chapter in these notes, we will show how to solve the PDE in (2.92) over the entire real line subject to an initial condition

$$u(x, 0) = f(x), \qquad \text{for all } x \in \mathbb{R},$$

for some given function $f\colon \mathbb{R} \to \mathbb{R}$, and some integrability conditions on u, $\frac{\partial u}{\partial x}$ and f.

The equation in (2.92) also describes the flow of heat in a cylindrical metal rod of constant cross–sectional area whose cylindrical boundary is insulated so that heat can only flow in or out of the rod through the cross sections a the ends of the rod (see Assignment #4). In this case $u(x,t)$ denotes the temperature in

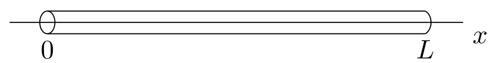

Figure 2.2.5: Heat Conduction in a Cylindrical Rod

the cross–section of the rod located at x and at time t, and the constant D is given by

$$D = \frac{\kappa}{c\rho},$$

where ρ is the density, c is the specific heat, and κ is the heat conductivity of the material the of the rod (see Assignment #4). Thus, (2.92) ia also called the **heat equation**. In this case D is called the **thermal diffusivity**.

In these notes we will see how to solve the heat equation in (2.92) subject the initial and boundary conditions

$$\begin{cases} u(x,0) = f(x), & \text{for } 0 < x < L; \\ u(0,t) = T_o(t), & \text{for } t > 0; \\ u(L,t) = T_L(t), & \text{for } t > 0, \end{cases}$$

where f, T_o and T_L are given functions of single variable. We will also solve the problem with the boundary conditions

$$\begin{cases} \frac{\partial u}{\partial x}(0,t) = 0, & \text{for } t > 0; \\ \frac{\partial u}{\partial x}(L,t) = 0, & \text{for } t > 0. \end{cases}$$

These conditions imply that heat cannot flow through the end cross–sections either; so that the rod is totally insulated.

2.3 Variational Problems

In the previous two sections we have seen how conservation principles give rise to problems involving PDEs. Another important source of PDE problems arises from the application of **variational principles**. A variational principle states that a configuration, or function, describing the state of a system must minimize, or maximize, certain quantity (e.g., energy). In this section we will see two applications of variational principles: the derivations of the minimal surface equation and the vibrating string equation.

2.3.1 Minimal Surfaces

Imagine you take a twisted wire loop, as that pictured in Figure 2.3.6, and dip into a soap solution. When you pull it out of the solution, a soap film spanning the wire loop develops. We are interested in understanding the mathematical properties of the film, which can be modeled by a smooth surface in three

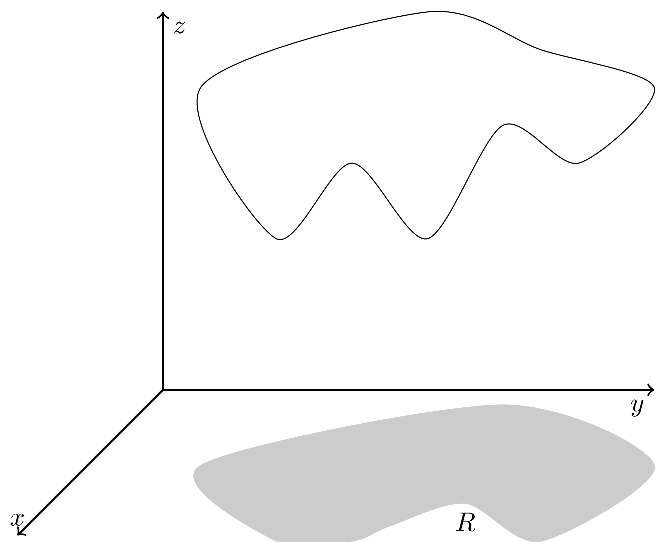

Figure 2.3.6: Wire Loop

dimensional space. Specifically, the shape of the soap film spanning the wire loop, can be modeled by the graph of a smooth function, $u\colon \overline{R} \to \mathbb{R}$, defined on the closure of a bounded region, R, in the xy–plane with smooth boundary ∂R. The physical explanation for the shape of the soap film relies on the variational principle that states that, at equilibrium, the configuration of the film must be such that the energy associated with the surface tension in the film must be the lowest possible. Since the energy associated with surface tension in the film is proportional to the area of the surface, it follows from the least–energy principle that a soap film must minimize the area; in other words, the soap film spanning the wire loop must have the shape of a smooth surface in space containing the wire loop with the property that it has the smallest possible area among all smooth surfaces that span the wire loop. In this section we will develop a mathematical formulation of this variational problem.

The wire loop can be modeled by the curve determined by the set of points:

$$(x, y, g(x, y)), \qquad \text{for } (x, y) \in \partial R,$$

where ∂R is the smooth boundary of a bounded open region R in the xy–plane (see Figure 2.3.6), and g is a given function defined in a neighborhood of ∂R, which is assumed to be continuous. A surface, S, spanning the wire loop can be modeled by the image of a C^1 map

$$\Phi\colon R \to \mathbb{R}^3$$

given by

$$\Phi(x,y) = (x, y, u(x,u)), \qquad \text{for all } x \in \overline{R}, \tag{2.93}$$

where $\overline{R} = R \cup \partial R$ is the closure of R, and

$$u\colon \overline{R} \to \mathbb{R}$$

is a function that is assumed to be C^2 in R and continuous on $\overline{R}$; we write

$$u \in C^2(R) \cap C(\overline{R}).$$

Let $\mathcal{A}_g$ denote the collection of functions $u \in C^2(R) \cap C(\overline{R})$ satisfying

$$u(x,y) = g(x,y), \quad \text{for all } (x,y) \in \partial R;$$

that is,

$$\mathcal{A}_g = \{u \in C^2(R) \cap C(\overline{R}) \mid u = g \text{ on } \partial R\}. \tag{2.94}$$

Next, we see how to compute the area of the surface $S_u = \Phi(R)$, where Φ is the map given in (2.93) for $u \in \mathcal{A}_g$, where $\mathcal{A}_g$ is the class of functions defined in (2.94).

The grid lines $x = c$ and $y = d$, for arbitrary constants c and d, are mapped by the parametrization Φ into curves in the surface S_u given by

$$y \mapsto \Phi(c, y)$$

and

$$x \mapsto \Phi(x, d),$$

respectively. The tangent vectors to these paths are given by

$$\Phi_y = \left(0, 1, \frac{\partial u}{\partial y}\right) \tag{2.95}$$

and

$$\Phi_x = \left(1, 0, \frac{\partial u}{\partial x}\right), \tag{2.96}$$

respectively. The quantity

$$\|\Phi_x \times \Phi_y\| \Delta x \Delta y \tag{2.97}$$

gives an approximation to the area of portion of the surface S_u that results from mapping the rectangle $[x, x + \Delta x] \times [y, y + \Delta y]$ in the region R to the surface S_u by means of the parametrization Φ given in (2.93). Adding up all the contributions in (2.97), while refining the grid, yields the following formula for the area S_u:

$$\text{area}(S_u) = \iint_R \|\Phi_x \times \Phi_y\|\, dxdy. \tag{2.98}$$

Using the definitions of the tangent vectors Φ_x and Φ_y in (2.95) and (2.96), respectively, we obtain that

$$\Phi_x \times \Phi_y = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right),$$

so that

$$\|\Phi_x \times \Phi_y\| = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2},$$

or

$$\|\Phi_x \times \Phi_y\| = \sqrt{1 + |\nabla u|^2},$$

where $|\nabla u|$ denotes the Euclidean norm of ∇u. We can therefore write (2.98) as

$$\text{area}(S_u) = \iint_R \sqrt{1 + |\nabla u|^2}\; dxdy. \tag{2.99}$$

The formula in (2.99) allows us to define a map

$$A\colon \mathcal{A}_g \to \mathbb{R}$$

by

$$A(u) = \iint_R \sqrt{1 + |\nabla u|^2}\; dxdy, \quad \text{for all } u \in \mathcal{A}_g, \tag{2.100}$$

which gives the area of the surface parametrized by the map $\Phi\colon \overline{R} \to \mathbb{R}^3$ given in (2.93) for $u \in \mathcal{A}_g$. We will refer to the map $A\colon \mathcal{A}_g \to \mathbb{R}$ defined in (2.100) as the area **functional**. With the new notation we can restate the variational problem of this section as follows:

Problem 2.3.1 (Variational Problem 1)**.** *Out of all functions in $\mathcal{A}_g$, find one such that*

$$A(u) \leqslant A(v), \quad \textit{for all } v \in \mathcal{A}_g. \tag{2.101}$$

That is, find a function in $\mathcal{A}_g$ that minimizes the area functional in the class $\mathcal{A}_g$.

Problem 2.3.1 is an instance of what has been known as Plateau's problem in the Calculus of Variations. The mathematical question surrounding Pateau's problem was first formulated by Euler and Lagrange around 1760. In the middle of the 19^{th} century, the Belgian physicist Joseph Plateu conducted experiments with soap films that led him to the conjecture that soap films that form around wire loops are of minimal surface area. It was not until 1931 that the American mathematician Jesse Douglas and the Hungarian mathematician Tibor Radó, independently, came up with the first mathematical proofs for the existence of minimal surfaces. In this section we will derive a necessary condition for the existence of a solution to Problem 2.3.1, which is expressed in terms of a PDE that $u \in \mathcal{A}_g$ must satisfy, the minimal surface equation.

Suppose we have found a solution, $u \in \mathcal{A}_g$, to Problem 2.3.1 in $u \in \mathcal{A}_g$. Let $\varphi \colon \overline{R} \to \mathbb{R}$ by a C^∞ function with compact support in R; we write $\varphi \in C_c^\infty(R)$ (see Assignment #5 for a construction of such function). It then follows that

$$u + t\varphi \in \mathcal{A}_g, \quad \text{for all } t \in \mathbb{R}, \tag{2.102}$$

since φ vanishes in a neighborhood of ∂R and therefore $u + t\varphi = g$ on ∂R. It follows from (2.102) and (2.101) that

$$A(u) \leqslant A(u + t\varphi), \quad \text{for all } t \in \mathbb{R}. \tag{2.103}$$

Consequently, the function $f \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = A(u + t\varphi), \quad \text{for all } t \in \mathbb{R}, \tag{2.104}$$

has a minimum at 0, by virtue of (2.104) and (2.104). It follows from this observation that, if f is differentiable at 0, then

$$f'(0) = 0. \tag{2.105}$$

We will see next that, since we are assuming that $u \in C^2(R) \cap C(\overline{R})$ and $\varphi \in C_c^\infty(R)$, f is indeed differentiable. To see why this is the case, use (2.104) and (2.100) to compute

$$f(t) = \iint_R \sqrt{1 + |\nabla(u + t\varphi)|^2}\; dxdy, \quad \text{for all } t \in \mathbb{R}, \tag{2.106}$$

where

$$\nabla(u + t\varphi) = \nabla u + t\nabla\varphi, \quad \text{for all } t \in \mathbb{R},$$

by the linearity of the differential operator ∇. It then follows that

$$\begin{aligned} \nabla(u + t\varphi) &= (\nabla u + t\nabla\varphi)\cdot(\nabla u + t\nabla\varphi) \\ &= \nabla u\cdot\nabla u + t\nabla u\cdot\nabla\varphi + t\nabla\varphi\cdot\nabla u + t^2\nabla\varphi\cdot\nabla\varphi \\ &= |\nabla u|^2 + 2t\nabla u\cdot\nabla\varphi + t^2|\nabla\varphi|^2, \end{aligned}$$

so that, substituting into (2.106),

$$f(t) = \iint_R \sqrt{1 + |\nabla u|^2 + 2t\nabla u\cdot\nabla\varphi + t^2|\nabla\varphi|^2}\; dxdy, \quad \text{for all } t \in \mathbb{R}. \tag{2.107}$$

Since the integrand in (2.107) is C^1, we can differentiate under the integral sign to get

$$f'(t) = \iint_R \frac{\nabla u\cdot\nabla\varphi + t|\nabla\varphi|^2}{\sqrt{1 + |\nabla u|^2 + 2t\nabla u\cdot\nabla\varphi + t^2|\nabla\varphi|^2}}\; dxdy, \tag{2.108}$$

for all $t \in \mathbb{R}$. Thus, f is differentiable and, substituting 0 for t in (2.108),

$$f'(0) = \iint_R \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}}\, dxdy. \tag{2.109}$$

Hence, if u is a minimizer of the area functional in $\mathcal{A}_g$, it follows from (2.104) and (2.109) that

$$\iint_R \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}}\, dxdy = 0, \quad \text{for all } \varphi \in C_c^\infty(R). \tag{2.110}$$

The statement in (2.110) provides a necessary condition for the existence of a minimizer of the area functional in $\mathcal{A}_g$. We will next see how (2.110) gives rise to a PDE that $u \in C^2(R) \cap C(\overline{R})$ must satisfy in order for it to be minimizer of the area functional in $\mathcal{A}_g$.

First, we "integrate by parts" (see Assignment #6) in (2.110) to get

$$-\iint_R \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \varphi\, dxdy + \int_{\partial R} \varphi \frac{\nabla u \cdot \vec{n}}{\sqrt{1 + |\nabla u|^2}}\, ds = 0, \tag{2.111}$$

for all $\varphi \in C_c^\infty(R)$, where the second integral in (2.111) is a path integral around the boundary of R. Since $\varphi \in C_c^\infty(R)$ vanishes in a neighborhood of the boundary of R, it follows from (2.111) that

$$\iint_R \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \varphi\, dxdy = 0, \quad \text{for all } \varphi \in C_c^\infty(R). \tag{2.112}$$

By virtue of the assumption that u is a C^2 functions, it follows that the divergence term of the integrand (2.112) is continuous on R, it follows from the statement in (2.112) that

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0, \quad \text{in } R. \tag{2.113}$$

(see Assignment #6).

The equation in (2.113) is a second order nonlinear PDE known as the **minimal surface equation**. It provides a necessary condition for a function $u \in C^2(R) \cap C(\overline{R})$ to be a minimizer of the area functional in $\mathcal{A}_g$. Since, we are also assuming that $u \in \mathcal{A}_g$, we get that must solve the boundary value problem (BVP):

$$\begin{cases} \nabla \cdot \left(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 & \text{in } R; \\ \\ u = g & \text{on } \partial R. \end{cases} \tag{2.114}$$

The BVP in (2.114) is called the **Dirichlet problem** for the minimal surface equation.

The PDE in (2.113) can also be written as

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \quad \text{in } R, \tag{2.115}$$

where the subscripted symbols read as follows:

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y},$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2},$$

and

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = u_{yx}. \tag{2.116}$$

The fact that the "mixed" second partial derivatives in (2.116) are equal follows from the assumption that u is a C^2 function.

When we study the classification of PDEs we will see that the equation in (2.115) is a nonlinear, second order, elliptic PDE.

2.3.2 The Linearized Minimal Surface Equation

For the case in which the wire loop in the previous section is very close to a horizontal plane (see Figure 2.3.7), it is reasonable to assume that, if $u \in \mathcal{A}_g$,

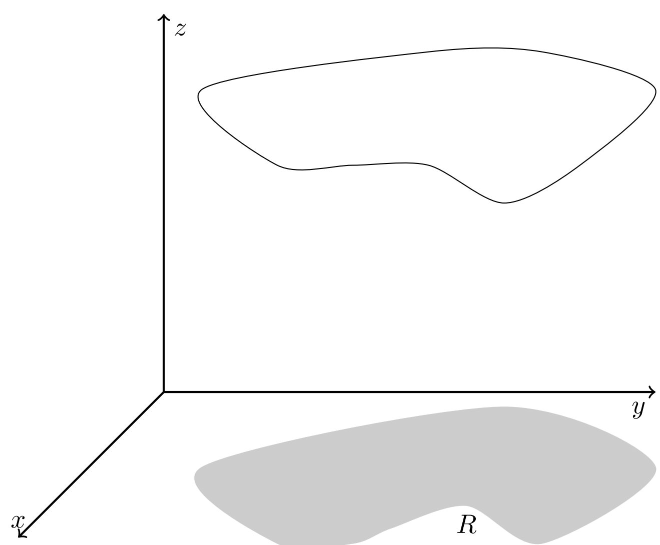

Figure 2.3.7: Almost Planar Wire Loop

$|\nabla u|$ is very small throughout R. We can therefore use the linear approximation

$$\sqrt{1+t} \approx 1 + \frac{1}{2}t, \quad \text{for small } |t|, \tag{2.117}$$

to approximate the area function in (2.100) by

$$A(u) \approx \iint_R \left[1 + \frac{1}{2}|\nabla u|^2\right] \, dxdy, \qquad \text{for all } u \in \mathcal{A}_g,$$

so that

$$A(u) \approx \text{area}(R) + \frac{1}{2}\iint_R |\nabla u|^2 \, dxdy, \qquad \text{for all } u \in \mathcal{A}_g. \tag{2.118}$$

The integral on the right–hand side of the expression in (2.118) is known as the **Dirichlet Integral**. We will use it in these notes to define the Dirichlet functional, $\mathcal{D} \colon \mathcal{A}_g \to \mathbb{R}$,

$$\mathcal{D}(u) = \frac{1}{2}\iint_R |\nabla u|^2 \, dxdy, \qquad \text{for all } u \in \mathcal{A}_g. \tag{2.119}$$

Thus, in view of (2.118) and (2.119),

$$A(u) \approx \text{area}(R) + \mathcal{D}(u), \qquad \text{for all } u \in \mathcal{A}_g. \tag{2.120}$$

Thus, according to (2.120), for wire loops close to a horizontal plane, minimal surfaces spanning the wire loop can be approximated by solutions to the following variational problem,

Problem 2.3.2 (Variational Problem 2)**.** *Out of all functions in $\mathcal{A}_g$, find one such that*

$$\mathcal{D}(u) \leqslant \mathcal{D}(v), \qquad \textit{for all } v \in \mathcal{A}_g. \tag{2.121}$$

It can be shown that a necessary condition for $u \in \mathcal{A}_g$ to be a solution to the Variational Problem 2.3.2 is that u solves the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } R; \\ u = g & \text{on } \partial R, \end{cases} \tag{2.122}$$

where

$$\Delta u = u_{xx} + u_{yy},$$

the two–dimensional Laplacian. The BVP in (2.122) is called the Dirichlet Problem for Laplace's equation.

2.3.3 Vibrating String

Consider a string of length L (imagine a guitar string or a violin string) whose ends are located at $x = 0$ and $x = L$ along the x–axis (see Figure 2.3.8). We assume that the string is made of some material of density $\rho(x)$ (in units of mass per length). Assume that the string is fixed at the end–points and is tightly stretched so that there is a constant tension, τ, acting tangentially along the string at all times. We would like to model what happens to the string

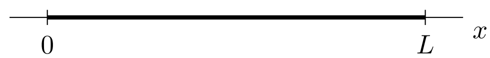

Figure 2.3.8: String of Length L at Equilibrium

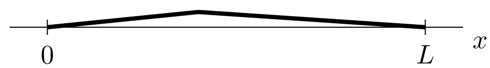

Figure 2.3.9: Plucked String of Length L

after it is plucked to a configuration like that pictured in Figure 2.3.9 and then released. We assume that the shape of the plucked string is described by a continuous function, f, of x, for $x \in [0, L]$. At any time $t \geqslant 0$, the shape of the string is described by a function, u, of x and t; so that $u(x, t)$ gives the vertical displacement of a point in the string located at x when the string is in the equilibrium position pictured in Figure 2.3.8, and at time $t \geqslant 0$. We then have that

$$u(x, 0) = f(x), \quad \text{ for all } x \in [0, L]. \tag{2.123}$$

In addition to the initial condition in (2.123), we will also prescribe the initial speed of the string,

$$\frac{\partial u}{\partial t}(x, 0) = g(x), \quad \text{ for all } x \in [0, L], \tag{2.124}$$

where g is a continuous function of x; for instance, if the plucked string is released from rest, then $g(x) = 0$ for all $x \in [0, L]$. We also have the boundary conditions,

$$u(0, t) = u(L, t) = 0, \text{ for all } t, \tag{2.125}$$

which model the assumption that the ends of the string do not move.

The question we would like to answer is: Given the initial conditions in (2.123) and (2.124), and the boundary conditions in (2.125), can we determine the shape of the string, $u(x, t)$, for all $x \in [0, L]$ and all times $t > 0$. We will answer this questions in a subsequent chapter in these notes. In this section, though, we will derive a necessary condition in the form of a PDE that u must satisfy in order for it to describe the motion of the vibrating string.

In order to find the PDE governing the motion of the string, we will formulate the problem as a variational problem. We will use Hamilton's Principle in Mechanics, or the **Principle of Least Action**. This principle states that the the path that configurations of a mechanical system take from time $t = 0$ to $t = T$ is such that a quantity called the **action** is minimized (or optimized) along the path. The action is defined by

$$A = \int_0^T [K(t) - V(t)]\, dt, \tag{2.126}$$

where $K(t)$ denotes the kinetic energy of the system at time t, and $V(t)$ its potential energy at time t. For the case of a string whose motion is described by small vertical displacements $u(x,t)$, for all $x \in [0,L]$ and all times t, the kinetic energy is given by

$$K(t) = \frac{1}{2}\int_0^L \rho(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2\,dx. \tag{2.127}$$

To see how (2.127) comes about, note that the kinetic energy of a particle of mass m is

$$K = \frac{1}{2}mv^2,$$

where v is the speed of the particle. Thus, for a small element of the string whose projection on the x–axis is the interval $[x, x+\Delta x]$, so that its approximate length is Δx, the kinetic energy is, approximately,

$$\Delta K \approx \frac{1}{2}\rho(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2. \tag{2.128}$$

Thus, adding up the kinetic energies in (2.128) over all elements of the string, and letting $\Delta x \to 0$, yields the expression in (2.127), which we rewrite as

$$K(t) = \frac{1}{2}\int_0^L \rho u_t^2\,dx, \quad \text{for all } t, \tag{2.129}$$

where u_t denotes the partial derivative of u with respect to t.

In order compute the potential energy of the string, we compute the work done by the tension, τ, along the string in stretching the string from its equilibrium length of L, to the length at time t given by

$$\int_0^L \sqrt{1+u_x^2}\,dx; \tag{2.130}$$

so that

$$V(t) = \tau\left[\int_0^L \sqrt{1+u_x^2}\,dx - L\right], \quad \text{for all } t. \tag{2.131}$$

Since we are considering small vertical displacements of the string, we can linearize the expression in in (2.130) by means of the linear approximation in (2.117) to get

$$\int_0^L \sqrt{1+u_x^2}\,dx \approx \int_0^L [1+\frac{1}{2}u_x^2]\,dx = L + \frac{1}{2}\int_0^L u_x^2\,dx,$$

so that, substituting into (2.131),

$$V(t) \approx \frac{1}{2}\int_0^L \tau u_x^2\,dx, \quad \text{for all } t. \tag{2.132}$$

Thus, in view of (2.126), we consider the problem of optimizing the quantity

$$A(u) = \int_0^T \int_0^L \left[\frac{1}{2}\rho u_t^2 - \frac{1}{2}\tau u_x^2\right] \, dxdt, \tag{2.133}$$

where we have substitute the expressions for $K(t)$ and $V(t)$ in (2.129) and (2.132), respectively, into the expression for the action in (2.126).

We will use the expression for the action in (2.133) define a functional in the class of functions $\mathcal{A}$ defined as follows: Let $R = (0, L) \times (0, T)$, the cartesian product of the open intervals $(0, L)$ and $(0, T)$. Then, R is an open rectangle in the xt–plane. We say that $u \in \mathcal{A}$ if $u \in C^2(R) \cap C(\overline{R})$, and u satisfies the initial conditions in (2.123) and (2.124), and the boundary conditions in (2.125). Then, the action functional,

$$A \colon \mathcal{A} \to \mathbb{R},$$

is defined by the expression in (2.133), so that

$$A(u) = \frac{1}{2} \iint_R \left[\rho u_t^2 - \tau u_x^2\right] \, dxdt, \qquad \text{for } u \in \mathcal{A}. \tag{2.134}$$

Next, for $\varphi \in C_c^\infty(R)$, note that $u + s\varphi \in \mathcal{A}$, since φ has compact support in R, and therefore φ and all its derivatives are 0 on ∂R. We can then define a real valued function $h \colon \mathbb{R} \to \mathbb{R}$ by

$$h(s) = A(u + s\varphi), \qquad \text{for } s \in \mathbb{R}, \tag{2.135}$$

Using the definition of the functional A in (2.134), we can rewrite $h(s)$ in (2.135) as

$$\begin{aligned} h(s) &= \frac{1}{2} \iint_R \left[\rho[(u + s\varphi)_t]^2 - \tau[(u + s\varphi)_x]^2\right] \, dxdt \\ &= \frac{1}{2} \iint_R \left[\rho[u_t + s\varphi_t]^2 - \tau[u_x + s\varphi_x]^2\right] \, dxdt, \end{aligned}$$

so that

$$h(s) = A(u) + s \iint_R [\rho u_t \varphi_t - \tau u_x \varphi_x] \, dxdt + s^2 A(\varphi), \tag{2.136}$$

for $s \in \mathbb{R}$, where we have used the definition of the action functional in (2.134).

It follows from (2.136) that h is differentiable and

$$h'(s) = \iint_R [\rho u_t \varphi_t - \tau u_x \varphi_x] \, dxdt + 2sA(\varphi), \qquad \text{for } s \in \mathbb{R}. \tag{2.137}$$

The principle of least action implies that, if u describes the shape of the string, then $s = 0$ must be ac critical point of h. Hence, $h'(0) = 0$ and (2.137) implies that

$$\iint_R [\rho u_t \varphi_t - \tau u_x \varphi_x] \, dxdt = 0, \qquad \text{for } \varphi \in C_c^\infty(R), \tag{2.138}$$

is a necessary condition for $u(x,t)$ to describe the shape of a vibrating string for all times t.

Next, we use the integration by parts formulas

$$\iint_R \psi \frac{\partial \varphi}{\partial x}\ dxdt = \int_{\partial R} \psi\varphi n_1\ ds - \iint_R \frac{\partial \psi}{\partial x}\varphi\ dxdt,$$

for C^1 functions ψ and φ, where n_1 is the first component of the outward unit normal, $\vec{n}$, on ∂R (wherever this vector is defined), and

$$\iint_R \psi \frac{\partial \varphi}{\partial t}\ dxdt = \int_{\partial R} \psi\varphi n_2\ ds - \iint_R \frac{\partial \psi}{\partial t}\varphi\ dxdt,$$

where n_2 is the second component of the outward unit normal, $\vec{n}$, to obtain

$$\iint_R \rho u_t \varphi_t\ dxdt = \int_{\partial R} \rho u_t \varphi n_2\ ds - \iint_R \frac{\partial}{\partial t}[\rho u_t]\varphi\ dxdt,$$

so that

$$\iint_R \rho u_t \varphi_t\ dxdt = -\iint_R \frac{\partial}{\partial t}[\rho u_t]\varphi\ dxdt, \tag{2.139}$$

since φ has compact support in R.

Similarly,

$$\iint_R \tau u_x \varphi_x\ dxdt = -\iint_R \frac{\partial}{\partial x}[\tau u_x]\varphi\ dxdt. \tag{2.140}$$

Next, substitute the results in (2.139) and (2.140) into (2.138) to get

$$\iint_R \left[\frac{\partial}{\partial t}[\rho u_t] - \frac{\partial}{\partial x}[\tau u_x]\right]\varphi\ dxdt = 0, \quad \text{for } \varphi \in C_c^\infty(R). \tag{2.141}$$

Thus, we obtain from (2.141) that

$$\rho\frac{\partial^2 u}{\partial t^2} - \tau\frac{\partial^2 u}{\partial x^2} = 0, \quad \text{in } R, \tag{2.142}$$

since we area assuming that u is C^2, ρ is a continuous function of x, and τ is constant.

The PDE in (2.142) is called the one–dimensional **wave equation**. It is sometimes written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho}\frac{\partial^2 u}{\partial x^2},$$

or

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \tag{2.143}$$

where

$$c^2 = \frac{\tau}{\rho},$$

for the case in which ρ and τ are assumed to be constant.

The wave equation in (2.142) or (2.143) is a second order, linear, hyperbolic PDE.

2.4 Modeling Small Amplitude Vibrations

In this section we present another derivation of the wave equation in (2.143) based on an application of Newton's laws of motion.

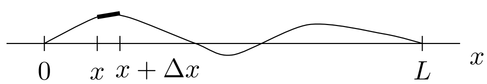

Figure 2.4.10: Vibrating String in Motion

Figure 2.4.10 shows a snapshot of the vibrating string at some time t. The figure also highlights a small section of the string that is above an interval $[x, x + \Delta x]$, for some $x \in (0, L)$. As in the derivation in Section 2.3.3, $u(x, t)$ is the vertical displacement of a point on the string located at x when the string is at equilibrium, and time t (we are assuming here that the vibrations have small amplitude; so that, we may assume that each particle along the string moves along a vertical direction); ρ is the density of the material making up the string (in units of mass per unit of length); and τ is the tension along the string in units of force. We assume that u has continuous partial derivatives,

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial x \partial t}, \quad \frac{\partial^2 u}{\partial t^2},$$

for all $x \in (0, L)$ and all $t > 0$. We introduce the angle $\theta(x, t)$ that the tangent line to the curve in Figure 2.4.10 at the point $(x, u(x))$ makes with the positive x–axis and note that

$$\tan(\theta(x, t)) = \frac{\partial u}{\partial x}(x, t), \qquad \text{for } x \in (0, L), \quad \text{and } t > 0,$$

which we can also write more succinctly as

$$\tan(\theta) = u_x. \tag{2.144}$$

It follows from (2.144) that θ has continuous partial derivatives as long as $\sec^2 \theta \neq 0$, and

$$\frac{\partial \theta}{\partial x} = \frac{u_{xx}}{\sec^2 \theta},$$

which we can rewrite as

$$\frac{\partial \theta}{\partial x} = \frac{u_{xx}}{1 + u_x^2}, \tag{2.145}$$

where we have used (2.144) again. We also obtain from (2.144) that

$$\cos(\theta) = \frac{1}{\sqrt{1 + u_x^2}}. \tag{2.146}$$

If we assume that Δx is very small, we may treat the section of the string above the interval $[x, x+\Delta x]$ as a particle of mass given approximately by

$$\rho(x)\Delta s,$$

where Δs is the arc–length of the section of the string, which is given, approximately, by

$$\Delta s \approx \sqrt{1+u_x^2}\ \Delta x.$$

Thus, the mass of the section of the string above the interval $[x, x+\Delta x]$ is, approximately,

$$\Delta m \approx \rho(x)\sqrt{1+u_x^2}\ \Delta x. \tag{2.147}$$

Newton's Second Law of Motion applied to a particle of mass Δm given in (2.147) reads

$$(\Delta m)\frac{\partial^2 u}{\partial t^2} = \text{ forces acting on the particle in vertical dirction.} \tag{2.148}$$

The forces acting on the particle in the vertical direction are the force of gravity and the vertical components of the tension, which acts tangentially to the string and away from the section above the interval $[x, x+\Delta x]$. There might be other forces acting on the string as well; for instance, a violin string might be acted on by the bow when playing. We will put together the forces other than tension into a function, $F(x,t)$, in units of force per length. Thus, the forces, other than tension, acting on the particle are

$$F(x,t)\Delta x. \tag{2.149}$$

The vertical component of the tension acting on the section of the string above the interval $[x, x+\Delta x]$ is

$$\tau\sin(\theta(x+\Delta x,t)) - \tau\sin(\theta(x,t)), \tag{2.150}$$

where we are assuming that τ is constant throughout the string.

Putting together (2.147), (2.148), (2.149) and (2.150),

$$\rho(x)\sqrt{1+u_x^2}\ \Delta x\frac{\partial^2 u}{\partial t^2} = \tau[\sin(\theta(x+\Delta x,t)) - \sin(\theta(x,t))] + F(x,t)\Delta x. \tag{2.151}$$

Next, divide the equation in (2.151) on both sides by $\Delta x \neq 0$, and letting $\Delta x \to 0$, we obtain that

$$\rho(x)\sqrt{1+u_x^2}\ \frac{\partial^2 u}{\partial t^2} = \tau\frac{\partial}{\partial x}[\sin(\theta(x,t))] + F(x,t),$$

which we can rewrite as

$$\rho(x)\sqrt{1+u_x^2}\ \frac{\partial^2 u}{\partial t^2} = \tau\cos(\theta)\frac{\partial \theta}{\partial x} + F(x,t);$$

so that, using (2.145),

$$\rho\sqrt{1+u_x^2}\ u_{tt} = \tau\cos(\theta)\frac{u_{xx}}{1+u_x^2} + F(x,t),$$

and, using (2.146),

$$\rho\sqrt{1+u_x^2}\ u_{tt} = \frac{\tau u_{xx}}{(1+u_x^2)^{3/2}} + F(x,t). \tag{2.152}$$

The PDE in (2.152), as it stands, is a nonlinear equation (we will see why this is the case in the next section on classification of PDEs). We can approximate the equation by a linear PDE by using the assumption that the vibrations of the string are on very small amplitude. In the case of very small amplitude vibrations, u_x, which is related to the tangent of the angle the string makes with a horizontal direction, is very small in magnitude. Thus, we can make the approximation

$$u_x \approx 0$$

in (2.152) to obtain

$$\rho u_{tt} = \tau u_{xx} + F(x,t). \tag{2.153}$$

Observe that the equation in (2.153) leads to the wave equation in (2.142) for the case $F(x,t) = 0$ for all $x \in [0,L]$ and $t > 0$. The equation in (2.142) will be called in this notes the one–dimensional, linear, homogeneous wave equation. We will refer to the equation in (2.153) with $F \not\equiv 0$ as the one–dimensional, linear, non–homogeneous wave equation. In these notes we will show how to construct solutions of those equations.

Chapter 3

How are PDEs Classified?

In the previous chapter we saw how various types of PDEs.

PDEs are classified according to order (the highest order of the derivative of the unknown functions involved in the equation). The Euler equations for an ideal, incompressible fluid,

$$\left\{\begin{array}{rcl} \dfrac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho & = & 0; \\ \rho \dfrac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \nabla)\vec{u} & = & -\nabla p + \vec{f}; \\ \nabla \cdot \vec{u} & = & 0, \end{array}\right. \tag{3.1}$$

are a system of first–order PDEs.

The 3–dimensional diffusion equation, or heat conduction equation,

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{3.2}$$

the two–dimensional Laplace's equation,

$$u_{xx} + u_{yy} = 0, \tag{3.3}$$

the one dimensional wave equations,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{3.4}$$

and the minimal surface equation,

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \tag{3.5}$$

are second order PDEs.

PDEs can further be classified as linear or nonlinear. For instance, the third equation in the system in (3.1), and the PDEs in (3.2), (3.3) and (3.4) are linear equations, while the first two equations in the system in (3.1) and the PDE in (3.5) are not linear. In the next section we will discuss properties of linear equations, and how those properties can be very helpful in the construction of solutions, and proofs of uniqueness for some initial/boundary value problems.

The PDE in (3.5) is in a general class of equations of the form

$$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d, \qquad (3.6)$$

for some continuous function a, b, c and d of the five variables x, y, u, u_x and u_y, generally. If the coefficient functions in (3.6) depend only on x and y, we get the general linear second order equation in two variables,

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = d(x, y). \qquad (3.7)$$

If the coefficient functions in (3.6) do not depend on the derivatives of the unknown function u, we obtain the **quasi–linear**, second order PDE

$$a(x, y, u)u_{xx} + b(x, y, u)u_{xy} + c(x, y, u)u_{yy} = d(x, y, u). \qquad (3.8)$$

In Section 3.2 we will discuss a further classification of the general second order PDE in (3.6) based on properties of certain curves associated with the equation known as **characteristic curves**. This will lead to the definition of **elliptic**, **hyperbolic** and **parabolic** PDEs. Laplace's equation,

$$u_{xx} + u_{yy} = 0, \qquad (3.9)$$

the one–dimensional wave equation,

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0, \qquad (3.10)$$

and the one–dimensional diffusion equations,

$$Du_{xx} - u_t = 0, \qquad (3.11)$$

are archetypes of these classes of equations, respectively.

3.1 Linearity

Laplace's equation (3.9), the one–dimensional wave equation (3.10), and the one–dimensional diffusion equations (3.11) are linear equations. To understand the use of this terminology in the context of PDEs, note that Laplace's equation (3.9) can also be written as

$$\Delta u = 0,$$

where $\Delta\colon C^2(R) \to C(R)$ defines a linear operator between the spaces of functions $C^2(R)$ and $C(R)$ given by

$$\Delta u = u_{xx} + u_{yy}, \quad \text{for all } u \in C^(R), \qquad (3.12)$$

for some open subset R of $\mathbb{R}^2$. The differential Δ defined in (3.12) is linear because of the linearity property of differentiation that we learned in Calculus; indeed, given functions $u, v \in C^2(R)$ and real constants c_1 and c_2, it follows from the linearity of differentiation that

$$\Delta(c_1 u + c_2 v) = c_1 \Delta u + c_2 \Delta v.$$

Similarly, the one–dimensional wave equation in (3.10) can be written as

$$-\Box u = 0,$$

where the linear operator $\Box\colon C^2(R) \to C(R)$, also known as the **d'Alembert operator**, is defined by

$$\Box u = \frac{1}{c^2} u_{tt} - u_{xx}, \quad \text{for all } u \in C^2(R),$$

where R is an open region in the xt–plane; and the one–dimensional diffusion equation in (3.11) can be written as

$$-Lu = 0,$$

where $L\colon C^2(R) \to C(R)$ defined by

$$Lu = u_t - Du_{xx}, \quad \text{for all } u \in C^2(R),$$

where R is an open region in the xt–plane, is also a linear operator.

By contrast, the map $N\colon C^1(R) \times C^1(R) \times C^1(R) \to C(R) \times C(R) \times C(R)$, given by

$$N(\vec{u}) = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}, \quad \text{for all } \vec{u} \in C^1(R) \times C^1(R) \times C^1(R),$$

where R is an open set in $\mathbb{R}^3$, is not linear (see Problem 1 in Assignment #9). Hence, the second PDE in the system (3.1) is not a linear equation.

In general, a linear PDE is an expression of the form

$$Lu = f, \tag{3.13}$$

where $L\colon \mathcal{U} \to \mathcal{F}$ is a linear differential operator from a linear space, $\mathcal{U}$, of differentiable functions to a linear space, $\mathcal{F}$, of continuous functions. An example of the equation in (3.13) is provided by Poisson's equation in Potential Theory,

$$-\Delta u = f, \quad \text{in } R, \tag{3.14}$$

where R is an open region in $\mathbb{R}^n$. In this case, the linear operator $L = -\Delta$ maps $C^2(R)$ to $C(R)$.

If $f = 0$ in the right–hand side of (3.13) we get the **homogeneous** PDE

$$Lu = 0. \tag{3.15}$$

The equation in (3.15) has the following very useful property known as the **principle of superposition**.

Proposition 3.1.1 (Principle of Superposition)**.** Let u and v denote two solutions of the homogeneous PDE in (3.15). Then, for any constants c_1 and c_2, the functions $c_1u + c_2v$ is also a solution of (3.15).

Proof: Since L is a linear differential operator, it follows that

$$L(c_1u + c_2v) = c_1Lu + c_2Lv.$$

Thus, if u and v solve (3.15), it follows that

$$L(c_1u + c_2v) = c_1 0 + c_2 0 = 0,$$

which shows that $c_1u + c_2v$ solves (3.15). ■

3.2 Classification of Second Order PDEs

Laplace's equation (3.9), the one–dimensional wave equation (3.10), and the one–dimensional diffusion equations (3.11) are second order PDEs. They are classified as **elliptic**, **hyperbolic** and **parabolic** PDEs, respectively. In this sections we study the rationale of that classification as it applies to the general second order PDE in two variables:

$$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = d. \quad (3.16)$$

We begin with the special case of the linear equation

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = d(x, y), \quad (3.17)$$

where a, b, c and d are continuous functions defined in some opens subset, R, of $\mathbb{R}^2$. The classification of the equations of the type in (3.16) or (3.17) is based on properties of curves in R associated with the equations; these curves are called **characteristic curves**. We begin with a curve, Γ, in R parametrized by a map $\gamma\colon I \to \mathbb{R}^2$,

$$t \mapsto \gamma(t) = (x(t), y(t)), \qquad \text{for } t \in I,$$

where I is some interval of real numbers; see Figure 3.2.1. Suppose we are trying to solve the linear PDE in (3.17) subject to information about u given on the curve Γ. Specifically, suppose we are given the values of u and its first derivatives on Γ; we can specify this conditions these condition on u by the equations

$$u(x(t), y(t)) = u_o(t), \quad \text{for } t \in I, \quad (3.18)$$

$$u_x(x(t), y(t)) = f(t), \quad \text{for } t \in I, \quad (3.19)$$

and

$$u_y(x(t), y(t)) = g(t), \quad \text{for } t \in I, \quad (3.20)$$

where u_o, f and g are given continuous functions on I. If we assume, in addition, that f and g are C^∞ functions, we can in theory obtain information about the

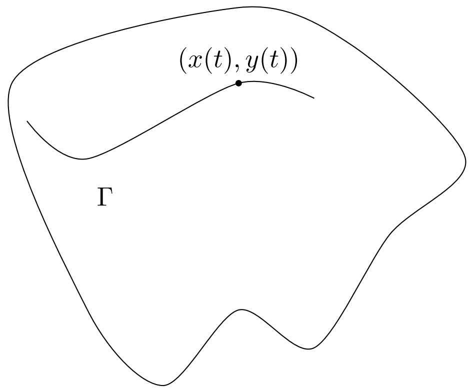

Figure 3.2.1: Characteristic Curves

second order derivatives, u_{xx}, u_{xy} and u_{yy}, of u (and higher order derivatives) on Γ. Is this can be done, we can attempt to construct a solution of the PDE in (3.17) by building Taylor series expansions around every point on Γ using the values of u and its derivatives based on the conditions in (3.18), (3.19) and (3.20) and derivatives of the expressions in (3.19) and (3.20). The first step in this construction consists of taking the derivatives of derivatives of the expressions in (3.19) and (3.20) and combining these with the information provided by the PDE (3.17) to obtain the linear system

$$\begin{cases} \dot{x}\, u_{xx} + \dot{y}\, u_{xy} & = \dot{f} \\ \qquad \dot{x}\, u_{xy} + \dot{y}\, u_{yy} & = \dot{g} \\ a\, u_{xx} + b\, u_{xy} + c\, u_{yy} & = d, \end{cases} \tag{3.21}$$

for the unknowns u_{xx}, u_{xy} and u_{yy} on Γ, where a dot on top of a variable denotes derivative with respect to t:

$$\dot{x} = \frac{dx}{dt}, \; \dot{y} = \frac{dy}{dt}, \; \dot{f} = \frac{df}{dt} \; \text{ and } \; \dot{g} = \frac{dg}{dt}.$$

Note that the system in (3.21) can be written in matrix form as

$$\begin{pmatrix} \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \\ a & b & c \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} \dot{f} \\ \dot{g} \\ d \end{pmatrix}. \tag{3.22}$$

The matrix equation in (3.22) can be solved for the second derivatives of u, in terms of the data $(\dot{f}, \dot{g}, d)$ on Γ, provided that the determinant of the matrix

$$\begin{pmatrix} \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \\ a & b & c \end{pmatrix} \tag{3.23}$$

is not zero. The case in which the determinant of the matrix in (3.23) yields the equation

$$a(\dot{y})^2 - b\dot{x}\dot{y} + c(\dot{x})^2 = 0. \tag{3.24}$$

Dividing the equation in (3.24) by $\dot{x}^2$and using the fact that

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx},$$

by the Chain Rule, we obtain the ordinary differential equation (ODE)

$$a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c = 0. \tag{3.25}$$

We shall refer to the ODE in (3.25) as the characteristic equation to the PDE in (3.17). Solutions to the ODE in (3.25) are called **characteristic curves** of the PDE in (3.17). Assuming that $a \neq 0$ in R, we can solve for $\dfrac{dy}{dx}$ in (3.25) to get

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.26}$$

We have three possibilities, depending on whether $b^2 - 4ac$ is positive, zero, or negative.

If $b^2 - 4ac > 0$, then the PDE in (3.17) has two families of characteristic curves given by the solutions to the two ODEs in (3.26). In this case we say that the PDE in (3.17) is **hyperbolic**.

If $b^2 - 4ac = 0$, then the PDE in (3.17) has one family of characteristic curves given by the solution to the ODE

$$\frac{dy}{dx} = \frac{b}{2a}.$$

In this case we say that the PDE in (3.17) is **parabolic**.

If $b^2 - 4ac < 0$, then the ODE in (3.25) has no real solutions. Thus, the PDE in (3.17) has no (real) characteristic curves. In this case we say that the PDE in (3.17) is **elliptic**.

Example 3.2.1 (The One–dimensional Wave Equation)**.** In the case of the linear second order equation

$$c^2 u_{xx} - u_{tt} = 0, \tag{3.27}$$

describing small amplitude vibrations of a string that we derived in Section 2.3.3 (see the PDE in (2.143), $a = c^2$, $b = 0$ and c in (3.25) is -1 (in this case t is playing the role of y). We therefore get that $b^2 - 4ac = -4(c^2)(-1) = 4c^2 > 0$; so that the equation in (3.27) is hyperbolic. For this PDE the equations for the characteristic curves in (3.26) yields

$$\frac{dt}{dx} = \pm\frac{2c}{2c^2},$$

or

$$\frac{dt}{dx} = \pm\frac{1}{c},$$

which we can rewrite as

$$\frac{dx}{dt} = \pm c. \tag{3.28}$$

The equations in (3.28) is a pair of ODEs that can be solved to yield the two families of curves

$$x = ct + \xi, \tag{3.29}$$

and

$$x = -ct + \eta, \tag{3.30}$$

where ξ and η are the parameters for each of the families of characteristic curves in (3.29) and (3.30). The family of characteristic curves described by the equations in (3.29) consists of parallel lines of (positive) slope $1/c$ in the xt–plane with x–intercept $\xi \in \mathbb{R}$. Some of those characteristic curves are shown in Figure 3.2.2.

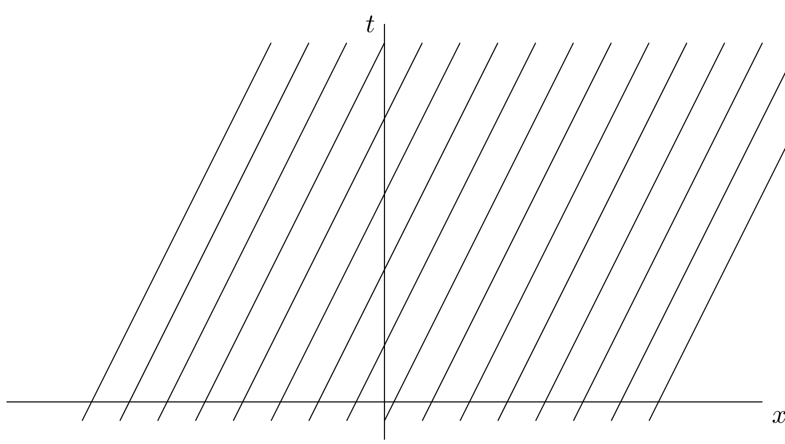

Figure 3.2.2: Characteristic Curves of $u_{tt} = c^2 u_{xx}$

The family of characteristic curves described by the equations in (3.29) consist of parallel lines of (negative) slope $-1/c$ in the xt–plane and x–intercept $\eta \imath \mathbb{R}$; some of theses curves are sketched in Figure 3.2.3.

Figure 3.2.4 contains a sketch of both sets of characteristic curve in the same graph. We will see in Example 4.1.1 of Section 4.1 how to use the two sets of characteristic curves in Figure 3.2.4 to construct a solution to the initial value problem to the one–dimensional wave equation.

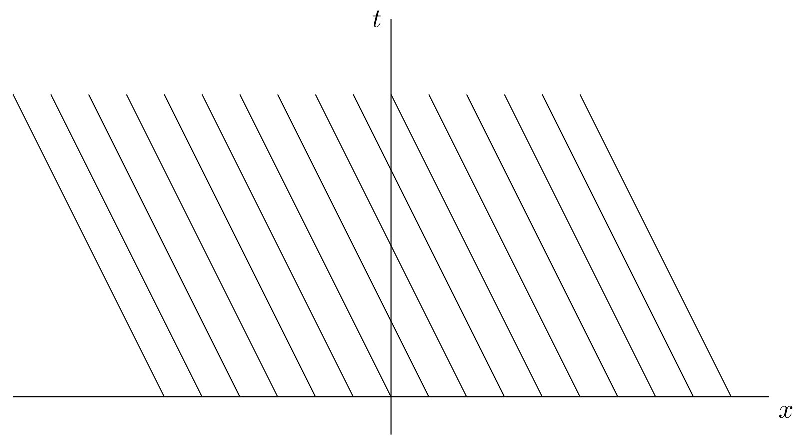

Figure 3.2.3: Characteristic Curves of $u_{tt} = c^2 u_{xx}$

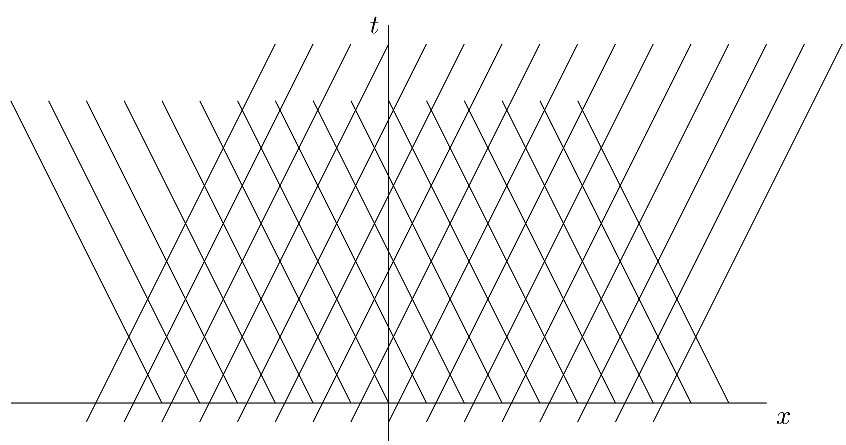

Figure 3.2.4: Characteristic Curves of $u_{tt} = c^2 u_{xx}$

Chapter 4

How are PDEs Solved?

There is really no general theory for solving any given PDE of the form in (2.1),

$$F(x, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots, u_{x_n x_n}, \ldots) = 0.$$

Approaches to the construction of solutions of PDE problems are determined by the types of PDEs and the geometric properties (e.g., symmetries) of the equations and/or domains in which the problems are posed. In this chapter we present some of those approaches. Emphasis will be placed on a few general principles that can aid us when looking for solutions of PDEs.

We will begin with an approach that uses knowledge of the characteristic curves of the equations. We will then look at approaches that exploit any symmetries that the equations or domains in the problems might have. We will then look at methods of solutions for linear equations based on the principle of superposition.

4.1 Using Characteristic Curves to Solve PDEs

In Section 3.2 we defined characteristic curves for second order PDEs in two variables, and saw how characteristic curves can be used to come up with a classification scheme for those equations. In this section we see how to use characteristic curves to construct solutions to certain types of PDEs in two variables. We begin with the example of the one–dimensional wave equation.

4.1.1 Solving the One–Dimensional Wave Equation

Example 4.1.1 (Solving the One–Dimensional Wave Equation)**.** We consider the initial value problem in the entire real:

$$\begin{cases} u_{xx} - \dfrac{1}{c^2} u_{tt} = 0, & \text{for } x \in \mathbb{R}, t > 0, \\ \\ u(x,0) = f(x), & \text{for all } x \in [0, L]; \\ u_t(x,0) = g(x), & \text{for all } x \in [0, L], \end{cases} \tag{4.1}$$

where f and g are given continuous functions defined in $\mathbb{R}$.

In Example 3.2.1 in the previous section we showed that the PDE in (4.1) has two families of characteristic curves given by

$$x = ct + \xi, \tag{4.2}$$

and

$$x = -ct + \eta. \tag{4.3}$$

These families of curves consist of parallel straight lines in the xt–plane of slope $1/c$ and of slope $-1/c$, respectively. We will see in the next section that characteristic curves carry information about solutions of the equation from one point on the curve to another point on the same curve. This suggests that we we consider the PDE in (4.1) along the curves given in (4.2) and (4.3). We can do this my considering the parameters ξ and η in (4.2) and (4.3) as a new set of variables,

$$\xi = x - ct, \tag{4.4}$$

and

$$\eta = x + ct. \tag{4.5}$$

If we are give a solution, u, to the PDE in (4.1), we can use the change of variables provided by (4.4) and (4.5) to define a function, v, of ξ and η in terms of u by means of

$$v(\xi, \eta) = u(x, t), \tag{4.6}$$

where x and t can be obtained in terms of ξ and θ by solving the linear system

$$\begin{cases} x - ct &= \xi; \\ x + ct &= \eta, \end{cases}$$

so that

$$\begin{aligned} x &= \frac{1}{2}\eta + \frac{1}{2}\xi; \\ t &= \frac{1}{2c}\eta - \frac{1}{2c}\xi. \end{aligned} \tag{4.7}$$

Alternatively, we can rewrite (4.6) as

$$u(x, t) = v(\xi, \eta), \tag{4.8}$$

where ξ and η are given by (4.4) and (4.5), respectively.

Assume that $u \in C^2(\mathbb{R}^2)$ solves the PDE in (4.1). We would like to derive a PDE satisfied by the function v defined in (4.6) and (4.7). The PDE that v will satisfy will be expressed in terms of the new variables ξ and η. In order to do this, we use (4.8) and the Chain Rule to get

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x}, \tag{4.9}$$

where

$$\frac{\partial \xi}{\partial x} = 1 \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 1, \tag{4.10}$$

by virtue of (4.4) and (4.5). Thus, substituting the expressions in (4.10) into (4.9),

$$u_x = v_\xi + v_\eta. \tag{4.11}$$

Next, take partial derivative with respect to x on both sides of (4.11) and use the Chain Rule to get

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x}[v_\xi] + \frac{\partial}{\partial x}[v_\eta] \\ &= v_{\xi\xi}\frac{\partial \xi}{\partial x} + v_{\xi\eta}\frac{\partial \eta}{\partial x} + v_{\eta\xi}\frac{\partial \xi}{\partial x} + v_{\eta\eta}\frac{\partial \eta}{\partial x}, \end{aligned}$$

so that, using the expressions in (4.10) and the fact that mixed partial derivatives, $v_{\xi\eta}$ and $v_{\eta\xi}$, of C^2 functions are equal,

$$u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}. \tag{4.12}$$

Similar calculations to those leading to (4.12) can be used to obtain an expression for u_{tt}. Indeed, take partial derivative with respect to t on both sides of (4.8) and use the Chain Rule to get

$$u_t = v_\xi \frac{\partial \xi}{\partial t} + v_\eta \frac{\partial \eta}{\partial t}, \tag{4.13}$$

where

$$\frac{\partial \xi}{\partial x} = -c \quad \text{and} \quad \frac{\partial \eta}{\partial x} = c, \tag{4.14}$$

by virtue of (4.4) and (4.5). Thus, substituting the expressions in (4.14) into (4.13),

$$u_t = -cv_\xi + cv_\eta. \tag{4.15}$$

Next, take partial derivative with respect to t on both sides of (4.15) and use the Chain Rule to get

$$\begin{aligned} u_{tt} &= \frac{\partial}{\partial t}[v_\xi] + \frac{\partial}{\partial t}[v_\eta] \\ &= -cv_{\xi\xi}\frac{\partial \xi}{\partial t} - cv_{\xi\eta}\frac{\partial \eta}{\partial t} + cv_{\eta\xi}\frac{\partial \xi}{\partial t} + cv_{\eta\eta}\frac{\partial \eta}{\partial t}; \end{aligned}$$

thus, using the expressions in (4.14), we get

$$u_{tt} \quad = \quad c^2 v_{\xi\xi} - c^2 v_{\xi\eta} - c^2 v_{\eta\xi} + c^2 v_{\eta\eta},$$

or

$$u_{tt} = c^2[v_{\xi\xi} - 2v_{\xi\eta} + v_{\eta\eta}], \tag{4.16}$$

where we have used the equality of the mixed second partial derivatives.

Since we are assuming that u solves the PDE

$$u_{xx} - \frac{1}{c^2} u_{tt} = 0, \tag{4.17}$$

it follows from (4.12), (4.16) and (4.17) that v solves the second order PDE

$$v_{\xi\eta} = 0. \tag{4.18}$$

Note that the PDE in (4.18) is also a hyperbolic, second order, linear PDE (in this case $a = c = 0$ and $b = 1$, so that $b^2 - 4ac = 1 > 0$). In contrasts with the hyperbolic PDE in (4.17), the PDE in (4.18) can be solved directly by integration. Indeed, writing (4.18) as

$$\frac{\partial}{\partial \eta}[v_\xi] = 0,$$

we see that

$$v_\xi = h(\xi), \tag{4.19}$$

where h is an arbitrary C^1 function of a single variable; and writing (4.19) as

$$\frac{\partial}{\partial \xi}[v(\xi, \eta)] = 0,$$

we see that

$$v(\xi, \eta) = F(\xi) + G(\eta), \tag{4.20}$$

where F is an antiderivative of h (i.e., $F' = h$), and G is an arbitrary C^2 function of a single variable.

The function v defined by the expression in (4.20), where F and G are arbitrary C^2 functions of a single variable, is the **general solution** to the PDE in (4.18). We can use it, along with (4.8), (4.4) and (4.5) to obtain the general solution to the one–dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad \text{for } x \in \mathbb{R}, \quad \text{and } t \in \mathbb{R}; \tag{4.21}$$

namely,

$$u(x, t) = F(x - ct) + G(x + ct), \tag{4.22}$$

where F and G are arbitrary C^2 functions of a single variable. The expression in (4.22) is known as d'Alembert's solutions to the one–dimensional wave equation.

We now use the general solution (4.22) to the one–dimensional wave equation in (4.21) to construct a solution to the initial value problem in (4.1). In this construction we will need to assume that f is C^2 and g is C^1.

Differentiate the expression for u in (4.22) with respect to t to obtain

$$u_t(x,t) = -cF'(x-ct) + cG'(x+ct), \tag{4.23}$$

where we have applied the Chain Rule. Next, apply the initial conditions in (4.1) to obtain the equations

$$\left\{ \begin{array}{rcl} F(x) + G(x) & = & f(x); \\ -cF'(x) + cG'(x) & = & g(x), \end{array} \right. \tag{4.24}$$

for all $x \in \mathbb{R}$, where we have used (4.22) and (4.23).

Next, differentiate the first equation in (4.24) and divide the second equation by c to get

$$\left\{ \begin{array}{rcl} F'(x) + G'(x) & = & f'(x); \\ -F'(x) + G'(x) & = & g(x)/c, \end{array} \right. \tag{4.25}$$

for all $x \in \mathbb{R}$, where we have used the differentiability assumptions on f.

Adding the equations in (4.25) then yields the following equation for G':

$$G'(x) = f'(x) + \frac{1}{2c} g(x), \qquad \text{for all } x \in \mathbb{R}. \tag{4.26}$$

Integrating the equation in (4.26) then yields

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(z)\ dz + C_1, \qquad \text{for all } x \in \mathbb{R}, \tag{4.27}$$

where C_1 is a constant of integration.

Similarly, subtracting the second equation in (4.25) from the first equation and integrating yields

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(z)\ dz + C_2, \qquad \text{for all } x \in \mathbb{R}, \tag{4.28}$$

where C_2 is a constant of integration.

Next, substitute the functions in (4.28) and (4.27) into the formula for $u(x,t)$ in (4.22) to get

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\ dz + C_3, \tag{4.29}$$

for all $x \in \mathbb{R}$, where $C_3 = C_1 + C_2$.

It follows from the first initial condition in (4.1) that the constant C_3 in (4.29) must be 0, so that

$$u(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z)\ dz, \qquad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}, \tag{4.30}$$

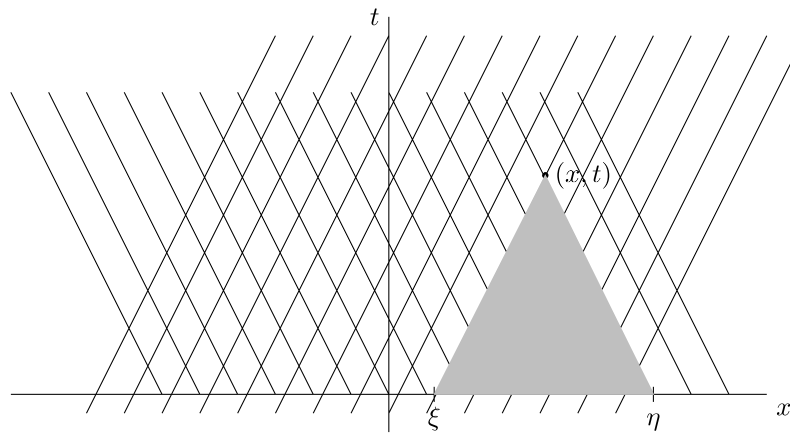

Figure 4.1.1: Characteristic Curves of $u_{tt} = c^2 u_{xx}$

solves the initial value problem (4.1) for the one–dimensional wave equation.

In order to understand what the solution to the IVP in (4.1) displayed in (4.30) is saying, refer to Figure 4.1.1. Suppose we want to compute the value of u at x and at time $t > 0$; that is, $u(x,t)$, where (x,t) is a point in the xt–plane. Two characteristic curves cross at that point: one with x–intercept labeled ξ in Figure 4.1.1, and the other with x–intercept labeled η in the figure. These correspond to the values $x - ct$ and $x + ct$, respectively. According to the expression for u in (4.29), the value of u at (x,t) is the average the values of the initial data f at those two points, plus t times the average of <u>all</u> the values of the initial speed, g, over the interval $[\xi, \eta]$.

For the special case in which the initial speed is zero throughout $\mathbb{R}$, we obtain from (4.30) the special form

$$u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)], \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}. \tag{4.31}$$

The function u in (4.31) is made up of two traveling wave forms: $\frac{1}{2}f(x-ct)$, which moves to the right with speed c, and $\frac{1}{2}f(x+ct)$, which moves to the left with speed c. We illustrate this for the spacial case in which the initial data f is in $C_c^\infty(\mathbb{R})$, with $\text{supp}(f) = [-1, 1]$; see Figure 4.1.2. Figure 4.1.3 shows the supports of the initial data and two of the traveling waves at some time $t > 0$ later with $ct > 2$. Figure 4.1.4 shows the two pulses traveling in opposite directions at that instant of time. Note that the two pulses in Figure 4.1.4 have half of the amplitude of the initial pulse in Figure 4.1.2.

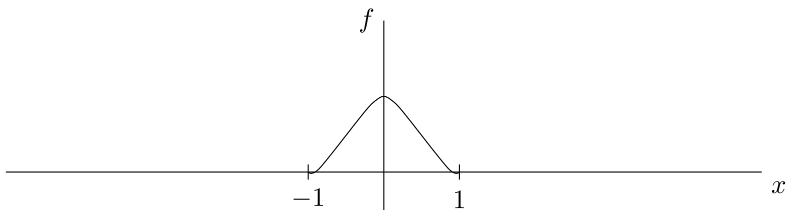

Figure 4.1.2: Initial Data

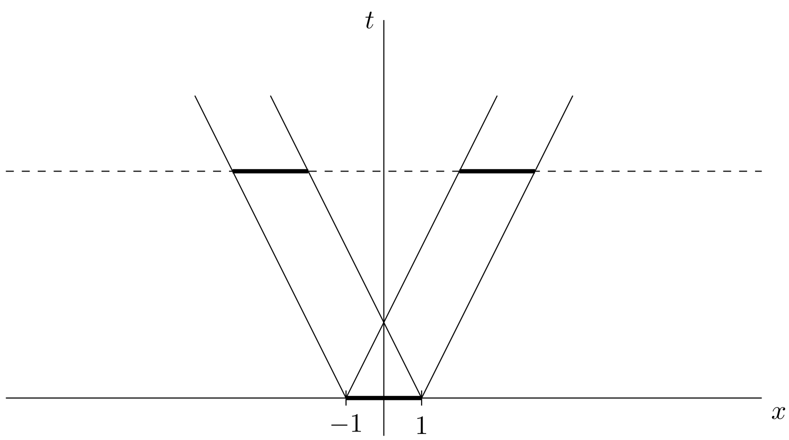

Figure 4.1.3: Traveling Waves

4.1.2 Solving First–Order PDEs

In this section we define characteristic curves for the first order equation in two variables

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u), \tag{4.32}$$

where a, b and c are C^∞ functions of three real variables, (x, y, z), where (x, y) lies in an open region, R, in $\mathbb{R}^2$. For the case in which coefficient functions, a, b and c, in (4.32) depend only on $(x, y) \in R$, the PDE in (4.32) turns into the linear PDE:

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y), \quad \text{for } (x,y) \in R. \tag{4.33}$$

We will first define the concept of characteristic curves for the PDE in (4.33). The discussion here is analogous to the discussion on characteristic curves for the second order equation in (3.17) on page 46. As in that discussion, the

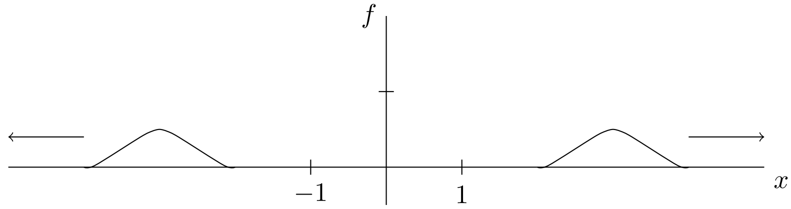

Figure 4.1.4: Traveling Pulses

starting point is smooth curve, Γ, in R parametrized by a map $\gamma\colon I \to \mathbb{R}^2$,

$$t \mapsto \gamma(t) = (x(t), y(t)), \qquad \text{for } t \in I,$$

where I is some interval of real numbers; see Figure 3.2.1. Suppose we are trying to solve the linear PDE in (4.33) subject to an "initial" condition on the curve Γ given by

$$u(x(t), y(t)) = f(t), \quad \text{for } t \in I, \tag{4.34}$$

where f is a known smooth function defined on I. The idea is that, given the information in (4.34), we can use that information together with the PDE in (4.33), to obtain the values of the derivatives, u_x and u_y, of u on Γ. Once these are obtained, we can differentiate (4.34) and the PDE in (4.33) to obtain information of the second derivatives on Γ. Since we are assuming that the coefficients, a, b and c, and the "initial" data, f, are C^∞ functions, we can, in theory, proceed in this fashion to obtain information about the higher order derivatives of u on Γ. If this can be done, we can attempt to construct a solution of the PDE in (4.33) by building Taylor series expansions around every point on Γ using the values of u and its derivatives. The first step in this construction is possible provided that the linear system

$$\begin{cases} \dot{x}\, u_x + \dot{y}\, u_y = \dot{f} \\ a\, u_x + b\, u_y = c, \end{cases} \tag{4.35}$$

for the unknowns u_x, and u_y on Γ can be solved. The system in (4.35) can be written in matrix form as

$$\begin{pmatrix} \dot{x} & \dot{y} \\ a & b \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \dot{f} \\ c \end{pmatrix}. \tag{4.36}$$

The matrix equation in (4.36) can be solved for the first derivatives of u, in terms of the data $\dot{f}$ on Γ, provided that the determinant of the matrix

$$\begin{pmatrix} \dot{x} & \dot{y} \\ a & b \end{pmatrix} \tag{4.37}$$

is not zero. The case in which the determinant of the matrix in (4.37) is zero yields the equation for the characteristic curves of the first order PDE in (4.33):

$$b\dot{x} - a\dot{y} = 0. \tag{4.38}$$

Observe that the ODE in (4.38) is equivalent to the system of first order ODEs:

$$\begin{cases} \dfrac{dx}{dt} = a(x,y); \\[2ex] \dfrac{dy}{dt} = b(x,y). \end{cases} \tag{4.39}$$

The system of ODEs in (4.39) defines the characteristic curves for the first–order linear PDE in (4.33). Since, we are assuming that a and b ar C^∞ functions, solutions to the system of first–order ODEs in (4.39) is guaranteed to have a unique solution around $t_o \in \mathbb{R}$ subject to the the initial condition $(x(t_o), y(t_o)) = (x_o, y_o)$. Thus, in theory, characteristic curves for the PDE in (4.33) can always be computed.

Suppose that we have computed the characteristic curves for the PDE in (4.33) according to the system of ODEs in (4.39). Let one of those characteristics be given by the parametrization

$$t \mapsto (x(t), y(t)), \qquad \text{for } t \in I, \tag{4.40}$$

where I is a maximal interval of existence. Suppose that u is a solution of the PDE in (4.32) and consider the values of u on the characteristic curve parametrized by (4.40),

$$u(x(t), y(t)), \qquad \text{for } t \in I. \tag{4.41}$$

It follows from (4.41) and the Chain Rule that

$$\frac{d}{dt}[u(x(t), y(t))] = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt},$$

so that, using the definition of the characteristic curves in (4.39),

$$\frac{d}{dt}[u(x(t), y(t))] = a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y),$$

by virtue of the PDE in (4.33). We have therefore shown that if u is a solution of the PDE in (4.33), then u must also solve the ODE

$$\frac{du}{dt} = c(x,y) \tag{4.42}$$

along the characteristic curves. This suggests a way to construct a solution to initial value problem for the PDE in (4.33) where the initial data is given on a curve that is not a characteristic curve. This approach is known as the **method of characteristic curves**.

Suppose we want to solve the IVP:

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = c(x,y) \quad \text{in } R; \\[2ex] u = f \quad \text{on } \Gamma, \end{cases} \tag{4.43}$$

where Γ is a curve in R that is not a characteristic curve. The method of characteristic curves consists of, first, finding the characteristic curves of the PDE in (4.43) by solving the system of ODEs in (4.39). Then, solve the ODE in (4.42). We illustrate this method by solving the following IVP for the one–dimensional **convection equation**.

Example 4.1.2 (One–Dimensional Convection Equation)**.** Consider the IVP

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0 & \text{for } x \in \mathbb{R} \text{ and } t > 0; \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{4.44}$$

where c is a nonzero constant and f is a given C^1 function defined in $\mathbb{R}$.

In this example t is playing the role of y, so that the equations for the characteristic curves in (4.39) become the single ODE

$$\frac{dx}{dt} = c, \tag{4.45}$$

which can be solved to yield

$$x = ct + \xi, \tag{4.46}$$

where ξ is a real parameter indexing the characteristic curves. For the case in which $c > 0$ the characteristic curves for the PDE in (4.44) are straight lines of positive slope $1/c$ in the xt–plane and x–intercept ξ. Some of these curves are sketched in Figure 4.1.5. Along each characteristic curve in (4.46), a solution

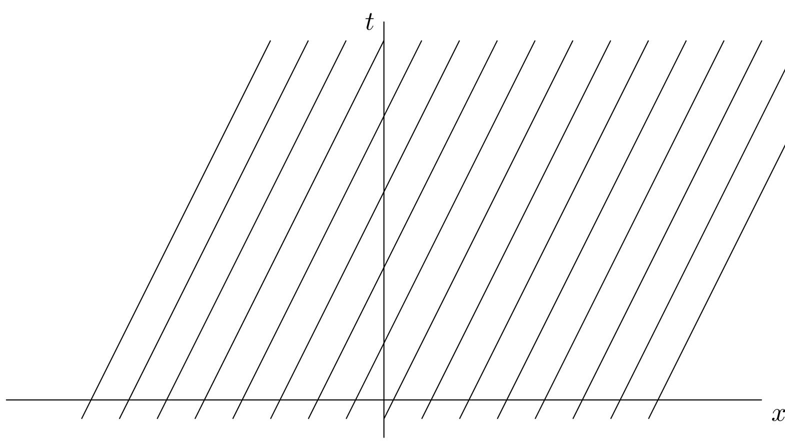

Figure 4.1.5: Characteristic Curves of $u_t + cu_x = 0$

to the PDE in (4.44) solves the ODE

$$\frac{du}{dt} = 0, \tag{4.47}$$

according to (4.42). Alternatively, we can obtain (4.47) by computing

$$\begin{aligned} \frac{d}{dt}[u(x(t),t)] &= u_x \cdot \frac{dx}{dt} + u_t \\ &= u_t + cu_x \\ &= 0, \end{aligned}$$

where we have used the Chain Rule, (4.45), and the assumption that u solves the PDE in (4.44).

We can solve the ODE in (4.47) to obtain that

$$u(x,t) = \text{constant along characteristic curves} \tag{4.48}$$

Since the characteristic curves in (4.46) are indexed by ξ, we can rewrite (4.48) as

$$u(x,t) = F(\xi), \tag{4.49}$$

where F is an arbitrary C^1 function of a real variable. Next, solve for ξ in (4.46) and substitute into (4.49) to obtain the general solutions to the PDE in (4.44),

$$u(x,t) = F(x - ct), \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R}. \tag{4.50}$$

For the case in which $c > 0$, (4.50) describes a traveling wave moving to the right with speed c.

Finally, using the initial condition in (4.44), we get that

$$F(x) = f(x), \quad \text{for all } x \in \mathbb{R},$$

so that

$$u(x,t) = f(x - ct), \quad \text{for } x \in \mathbb{R} \text{ and } t \in \mathbb{R},$$

is a solution to the IVP in (4.44).

The method of characteristic curves illustrated thus far also applied to the quasi–linear equation in (4.32). In this case, the equations to the characteristic curves read

$$\left\{ \begin{aligned} \frac{dx}{dt} &= a(x,y,u); \\ \frac{dy}{dt} &= b(x,y,u). \end{aligned} \right. \tag{4.51}$$

Along characteristic curves u solves the ODE

$$\frac{du}{dt} = c(x,y,u). \tag{4.52}$$

In general, we might not be able to obtain an explicit formula for a solution of the PDE in (4.32) based on the system (4.50)–(4.52). But, in some cases, we might be able to obtain an expression that gives $u(x,y)$ implicitly. We illustrate this in the following example.

Example 4.1.3. Find a solution to the initial value problem

$$\begin{cases} u_t + uu_x = 0, & \text{for } x \in \mathbb{R}, t > 0; \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{4.53}$$

Here t is playing the role of y in the general discussion. In this case, the equation for the characteristic curves is

$$\frac{dx}{dt} = u. \tag{4.54}$$

In order to solve the ODE in (4.54) we need information on the function u, which is what ultimately we are trying too determine. The information is provided by the differential equation that u satisfies along characteristic curves; namely,

$$\frac{du}{dt} = 0,$$

which implies that u must be constant along characteristic curves. Thus, we can set

$$u = F(\xi), \tag{4.55}$$

where ξ is a parameter indexing the characteristic curves, and F is an arbitrary C^1 function of a single variable. Substituting the expression for u in (4.55) into the equation for the characteristic curves in (4.54) yields

$$\frac{dx}{dt} = F(\xi),$$

which can be solved to yield the equation for the characteristic curves of the PDE in (4.53):

$$x = F(\xi)t + \xi. \tag{4.56}$$

Observe that in this case the characteristic curves are straight lines in the xt–plane with x–intercept ξ and slope $1/F(\xi)$. Note that the slopes of the characteristic curves depend on the value of the solution on the characteristic curves, according to (4.55).

We can solve for ξ in (4.56) and use (4.55) to get

$$\xi = x - u(x,t)t$$

and then substitute this value into (4.55) to obtain an implicit formula for $u(x,t)$:

$$u(x,t) = F(x - u(x,t)t), \quad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0. \tag{4.57}$$

Next, use the initial condition in (4.53) to obtain from (4.57) that

$$F(x) = f(x), \quad \text{for all } x \in \mathbb{R},$$

so that

$$u(x,t) = f(x - u(x,t)t), \quad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0,$$

provides an expression that defines $u(x,t)$ implicitly.

In the remainder of this section, we present more examples on the use of characteristic curves to solve first order PDEs.

Example 4.1.4. Find a solution to the initial value problem

$$\begin{cases} u_t + u_x = u, & \text{for } x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & \text{for } x \in \mathbb{R}, \end{cases} \tag{4.58}$$

where f is a given C^1 function.

The equation for the characteristic curves in this example is

$$\frac{dx}{dt} = 1,$$

which can be solved to yield

$$x = t + \xi. \tag{4.59}$$

Now, along characteristic curves, u solves the ODE

$$\frac{du}{dt} = u;$$

so that

$$u = F(\xi)e^t, \tag{4.60}$$

where F is a C^1 function of a real variable, and ξ is the parameter indexing the characteristic curves in (4.59).

Next, solve for ξ in (4.59) and substitute into (4.60) to get the general solution,

$$u(x,t) = F(x-t)e^t, \quad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0, \tag{4.61}$$

for the PDE in (4.58), where F is an arbitrary C^1 function. The initial condition in (4.58) can now be used to obtain from (4.61) that

$$F(x) = f(x), \quad \text{for all } x \in \mathbb{R}.$$

It then follows from (4.61) that

$$u(x,t) = f(x-t)e^t, \quad \text{for } x \in \mathbb{R} \text{ and } t \geqslant 0,$$

solves the initial value problem in (4.58).

Example 4.1.5. Find the general solution to the linear partial differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u, \quad \text{for } (x,y) \in \mathbb{R}^2. \tag{4.62}$$

The equations for the characteristic curves are

$$\begin{cases} \dfrac{dx}{dt} = x; \\ \\ \dfrac{dy}{dt} = y. \end{cases} \tag{4.63}$$

Using the Chain Rule, we obtain from (4.63) the ODE

$$\frac{dy}{dx} = \frac{y}{x}, \quad \text{for } x \neq 0. \tag{4.64}$$

The ODE in (4.64) can be solved by separating variables to yield

$$y = \xi x, \tag{4.65}$$

where ξ is a real parameter. Thus, the characteristic curves for the PDE in (4.62) is a pencil of straight lines through the origin in $\mathbb{R}^2$.

Now, along the characteristic curves for the PDE in (4.62), u solves the ODE

$$\frac{du}{dt} = 2u. \tag{4.66}$$

Next, combine the ODE in (4.66) with the first ODE in (4.63) to obtain the ODE

$$\frac{du}{dx} = \frac{2u}{x}, \quad \text{for } x \neq 0. \tag{4.67}$$

The ODE in (4.67) can be solved by separation of variables to yield

$$u = F(\xi)x^2, \tag{4.68}$$

where F is an arbitrary C^1 function, and ξ is the parameter indexing the characteristic curves in (4.65).

Solving for ξ in (4.65) and substituting into (4.68) then yields the general solution,

$$u(x, y) = F\left(\frac{y}{x}\right) x^2, \quad \text{for } x \neq 0.$$

4.2 Using Symmetry to Solve PDEs

A partial differential equation is said to be **invariant** under a group of transformations if its form does not change after a changing variables according to the transformations in the group. We illustrate this idea by looking at symmetric solutions to Laplace's equation in $\mathbb{R}^2$.

4.2.1 Radially Symmetric Solutions to Laplace's Equation

Suppose that u is a C^2 solution of Laplace's equation in $\mathbb{R}^2$,

$$u_{xx} + u_{yy} = 0. \tag{4.69}$$

We consider what happens to the equation in (4.69) when we change to a new set of variables, $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$, given by a one–parameter group of rotations given by the matrices

$$M_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}; \tag{4.70}$$

that is, rotations in the counterclockwise sense by an angle ϕ. We set

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = M_\phi \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.71}$$

or

$$\left\{ \begin{array}{rcl} \xi & = & x\cos\phi - y\sin\phi; \\ \eta & = & x\sin\phi + y\cos\phi, \end{array} \right. \tag{4.72}$$

in view of (4.70) and (4.71). The equations in (4.72) can be solved for x and y in terms of ξ and η by inverting the matrix in (4.70),

$$M_\phi^{-1} = M_{-\phi} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix},$$

so that

$$\left\{ \begin{array}{rcl} x & = & \xi\cos\phi + \eta\sin\phi; \\ y & = & -\xi\sin\phi + \eta\cos\phi. \end{array} \right. \tag{4.73}$$

In view of the equations in (4.73), we can think of u as a function of ξ and η, which we will denote by $v(\xi,\eta)$; so that

$$v(\xi,\eta) = u(x,y), \tag{4.74}$$

where x and y on the right–hand side of (4.74) are given in terms of ξ and η in (4.73).

Applying the Chain Rule, we obtain from (4.74) that

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x},$$

where

$$\frac{\partial \xi}{\partial x} = \cos\phi \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \sin\phi, \tag{4.75}$$

in view of the equations in (4.72). Thus,

$$u_x = \cos\phi\ v_\xi + \sin\phi\ v_\eta. \tag{4.76}$$

Similar calculations using (4.74) and (4.72) yield

$$u_y = -\sin\phi\ v_\xi + \cos\phi\ v_\eta. \tag{4.77}$$

Next, differentiate on both sides of (4.76) with respect to x and apply the Chain Rule to get

$$u_{xx} \;=\; \cos\phi\left[v_{\xi\xi}\frac{\partial \xi}{\partial x} + v_{\xi\eta}\frac{\partial \eta}{\partial x}\right] + \sin\phi\left[v_{\eta\xi}\frac{\partial \xi}{\partial x} + v_{\eta\eta}\frac{\partial \eta}{\partial x}\right],$$

so that, using (4.75) and the fact that the mixed second partial derivatives of C^2 functions are equal,

$$u_{xx} \;=\; \cos^2\phi\ v_{\xi\xi} + 2\sin\phi\cos\phi\ v_{\xi\eta} + \sin^2\phi\ v_{\eta\eta}. \tag{4.78}$$

Similarly, taking the partial derivative with respect to y on both sides of (4.77), and using

$$\frac{\partial \xi}{\partial y} = -\sin\phi \quad \text{and} \quad \frac{\partial \eta}{\partial y} = \cos\phi,$$

which follow from (4.72), we obtain from (4.77) that

$$u_{yy} = \sin^2\phi\ v_{\xi\xi} - 2\sin\phi\cos\phi\ v_{\xi\eta} + \cos^2\phi\ v_{\eta\eta}. \tag{4.79}$$

Thus, adding the expressions in (4.78) and (4.79),

$$u_{xx} + u_{yy} = v_{\xi\xi} + v_{\eta\eta}.$$

Hence, if u solves Laplace's equation in (4.69), then v solves the equation

$$v_{\xi\xi} + v_{\eta\eta} = 0,$$

which has the same form as Laplace's equation. We therefore conclude that Laplace's equation in (4.69) is invariant under rotations. This suggests that we look for solutions of (4.69) that are functions of a combination of the independent variables that is independent of the rotation parameter ϕ. To obtain such a combination, use (4.72) to compute

$$\begin{aligned} \xi^2 + \eta^2 &= (x\cos\phi - y\sin\phi)^2 + (x\sin\phi + y\cos\phi)^2 \\ &= x^2\cos^2\phi - 2xy\cos\phi\sin\phi + y^2\sin^2\phi \\ &\qquad + x^2\sin^2\phi + 2xy\sin\phi\cos\phi + y^2\cos^2\phi \\ &= x^2 + y^2, \end{aligned}$$

so that x^2+y^2 or $\sqrt{x^2+y^2}$ are combinations of the independent variables, x and y, that do not depend on ϕ, the rotation parameter; that is, they are rotationally invariant. We will therefore look for solutions of the Laplace's equation in (4.69) that are of the form

$$u(x, y) = f(\sqrt{x^2+y^2}), \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{4.80}$$

where f is a C^2 function of a single variable. Functions of the form in (4.80) are said to be **radially symmetric**.

Example 4.2.1 (Radially Symmetric Solutions of Laplace's Equation in $\mathbb{R}^2$)**.** Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$. Find all radially symmetric solutions of (4.69) in R.

Solution: We look for solutions of

$$u_{xx} + u_{yy} = 0, \quad \text{in } \Omega, \tag{4.81}$$

of the form

$$u(x, y) = f(r), \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{4.82}$$

where

$$r = \sqrt{x^2 + y^2}, \tag{4.83}$$

and $f\colon (0,\infty) \to \mathbb{R}$ is a C^2 function.

Write the expression in (4.83) $r^2 = x^2 + y^2$ and differentiate on both sides with respect to x, applying the Chain Rule to get

$$2r\frac{\partial r}{\partial x} = 2x,$$

from which we get that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \text{for } r > 0. \tag{4.84}$$

Similar calculations show that

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \text{for } r > 0, \tag{4.85}$$

Next, use the Chain Rule to obtain from (4.82) that

$$u_x = f'(r)\frac{\partial r}{\partial x},$$

so that, by virtue of (4.84),

$$u_x = \frac{x}{r}f'(r), \quad \text{for } r > 0. \tag{4.86}$$

Similar calculations using (4.82) and (4.85) yield

$$u_y = \frac{y}{r}f'(r), \quad \text{for } r > 0. \tag{4.87}$$

Next, use the Product Rule, the Quotient Rule, and the Chain Rule to obtain from (4.86) that

$$u_{xx} = \frac{1}{r}f'(r) + x\frac{rf''(r)\frac{\partial r}{\partial x} - f'(r)\frac{\partial r}{\partial x}}{r^2};$$

thus, using (4.84),

$$u_{xx} = \frac{1}{r}f'(r) + \frac{x^2}{r^2}f''(r) - \frac{x^2}{r^3}f'(r), \quad \text{for } r > 0. \tag{4.88}$$

Similar calculations, using (4.85) and (4.87) yield

$$u_{yy} = \frac{1}{r}f'(r) + \frac{y^2}{r^2}f''(r) - \frac{y^2}{r^3}f'(r), \quad \text{for } r > 0. \tag{4.89}$$

Next, add the expressions in (4.88) and (4.89) to obtain

$$u_{xx} + u_{yy} = \frac{2}{r}f'(r) + \frac{x^2 + y^2}{r^2}f''(r) - \frac{x^2 + y^2}{r^3}f'(r), \quad \text{for } r > 0,$$

or using the fact that $x^2 + y^2 = r^2$,

$$u_{xx} + u_{yy} = \frac{2}{r}f'(r) + f''(r) - \frac{1}{r}f'(r), \quad \text{for } r > 0,$$

or

$$u_{xx} + u_{yy} = f''(r) + \frac{1}{r}f'(r), \quad \text{for } r > 0. \tag{4.90}$$

It follows from (4.90) that, if u in (4.82) solves Laplace's equation in R, then f solves the second order ODE

$$f''(r) + \frac{1}{r}f'(r) = 0, \quad \text{for } r > 0,$$

or

$$rf''(r) + f'(r) = 0, \quad \text{for } r > 0,$$

which can be rewritten as

$$\frac{d}{dr}[rf'(r)] = 0, \quad \text{for } r > 0. \tag{4.91}$$

Integrating the equation in (4.91) yields

$$rf'(r) = c_1, \quad \text{for } r > 0,$$

and some constant c_1, or

$$f'(r) = \frac{c_1}{r}, \quad \text{for } r > 0, \tag{4.92}$$

and some constant c_1. Integrating the equation in (4.92) yields

$$f(r) = c_1 \ln r + c_2, \quad \text{for } r > 0, \tag{4.93}$$

and constants c_1 and c_2.

It follows from (4.82) and (4.93) that radially symmetric solutions of (4.81) are of the form

$$u(x, y) = c_1 \ln \sqrt{x^2 + y^2} + c_2, \quad \text{for } (x, y) \neq (0, 0), \tag{4.94}$$

and constants c_1 and c_2. □

Example 4.2.2 (The Dirichlet Problem in an Annulus)**.** For positive numbers, r_1 and r_2, with $r_1 < r_2$, define Ω to be the annulus

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid r_1 < \sqrt{x^2 + y^2} < r_2\}. \tag{4.95}$$

Denote by C_r the circle of radius r centered at the origin.

Solve the boundary value problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } \Omega; \\ u = a, & \text{on } C_{r_1}; \\ u = b, & \text{on } C_{r_2}, \end{cases} \tag{4.96}$$

where a and b are real constants.

Solution: Since the annulus Ω in (4.95) has radial symmetry, and the boundary conditions in (4.96) are also radially symmetric, it makes sense to look for radially symmetric solutions of problem (4.96). According to the result of Example 4.2.1, these are of the form given in (4.94); namely

$$u(x,y) = c_1 \ln\sqrt{x^2+y^2} + c_2, \quad \text{for } (x,y)\in\Omega, \tag{4.97}$$

for some constants c_1 and c_2.

The boundary conditions In (4.96) then imply that

$$c_1 \ln r_1 + c_2 = a \tag{4.98}$$

and

$$c_1 \ln r_2 + c_2 = b, \tag{4.99}$$

in view of (4.97). Solving the system of equations in (4.98) and (4.99) for c_1 and c_2 yields

$$c_1 = \frac{b-a}{\ln(r_2/r_1)},$$

and

$$c_2 = \frac{a\ln r_2 - b\ln r_1}{\ln(r_2/r_1)}.$$

Substituting these values for c_1 and c_2 into (4.97) yields a solution,

$$u(x,y) = \frac{b-a}{\ln(r_2/r_1)}\ln\sqrt{x^2+y^2} + \frac{a\ln r_2 - b\ln r_1}{\ln(r_2/r_1)}, \quad \text{for } (x,y)\in\Omega, \tag{4.100}$$

to the BVP in (4.96). The result of Problem 5 in Assignment #5 then shows that the function u given in (4.100) is the solution of the BVP in (4.96). □

4.2.2 Dilation Invariant Solutions to Laplace's Equation

In this section we explore the effect of the change of variables

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \tag{4.101}$$

for nonzero real constants α and β, on the two–dimensional Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad \text{in } \mathbb{R}^2. \tag{4.102}$$

The change of variables in (4.101) corresponds to

$$\begin{cases} \xi &= \alpha x;\\ \eta &= \beta y, \end{cases} \tag{4.103}$$

or

$$\begin{cases} x &= \xi/\alpha;\\ y &= \eta/\beta. \end{cases} \tag{4.104}$$

Setting

$$v(\xi,\eta) = u(x,y), \tag{4.105}$$

where x and y are given in terms of ξ and η by the equations in (4.104), we compute, using the Chain Rule, we obtain from (4.105) that

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x},$$

where, by virtue of the equations in (4.103),

$$\frac{\partial \xi}{\partial x} = \alpha \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 0,$$

so that

$$u_x = \alpha v_\xi. \tag{4.106}$$

Similarly,

$$u_y = \beta v_\eta. \tag{4.107}$$

Next, differentiate on both sides of (4.106) and apply the Chain Rules as in the previous calculations to obtain

$$u_{xx} = \alpha^2 v_{\xi\xi}. \tag{4.108}$$

Similarly, we obtain from (4.107) that

$$u_{yy} = \beta^2 v_{\eta\eta}. \tag{4.109}$$

Adding (4.108) and (4.109) we obtain

$$u_{xx} u_{yy} = \alpha^2 v_{\xi\xi} + \beta^2 v_{\eta\eta}. \tag{4.110}$$

Thus, if u solves Laplace's equation in (4.102), we obtain from (4.110) that

$$\alpha^2 v_{\xi\xi} + \beta^2 v_{\eta\eta} = 0. \tag{4.111}$$

It follows from (4.111) that Laplace's equation in $\mathbb{R}^2$ is invariant under the scaling transformation in (4.101), provided that $\alpha^2 = \beta^2$. We will therefore set $\alpha = \beta = \lambda$ in (4.101) to get

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = D_\lambda \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.112}$$

where D_λ denotes the scalar matrix

$$D_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

for a nonzero parameter λ. The transformations in (4.112) form a one–parameter family of dilations corresponding to the change of variables

$$\left\{ \begin{array}{rcl} \xi & = & \lambda x; \\ \eta & = & \lambda y. \end{array} \right. \tag{4.113}$$

Note from (4.113) that a combination of the variables that is independent of the dilation parameter λ is

$$\frac{\eta}{\xi} = \frac{y}{x}, \qquad \text{for } x \neq 0.$$

This suggests that we look for solutions to Laplace's equation in $\mathbb{R}^2$ of the form

$$u(x,y) = f\left(\frac{y}{x}\right), \qquad \text{for } x \neq 0, \tag{4.114}$$

where f is a C^2 function of a single variable.

Set

$$s = \frac{y}{x}, \qquad \text{for } x \neq 0, \tag{4.115}$$

so that, in view of (4.114)

$$u(x,y) = f(s), \tag{4.116}$$

where s is given by (4.115).

We look for a solution to Laplace's equation in $\mathbb{R}^2$ of the form in (4.116) where f is a C^2 function and s is as given in (4.115). Thus, assume that u solves (4.102) and compute

$$u_x = f'(s)\frac{\partial s}{\partial x}, \tag{4.117}$$

where we have used the Chain Rule and where

$$\frac{\partial s}{\partial x} = -\frac{y}{x^2}, \qquad \text{for } x \neq 0,$$

by virtue of (4.115), so that

$$\frac{\partial s}{\partial x} = -\frac{s}{x}, \qquad \text{for } x \neq 0. \tag{4.118}$$

Substituting (4.118) into the right–hand side of (4.117) then yields

$$u_x = -\frac{1}{x}sf'(s), \qquad \text{for } x \neq 0. \tag{4.119}$$

Next, differentiate with respect to x on both sides of (4.119) to get

$$u_{xx} = \frac{1}{x^2}sf'(s) - \frac{1}{x}\frac{\partial s}{\partial x}f'(s) - \frac{1}{x}sf''(s)\frac{\partial s}{\partial x}, \qquad \text{for } x \neq 0, \tag{4.120}$$

where we have used the Product Rule and the Chain Rule. Then, substitute (4.118) into the right–hand side of (4.120) to get

$$u_{xx} = \frac{2s}{x^2}f'(s) + \frac{s^2}{x^2}f''(s), \qquad \text{for } x \neq 0. \tag{4.121}$$

Next, apply the Chain Rule to obtain from (4.116) that

$$u_y = f'(s)\frac{\partial s}{\partial y}, \tag{4.122}$$

where

$$\frac{\partial s}{\partial y} = \frac{1}{x}, \qquad \text{for } x \neq 0. \tag{4.123}$$

It then follows from (4.122) and (4.123) that

$$u_y = \frac{1}{x} f'(s), \qquad \text{for } x \neq 0. \tag{4.124}$$

Differentiate on both sides of (4.124) with respect to y, apply the Chain Rule, and use (4.123) to obtain

$$u_{yy} = \frac{1}{x^2} f''(s), \qquad \text{for } x \neq 0. \tag{4.125}$$

Next, add the expressions in (4.121) and (4.125) to get

$$u_{xx} + u_{yy} = \frac{2s}{x^2} f'(s) + \frac{1+s^2}{x^2} f''(s), \qquad \text{for } x \neq 0. \tag{4.126}$$

It follows from (4.126) that, if u solves Laplace's equation in $\mathbb{R}^2$, then f solves the second order ODE

$$\frac{2s}{x^2} f'(s) + \frac{1+s^2}{x^2} f''(s) = 0, \qquad \text{for } x \neq 0,$$

or

$$(1+s^2) f''(s) + 2s f'(s) = 0. \tag{4.127}$$

In order to solve the ODE in (4.127), set

$$v(s) = f'(s), \tag{4.128}$$

so that

$$(1+s^2)\frac{dv}{ds} + 2sv = 0. \tag{4.129}$$

The first order ODE in (4.129) can be solved by separating variables to yield

$$\int \frac{1}{v}\, dv = -\int \frac{2s}{1+s^2}\, ds,$$

or

$$\ln|v| = \ln\left(\frac{1}{1+s^2}\right) + c_o, \tag{4.130}$$

for some constant c_o.

Exponentiating on both sides of (4.130) and using the continuity of v we obtain

$$v(s) = \frac{c_1}{1+s^2}, \qquad \text{for } s \in \mathbb{R}, \tag{4.131}$$

and some constant c_1. It follows from (4.128) and (4.131) that

$$f'(s) = \frac{c_1}{1+s^2}, \qquad \text{for } s \in \mathbb{R},$$

and some constant c_1, which can be integrated to yield

$$f(s) = c_1 \arctan(s) + c_2, \quad \text{for } s \in \mathbb{R}, \tag{4.132}$$

and constants c_1 and c_2.

It follows from (4.114) and (4.132) that dilation–invariant solutions of Laplace's equation in $\mathbb{R}^2$ are of the form

$$u(x, y) = c_1 \arctan\left(\frac{y}{x}\right) + c_2, \quad \text{for } x \neq 0, \tag{4.133}$$

and constants c_1 and c_2. The result in (4.133) states that dilation–invariant harmonic functions in $\mathbb{R}^2$ are linear functions of the angle, θ, the the point (x, y), for $(x, y) \neq (0, 0)$, makes with the positive x–axis:

$$u = c_1\theta + c_2,$$

for constants c_1 and c_2.

4.2.3 Dilation Invariant Solutions of the Diffusion Equation

In this section we look for dilation–invariant solutions of the one–dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{4.134}$$

where $D > 0$ is the diffusivity constant. We proceed as in Section 4.2.2 by finding conditions on parameters α and β so that the diffusion equation in (4.134) is invariant under the change of variables

$$\left\{ \begin{array}{rcl} \xi & = & \alpha x; \\ \tau & = & \beta t, \end{array} \right. \tag{4.135}$$

where $\alpha\beta \neq 0$.

Write

$$v(\xi, \tau) = u(x, t), \tag{4.136}$$

where x and t are given in terms of ξ and τ by inverting the system in (4.136),

$$\left\{ \begin{array}{rcl} x & = & \xi/\alpha; \\ t & = & \tau/\beta, \end{array} \right.$$

and use the Chain Rule to compute

$$u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\tau \frac{\partial \tau}{\partial x},$$

$$\frac{\partial \xi}{\partial x} = \alpha \quad \text{and} \quad \frac{\partial \eta}{\partial x} = 0,$$

so that

$$u_x = \alpha v_\xi. \tag{4.137}$$

Similarly,

$$u_t = \beta v_\tau. \tag{4.138}$$

Next, differentiate on both sides of (4.137) and apply the Chain Rules as in the previous calculations to obtain

$$u_{xx} = \alpha^2 v_{\xi\xi}. \tag{4.139}$$

Using the expressions in (4.138) and (4.139) we obtain

$$u_t - D\, u_{xx} = \beta v_\tau - D\alpha^2 v_{\xi\xi},$$

so that, if u solves the diffusion equation in (4.134),

$$\beta v_\tau - D\alpha^2 v_{\xi\xi} = 0. \tag{4.140}$$

Hence, the diffusion equation in (4.134) is invariant under the change of variables in (4.136) provided that

$$\beta = \alpha^2. \tag{4.141}$$

It follows from (4.140) and (4.141) that the diffusion equation in (4.134) is invariant under the dilation

$$\begin{cases} \xi &= \alpha x; \\ \tau &= \alpha^2 t. \end{cases} \tag{4.142}$$

It follows from (4.142) that combinations of the variables that are independent of the dilation parameter, α, are

$$\frac{\xi^2}{\tau} = \frac{x^2}{t} \quad \text{or} \quad \frac{\xi}{\sqrt{\tau}} = \frac{x}{\sqrt{t}}, \quad \text{for } \tau > 0 \text{ and } t > 0.$$

Thus, in order to find dilation–invariant solutions of the one–dimensional diffusion equation, we look for solutions of the form

$$u(x,t) = f\left(\frac{x}{\sqrt{t}}\right), \quad \text{for } t > 0, \tag{4.143}$$

where f is a C^2 function of a single variable.

Set

$$s = \frac{x}{\sqrt{t}}, \quad \text{for } t > 0, \tag{4.144}$$

so that, in view of (4.143)

$$u(x,t) = f(s), \tag{4.145}$$

where s is given by (4.144).

Differentiate on both sides of (4.145) with respect to x, using the Chain Rule, to get

$$u_x = f'(s)\frac{\partial s}{\partial x},$$

where

$$\frac{\partial s}{\partial x} = \frac{1}{\sqrt{t}}, \qquad \text{for } t > 0, \tag{4.146}$$

by virtue of (4.144), so that

$$u_x = \frac{1}{\sqrt{t}} f'(s), \qquad \text{for } t > 0. \tag{4.147}$$

Differentiate with respect to x on both sides of (4.147), use the Chain Rule, and the result in (4.146) to get

$$u_{xx} = \frac{1}{t} f''(s), \qquad \text{for } t > 0. \tag{4.148}$$

Next, differentiate on both sides of (4.145) with respect to t, using the Chain Rule, to get

$$u_t = f'(s)\frac{\partial s}{\partial t}, \tag{4.149}$$

where, by virtue of (4.144),

$$\frac{\partial s}{\partial t} = -\frac{x}{2t\sqrt{t}},$$

or, using (4.144),

$$\frac{\partial s}{\partial t} = -\frac{s}{2t}, \qquad \text{for } t > 0. \tag{4.150}$$

Substitute the result in (4.150) into the right–hand side of (4.149) to get

$$u_t = -\frac{s}{2t} f'(s), \qquad \text{for } t > 0. \tag{4.151}$$

It follows from (4.148) and (4.151) that, if u given in (4.145) solves the diffusion equation in (4.134), then f solves the ODE

$$-\frac{s}{2t} f'(s) = \frac{D}{t} f''(s), \qquad \text{for } t > 0,$$

or

$$f''(s) + \frac{s}{2D} f'(s) = 0 \tag{4.152}$$

In order to solve the ODE in (4.152), set

$$v(s) = f'(s), \tag{4.153}$$

so that

$$\frac{dv}{ds} + \frac{s}{2D} v = 0. \tag{4.154}$$

The first order ODE in (4.154) can be solved by separating variables to yield

$$\int \frac{1}{v}\;dv = -\int \frac{s}{2D}\;ds,$$

or

$$\ln|v| = -\frac{s^2}{4D} + c_o, \tag{4.155}$$

for some constant c_o.

Exponentiating on both sides of (4.155) and using the continuity of v we obtain

$$v(s) = c_1 e^{-s^2/4D}, \quad \text{for } s \in \mathbb{R}, \tag{4.156}$$

and some constant c_1. It follows from (4.156) and (4.153) that

$$f'(s) = c_1 e^{-s^2/4D}, \quad \text{for } s \in \mathbb{R},$$

and some constant c_1, which can be integrated to yield

$$f(s) = c_1 \int_0^s e^{-z^2/4D}\;dz + c_2, \quad \text{for } s \in \mathbb{R}, \tag{4.157}$$

and constants c_1 and c_2. It follows from (4.143) and (4.157) that dilation–invariant solutions of one–dimensional diffusion equation in (4.134) are of the form

$$u(x,t) = c_1 \int_0^{x/\sqrt{t}} e^{-z^2/4D}\;dz + c_2, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{4.158}$$

and constants c_1 and c_2.

Chapter 5

Solving Linear PDEs

In Chapter 4 we saw two general approaches for finding solutions to first or second order PDEs: using characteristic curves and looking for symmetric solutions. In theory, these methods could be applied to nonlinear or linear equations. In this chapter we explore methods that exploit the special structure provided by linear PEDs. In Section 3.1 we saw the Principle of Superposition (Proposition 3.1.1 on page 46 in these notes), which states that linear combinations of solutions to the homogeneous linear PDE

$$Lu = 0,$$

where L is a linear differential operator, are also solutions. Thus, in principle, we can use superposition to construct solutions of linear PDEs satisfying certain conditions by putting together known solutions. We will see in this chapter that this procedure can be carried out by adding together infinitely many solutions in the form of a series or and integral transform. We will begin by solving the vibrating string equation that we derived in in Section 2.3.3 and Section 2.4 using separation of variables and Fourier series expansions. In this chapter, we will also look at a a special solutions that can be used as building blocks to obtain solutions to initial and/or boundary value problems for a large class of linear PDEs. These special solutions are known as **Fundamental Solutions**.

5.1 Solving the Vibrating String Equation

In this section we construct a solution of the initial–boundary value problem for the one–dimensional, linear, homogeneous wave equation in (2.143); namely,

$$u_{tt} - c^2 u_{xx} = 0, \quad \text{for } x \in (0, L) \text{ and } t > 0, \tag{5.1}$$

where c is a positive constant, subject to the boundary conditions

$$u(0, t) = 0, \quad \text{for } t > 0, \tag{5.2}$$

and

$$u(L,t) = 0, \qquad \text{for } t > 0, \tag{5.3}$$

and the initial conditions

$$u(x,0) = f(x), \qquad \text{for } x \in [0,L], \tag{5.4}$$

and

$$u_t(x,0) = 0, \qquad \text{for } x \in [0,L], \tag{5.5}$$

where $f\colon [0,L] \to \mathbb{R}$ is a given, real valued, continuous function defined on the interval $[0,L]$.

We recall that the constant c in (5.1) is given by

$$c^2 = \frac{\tau}{\rho}, \tag{5.6}$$

where ρ is the constant density of the string in units of mass per length, and τ is the constant tension of the string in units of force.

The PDE in (5.1), together with the boundary conditions in (5.2) and (5.3), and the initial conditions in (5.4) and (5.5), models the small amplitude vibrations of a taut string of length L that is fixed at the end–points 0 and L of the interval $[0,L]$, that is plucked from rest from an initial shape given by the graph of the function f. In this section we construct a solution of this initial–boundary value problem, which we write in concise form as

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{for } x \in (0,L) \text{ and } t > 0; \\ u(0,t) = u(L,t) = 0, & \text{for } t \geqslant 0; \\ u(x,0) = f(x), & \text{for } x \in [0,L]; \\ u_t(x,0) = 0, & \text{for } x \in [0,L]. \end{cases} \tag{5.7}$$

At this point, we will assume that f is a continuous function that satisfies

$$f(0) = 0 \quad \text{and} \quad f(L) = 0.$$

As we refine the construction of a solution $u\colon [0,L] \times [0,\infty) \to \mathbb{R}$ of the system in (5.7), we will make further assumptions on f.

We look for solutions, $u\colon [0,L] \times [0,\infty) \to \mathbb{R}$, of the partial differential equation in (5.1) that have continuous, second partial derivatives with respect to t and with respect to x. We first consider functions of a special form

$$u(x,t) = y(x)h(t), \qquad \text{for } x \in [0,L], \text{ and } t \geqslant 0, \tag{5.8}$$

where $y \in C^2([0,L],\mathbb{R})$ and $h \in C^2([0,\infty),\mathbb{R})$; that is $y\colon [0,L] \to \mathbb{R}$ is a real valued function defined on $[0,L]$ that has continuous second derivative in $(0,L)$, which can be extended to a continuous function on $[0,L]$; similarly, $h\colon [0,\infty) \to \mathbb{R}$ is a real valued, continuous function defined on $[0,\infty)$ that is twice differentiable in $(0,\infty)$ with continuous second derivative that can be extended to a continuous function on $[0, \infty)$. We will also require that these solutions satisfy the boundary conditions in (5.2) and (5.3).

Once we find solutions of (5.1) of the form given in (5.8) that satisfy the boundary conditions in (5.2) and (5.3), we take advantage of the fact that the PDE in (5.1) is linear to construct linear combinations of these solutions, which will also be solutions, to approximate solutions that satisfy the initial conditions in (5.4) and (5.5). This is the general outline of the procedure to be used in this section, which is know in the literature as the method of **separation of variables**.

5.1.1 Separation of Variables

Suppose that $u\colon [0, L]\times[0, \infty) \to \mathbb{R}$ is of the form in (5.8), where $y \in C^2([0, L], \mathbb{R})$ and $h \in C^2([0, \infty), \mathbb{R})$. Then, the second partial derivatives, u_{tt} and u_{xx}, of u are given by

$$u_{xx}(x, t) = y''(x)h(t), \quad \text{for } x \in (0, L), \text{ and } t > 0, \tag{5.9}$$

and

$$u_{tt}(x, t) = y(x)h''(t), \quad \text{for } x \in (0, L), \text{ and } t > 0. \tag{5.10}$$

If we require that u be a solution of the PDE in (5.1), substituting the expression for u_{xx} in (5.9) and for u_{tt} in (5.10) into (5.1) then yields

$$y(x)h''(t) = c^2y''(x)h(t), \quad \text{for } x \in (0, L), \text{ and } t > 0. \tag{5.11}$$

Next, divide on both sides of the equation in (5.11) by $c^2y(x)h(t)$ (assuming that $c^2y(x)h(t) \neq 0$) to obtain

$$\frac{h''(t)}{c^2h(t)} = \frac{y''(x)}{y(x)}, \quad \text{for } x \in (0, L), \text{ and } t > 0, \tag{5.12}$$

provided that $y(x) \neq 0$ and $h(t) \neq 0$.

For the expression in (5.12) to be true for all $x \in (0, L)$ and $t > 0$ at which the denominators in (5.12) are not 0, it must be the case that both sides of the equation in (5.12) must be equal to a constant. The reason for this is that the variables x and t vary independently of each other. Denoting that constant by $-\lambda$, we obtain that

$$\frac{y''(x)}{y(x)} = -\lambda,$$

and

$$\frac{h''(t)}{c^2h(t)} = -\lambda,$$

from which we obtain that pair of ordinary differential equations

$$y''(x) + \lambda y(x) = 0, \quad \text{for } x \in (0, L), \tag{5.13}$$

and

$$h''(t) + \lambda c^2 h(t) = 0, \qquad \text{for } t > 0. \tag{5.14}$$

Next, if u is to satisfy the boundary conditions in (5.2) and (5.3), it must be the case that

$$y(0)h(t) = 0 \ \text{ and } \ y(L)h(t) = 0, \qquad \text{for all } t \geqslant 0. \tag{5.15}$$

At this point we make the observation that the function defined by

$$u(x,t) = 0, \qquad \text{for all } x \in [0,L] \text{ and } t \geqslant 0, \tag{5.16}$$

is a solution of the PDE in (5.1). We call it the **trivial solution** of the equation. We also note that the function in (5.16) satisfies the boundary conditions in (5.2) and (5.3) and the initial condition in (5.5). However, for a general function $f\colon [0,L] \to \mathbb{R}$, the trivial function in (5.16) does not satisfy the initial condition in (5.4). Thus, if we are to construct a solution of the initial–boundary problem in (5.7), it cannot be the trivial solution. Hence, we look for nontrivial solutions of the PDE in (5.1) of the form in (5.8).

Consequently, if $u \in C^2([0,L] \times [0,\infty), \mathbb{R})$ of the form given in (5.8) is a nontrivial solution of the PDE in (5.1) that also satisfies the boundary conditions in (5.15), then it must be the case that

$$y(0) = 0 \ \text{ and } \ y(L) = 0. \tag{5.17}$$

To see why this is the case, use the assumption that u is nontrivial to conclude that there exists $t_o \geqslant 0$ such that $h(t_o) \neq 0$ and, according to (5.15),

$$y(0)h(t_o) = 0 \ \text{ and } \ y(L)h(t_o) = 0. \tag{5.18}$$

We can see now that (5.17) follows from (5.18) and the fact that $h(t_o) \neq 0$.

We therefore look for nontrivial solutions of the the two–point, boundary value problem (BVP)

$$\begin{cases} y'' + \lambda y = 0, & \text{for } x \in (0,L); \\ y(0) = 0; \\ y(L) = 0. \end{cases} \tag{5.19}$$

The BVP in (5.19) is an example of an **eigenvalue problem**. We need to determine values of λ for which the BVP has nontrivial solutions. These values are called eigenvalues. The corresponding nontrivial solutions will be called **eigenfunctions**.

We consider three possibilities for λ in the BVP in (5.19): $\lambda = 0$, $\lambda < 0$, or $\lambda > 0$.

If $\lambda = 0$, the ODE in (5.19) becomes

$$y'' = 0,$$

which has general solution

$$y(x) = ax + b, \qquad \text{for } x \in \mathbb{R}, \tag{5.20}$$

and for some constants a and b.

Since the function y given in (5.20) also needs to satisfy the boundary condition

$$y(0) = 0$$

in the BVP (5.19), it follows from (5.20) that

$$b = 0,$$

from which we get that

$$y(x) = ax, \quad \text{for } x \in \mathbb{R}. \tag{5.21}$$

By the same token, since y also has to satisfy the boundary condition

$$y(L) = 0$$

in (5.19), we get from (5.21) that

$$aL = 0,$$

we get that

$$a = 0,$$

since $L > 0$. Consequently, in view of (5.21),

$$y(x) = 0, \quad \text{for } x \in \mathbb{R},$$

which shows that y must be the trivial solution. Consequently, if $\lambda = 0$, the BVP in (5.19) has only the trivial solution. Therefore, $\lambda = 0$ is not an eigenvalue of the BVP (5.19).

Next, assume that $\lambda < 0$ in BVP (5.19), and set $\lambda = -\mu^2$, where $\mu > 0$. Then, the ODE in (5.19) becomes

$$y'' - \mu^2 y = 0. \tag{5.22}$$

The characteristic equation of the ODE in (5.22) (see Appendix A.1) is

$$m^2 - \mu^2 = 0,$$

which has roots μ and $-\mu$; hence, according to (A.5) in Appendix A.1, the general solution of the ODE in (5.22) is

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}, \quad \text{for } x \in \mathbb{R}, \tag{5.23}$$

for constants c_1 and c_2.

The boundary conditions in the BVP in (5.19) imply from (5.23) that

$$\begin{cases} c_1 + c_2 &= 0; \\ c_1 e^{\mu L} + c_2 e^{-\mu L} &= 0, \end{cases}$$

which we can write in matrix form as

$$\begin{pmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.24}$$

The homogeneous system in (5.24) has only the trivial solution if and only if

$$\det \begin{pmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{pmatrix} \neq 0, \tag{5.25}$$

where

$$\det \begin{pmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{pmatrix} = e^{-\mu L} - e^{\mu L}.$$

$$\det \begin{pmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{pmatrix} = 0$$

if and only if

$$e^{\mu L} = e^{-\mu L}$$

if and only if

$$e^{2\mu L} = 1$$

if and only if

$$2\mu L = 0,$$

from which we conclude that $\mu = 0$, which contradicts the assumption that $\mu > 0$. Hence, (5.25) must hold true, from which we conclude that the system in (5.24) has only the trivial solution

$$c_1 = c_2 = 0;$$

so that, in view of (5.23),

$$y(x) = 0, \qquad \text{for all } x \in \mathbb{R},$$

which shows that, if $\lambda < 0$, then the BVP in (5.19) has only the trivial solution.

It remains to consider the case $\lambda > 0$. Thus, assume that $\lambda = \omega^2$, where $\omega > 0$. In this case, the ODE in BVP (5.19) becomes

$$y'' + \omega^2 y = 0, \tag{5.26}$$

which has characteristic equation

$$m^2 + \omega^2 = 0,$$

with complex conjugate roots ωi and $-\omega i$. Consequently, according to (A.5) in Appendix A.1, the general solution of the ODE in (5.26) is given by

$$y(x) = c_1 \cos \omega x + c_2 \sin \omega x, \qquad \text{for all } x \in \mathbb{R}. \tag{5.27}$$

The boundary condition $y(0) = 0$ in BVP (5.19) implies from (5.27) that $c_1 = 0$; so that, in view of (5.27),

$$y(x) = c_2 \sin \omega x, \quad \text{for all } x \in \mathbb{R}. \tag{5.28}$$

The second boundary condition in (5.19) implies from (5.28) that

$$c_2 \sin \omega L = 0. \tag{5.29}$$

It follows from (5.28) that $c_2 \neq 0$; otherwise y would be trivial. Hence, we obtain from (5.29) that

$$\sin \omega L = 0. \tag{5.30}$$

The equation in (5.30) has infinitely many solutions given by

$$\omega L = n\pi, \quad \text{for } n \in \mathbb{Z},$$

from which we get that

$$\omega = \frac{n\pi}{L}, \quad \text{for } n \in \mathbb{Z}. \tag{5.31}$$

Since we are assuming that $\omega > 0$, we have to exclude the negative solutions and 0 from the list in (5.31); so that,

$$\omega = \frac{n\pi}{L}, \quad \text{for } n = 1, 2, 3, \ldots \tag{5.32}$$

Next, since $\lambda = \omega^2$, we have shown that the eigenvalue problem in (5.19) has an infinite sequence, (λ_n), of positive eigenvalues given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \text{for } n = 1, 2, 3, \ldots \tag{5.33}$$

To each of the eigenvalues, λ_n, in (5.33) there correspond eigenfunctions, $y_n \colon [0, L] \to \mathbb{R}$, defined by

$$y_n(x) = c_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{for } x \in [0, L] \text{ and } n = 1, 2, 3, \ldots, \tag{5.34}$$

according to (5.28), for arbitrary non–zero constants c_n.

Next, we turn to the ODE in (5.14) where λ is replaced by λ_n for $n \in \mathbb{N}$. We therefore get a sequence of ODEs

$$h''(t) + \lambda_n c^2 h(t) = 0, \quad \text{for } t > 0 \text{ and } n = 1, 2, 3, \ldots \tag{5.35}$$

The characteristic equation for each of the ODEs in (5.35) is

$$m^2 + \lambda_n c^2, \quad \text{for } n = 1, 2, 3, \ldots,$$

which has complex conjugate roots $\sqrt{\lambda_n} ci$ and $-\sqrt{\lambda_n} ci$. Consequently, by (A.6) in Appendix A, the general solutions of the ODEs in (5.35) are given by

$$h_n(t) = a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right), \tag{5.36}$$

for $t \geqslant 0$, $n = 1, 2, 3, \ldots$, and arbitrary constants a_n and b_n, for $n \in \mathbb{N}$, where we have used (5.33).

Thus, in view of (5.8), (5.34) and (5.36), for each $n \in \mathbb{N}$, there is a solution of the wave equation in (5.1) of the form

$$u_n(x,t) = y_n(x)h_n(t), \quad \text{for } x \in [0,L] \text{ and } t \geqslant 0,$$

or, takin $c_n = 1$ for all $n \in \mathbb{N}$ in (5.34),

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right), \tag{5.37}$$

for $x \in [0,L]$, $t \geqslant 0$, $n = 1, 2, 3, \ldots$, and arbitrary constants a_n and b_n, for $n \in \mathbb{N}$, which also satisfies the boundary conditions in the initial–boundary value problem in (5.7).

Now, each of the functions in (5.37) does not, in general, satisfy the initial conditions in (5.7); unless, for exmaple, the function f in (5.7) is a of a very spacial form:

$$f(x) = a_1 \sin\left(\frac{\pi x}{L}\right), \quad \text{for } x \in [0,L],$$

where $|a_1|$ is very small, $u_1(x,t)$ in (5.37) with $b_1 = 0$, no function in (5.37) by itself will yield a solution of the initial–boundary value problem (5.7). However, since the PDE in (5.7) is linear, sums of the functions in (5.37) are also solutions of the PDE. We therefore consider next functions of the form

$$u_N(x,t) = \sum_{n=1}^{N} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right), \tag{5.38}$$

for $x \in [0,L]$, $t \geqslant 0$, $N = 1, 2, 3, \ldots$, and arbitrary constants a_n and b_n, for $n \in \mathbb{N}$. We note that each of the functions in (5.38) satisfies the boundary conditions in problem (5.7), since

$$u_N(0,t) = 0 \quad \text{and } u_N(L,t) = 0, \quad \text{for all } t \geqslant 0.$$

Next, we explore whether functions in (5.38) can satisfy the initial conditions in (5.7) as well.

From (5.38) we can compute the partial derivative

$$\frac{\partial}{\partial t}[u_N(x,t)] = \sum_{n=1}^{N} \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right)\left(-a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right), \tag{5.39}$$

for $x \in [0,L]$ and $t \geqslant 0$. Thus, if we set $b_n = 0$ for all n, we get from (5.39) that

$$\frac{\partial}{\partial t}[u_N(x,0)] = 0, \quad \text{for all } x \in [0,L]. \tag{5.40}$$

It follows from (5.40) that the functions

$$u_N(x,t) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \text{for } x \in [0,L],\ t \geqslant 0, \tag{5.41}$$

$N = 1, 2, 3, \ldots$, and arbitrary constants a_n, satisfy the second of the initial conditions in the problem (5.7), as well as the boundary conditions and the PDE in that problem. However, in general, the functions in (5.41) do not satisfy first initial condition in (5.7), unless

$$\sum_{n=1}^{N} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x), \quad \text{for } x \in [0, L], \tag{5.42}$$

for some $N \in \mathbb{N}$ and some constants a_n. Only if the function f is a linear combination of the trigonometric functions on the left–hand side of (5.42), will the function given in (5.41) be a solution of the initial–boundary value problem for the one–dimensional wave equation in (5.7). For a large class of functions $f \in C([0, L], \mathbb{R})$, though, we will be able to express the f as a limit linear combinations of trigonometric functions as in the left–hand side of (5.42).

Set

$$S_N(x) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{for } x \in [0, L], \tag{5.43}$$

where the coefficients c_n will be chosen in a special way, depending on f, to be described in the next section. We would like to know when

$$\lim_{N\to\infty} S_N(x) = f(x), \quad \text{for } x \in [0, L]. \tag{5.44}$$

If the limit on the left–hand side of the (5.44) exists, we denote it by

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{for } x \in [0, L], \tag{5.45}$$

and call it a **Fourier series**, or a **Fourier series expansion**. Thus, in order to construct a solution of the initial–boundary value problem in (5.7), we first need to answer the question of whether or not the function $f \in C([0, L], \mathbb{R})$ has a Fourier series expansion; that is, the limit expression in (5.44), where S_N is given in (5.43), holds true (in some appropriate way), or

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x), \quad \text{for } x \in [0, L], \tag{5.46}$$

for a special class of coefficients, c_n, determined by f. We will answer this question in the following section.

5.1.2 Fourier Series Expansions

Let's assume for a moment that the series on the left–hand side of (5.46) converges uniformly to to f on $[0, L]$, so that term–by–term integration of the series is justified. For each $m \in \mathbb{N}$, multiply both sides of the equation in (5.46) by

$$\sin\left(\frac{m\pi x}{L}\right)$$

for some $m \in \mathbb{N}$ to get

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = f(x) \sin\left(\frac{m\pi x}{L}\right), \quad \text{for } x \in [0, L],$$

and integrate on both sides from 0 to L to get

$$\int_0^L \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx. \tag{5.47}$$

The assumption that the series in the integrand on the left–hand side of (5.47) justifies the term–by–term integration of the series to get

$$\sum_{n=1}^{\infty} c_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx. \tag{5.48}$$

To evaluate the integrals on the left–hand side on (5.48), we use the trigonometric identity

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right], \quad \text{for } A, B \in \mathbb{R}. \tag{5.49}$$

Applying (5.49) to the integrands on the left–hand side of (5.48), we get that

$$\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) = \frac{1}{2}\left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right], \tag{5.50}$$

for $m, n \in \mathbb{N}$. We integrate each of the terms on the right–hand side of (5.50) from 0 to L separately. First, integrate the right–most term in (5.50) to get

$$\int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right) dx = \left[\frac{L}{(n+m)\pi} \sin\left(\frac{(n+m)\pi x}{L}\right)\right]_0^L = 0,$$

for all $n, m \in \mathbb{N}$; so that, in view of (5.50),

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) dx, \tag{5.51}$$

for $n, m \in \mathbb{N}$.

To evaluate the integral on the right–hand side of (5.51), we consider the cases $n \neq m$ and $n = m$ separately.

Suppose that $n \neq m$ and evaluate

$$\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) dx = \left[\frac{L}{(n-m)\pi} \sin\left(\frac{(n+m)\pi x}{L}\right)\right]_0^L = 0;$$

so that, in view of (5.51),

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0, \quad \text{if } n \neq m. \tag{5.52}$$

On the other hand, if $n = m$, it follows from (5.51) that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \frac{L}{2}, \quad \text{if } n = m. \tag{5.53}$$

Combining (5.52) and (5.53) we then have that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\,dx = \begin{cases} \dfrac{L}{2}, & \text{if } n = m; \\ \\ 0, & \text{if } n \neq m. \end{cases} \tag{5.54}$$

Using the result in (5.54), we obtain from (5.48) that

$$c_m \frac{L}{2} = \int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)\,dx,$$

from which we get the formula

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad \text{for } n \in \mathbb{N}, \tag{5.55}$$

for computing the coefficients in the Fourier series expansion for f in (5.46).

Given $f \in C([0,L],\mathbb{R})$, define

$$S_N(x) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{for } x \in [0,L], \tag{5.56}$$

where the coefficients c_n, for $n \in \mathbb{N}$, are given in (5.55).

In the remainder of this section, we determine conditions on $f \in C([0,L],\mathbb{R})$ for which

$$\lim_{N\to\infty} S_N(x) = f(x), \quad \text{for } x \in [0,L], \tag{5.57}$$

where S_N is given in (5.56), and for which the convergence in (5.57) is uniform, so that the calculations leading to (5.55) are justified.

If the statement in (5.57) is true, we write

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x), \quad \text{for } x \in [0,L], \tag{5.58}$$

where the coefficients c_n, for $n \in \mathbb{N}$, are given in (5.55).

The expression on the left–hand side of (5.58) is an example of a Fourier series expansion.

In general, assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded, periodic function of period $2L$, where $L > 0$. Assume also that f is integrable over $[-L,L]$; so that,

$$\int_{-L}^{L} |f(x)|\,dx < \infty, \tag{5.59}$$

where the integral in (5.59) denotes the Riemann integral. If this is the case, we can the define Fourier coefficients of f as follows:

$$a_o = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx; \tag{5.60}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx, \quad \text{for } n = 1, 2, 3, \ldots; \tag{5.61}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx, \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.62}$$

The Fourier series expansion of f is the trigonometric series

$$a_o + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right], \quad \text{for } x \in [-L, L], \tag{5.63}$$

where the coefficients a_n, for $n = 0, 1, 2, 3, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are defined in (5.60), (5.61) and (5.62).

Example 5.1.1. Let $f\colon [0, L] \to \mathbb{R}$ be the function giving the first initial condition in the problem (5.7). Since we are assuming that $f(0) = f(L) = 0$, f can be extended to a continuous, odd, periodic function of period $2L$ be defining

$$f(x) = -f(-x), \quad \text{for } -L \leqslant x < 0,$$

and

$$f(x + 2L) = f(x), \quad \text{for all } x \in \mathbb{R}.$$

In this case, the formula for the Fourier coefficients of f in (5.60), (5.61) and (5.62 yield

$$a_n = 0, \quad \text{for } n = 0, 1, 2, 3, \ldots$$

and

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx, \quad \text{for } n = 1, 2, 3, \ldots, \tag{5.64}$$

since f is odd and therefore the integrand in (5.62) is even.

Thus, in this case, the Fourier series expansion of f yields

$$\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right), \quad \text{for } x \in [0, L],$$

where the coefficients b_n, for $n \in \mathbb{N}$ are given in (5.64). This is the same expansion given in (5.46) with c_n in place of b_n.

For each $N = 1, 2, 3, \ldots$, put

$$S_{_N}(x) = a_o + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad \text{for } x \in [-L, L], \quad (5.65)$$

where a_n, for $n = 0, 1, 2, 3, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of f in (5.60), (5.61) and (5.62.

We would like to know which conditions on the function f will guarantee that

$$\lim_{N\to\infty} S_{_N}(x) = f(x), \quad \text{for } x \in [0, L], \quad (5.66)$$

where $S_{_N}$ is given by (5.65). If (5.66) holds true, we say that the Fourier series expansion for f given in (5.63) converges to f **pointwisely**, and write

$$a_o + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] = f(x), \quad \text{for } x \in [-L, L]. \quad (5.67)$$

We would also like to determine conditions on f under which the convergence in (5.66) is uniform.

Substitute the expressions for a_n, for $n = 0, 1, 2, 3, \ldots$, and b_n, for $n \in \mathbb{N}$, given in (5.60)–(5.62 into the expression for $S_{_N}$ in (5.65) to get

$$\begin{aligned} S_{_N}(x) \quad = \quad & \frac{1}{2L}\int_{-L}^{L} f(y)\, dy \\ & + \sum_{n=1}^{N} \left(\frac{1}{L}\int_{-L}^{L} f(y)\cos\left(\frac{n\pi y}{L}\right) dy \right) \cos\left(\frac{n\pi x}{L}\right) \\ & + \sum_{n=1}^{N} \left(\frac{1}{L}\int_{-L}^{L} f(y)\sin\left(\frac{n\pi y}{L}\right) dy \right) \sin\left(\frac{n\pi x}{L}\right), \end{aligned} \quad (5.68)$$

for $x \in [0, L]$, where we have used y as the variable of integration in the definition of the Fourier coefficients because the variable x is being used as the argument of $S_{_N}$.

Next, interchange summation and integration and use the distributive property to in the last two terms on the right–hand side of (5.68) to get

$$\frac{1}{L}\int_{-L}^{L} f(y) \sum_{n=1}^{N} \left[\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi y}{L}\right) + \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right) \right] dy. \quad (5.69)$$

The term in the summation in (5.69) can be simplifies using the trigonometric identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B, \quad \text{for } A, B \in \mathbb{R}$$

to yield

$$\frac{1}{L}\int_{-L}^{L} f(y)\sum_{n=1}^{N}\cos\left(\frac{n\pi(x-y)}{L}\right)\,dy. \tag{5.70}$$

Consequently, the expression in (5.68) can be rewritten as

$$S_N(x) = \frac{1}{L}\int_{-L}^{L} f(y)\left(\frac{1}{2}+\sum_{n=1}^{N}\cos\left(\frac{n\pi(x-y)}{L}\right)\right)dy, \quad \text{for } x\in[0,L], \tag{5.71}$$

which can be rewritten in terms of the function

$$D_N(\theta) = \frac{1}{2}+\sum_{n=1}^{N}\cos\left(\frac{n\pi\theta}{L}\right), \quad \text{for } \theta\in\mathbb{R}, \tag{5.72}$$

as

$$S_N(x) = \frac{1}{L}\int_{-L}^{L} f(y)D_N(x-y)\,dy, \quad \text{for } x\in[0,L]. \tag{5.73}$$

The function defined in (5.72) is called the **Dirichlet kernel**. Observe that D_N is even and periodic of period $2L$. Furthermore, it can be shown that

$$D_N(\theta) = \frac{\sin\left[\left(N+\frac{1}{2}\right)\frac{\pi\theta}{L}\right]}{2\sin\left(\frac{\pi\theta}{2L}\right)}, \quad \text{for } \theta\neq 0. \tag{5.74}$$

(See Appendix C.1 for a derivation of the result in (5.74)).

Note that, according to (5.72),

$$D_N(0) = N+\frac{1}{2}. \tag{5.75}$$

Using (5.72) we can compute

$$\int_{-L}^{L} D_N(\theta)\,d\theta = L, \quad \text{for all } N\in\mathbb{N}. \tag{5.76}$$

Next, make the change of variables

$$z = x-y$$

in the integral on the right–hand side of (5.73) to get

$$S_N(x) = \frac{1}{L}\int_{x+L}^{x-L} f(x-z)D_N(z)\,(-dz), \quad \text{for } x\in[-L,L],$$

or

$$S_N(x) = \frac{1}{L}\int_{x-L}^{x+L} f(x-z)D_N(z)\,dz, \quad \text{for } x\in[-L,L];$$

so that,

$$S_N(x) = \frac{1}{L}\int_{-L}^{L} f(x-z)D_N(z)\,dz, \quad \text{for } x \in [-L, L], \tag{5.77}$$

where we have used the $2L$–periodicity of the integrand.

We will first prove, using the representation in (5.77), that for the case in which f is assumed to be a C^1 function, (5.66) holds true; that is, the trigonometric expansion of f in (5.65), where the Fourier coefficients a_n, for $n = 0, 1, 2, 3, \ldots$, and b_n, for $n \in \mathbb{N}$, given in (5.60)–(5.62, converges to f pointwisely in $[-L, L]$. We will state this fact as a theorem.

Theorem 5.1.2 (Point–wise Convergence.)**.** Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a bounded, periodic function of period $2L$, and let

$$S_N(x) = a_o + \sum_{n=1}^{N}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right], \tag{5.78}$$

for $x \in [-L, L]$ and $N = 1, 2, 3, \ldots$, where a_n, for $n = 0, 1, 2, 3, \ldots$, and b_n, for $n \in \mathbb{N}$, are given in (5.60)–(5.62). Then,

$$\lim_{N\to\infty} S_N(x) = f(x), \quad \text{for } x \in [-L, L]. \tag{5.79}$$

In the proof of Theorem 5.1.2 we will use the following result known as the Riemann–Lebesgue Lemma. We state here a spacial form of the result in the context of $2L$–periodic functions and the Fourier coefficients.

Theorem 5.1.3. Assume that $F\colon \mathbb{R} \to \mathbb{R}$ is bounded, $2L$–periodic, and integrable over $[-L, L]$. Let a_n, for $n = 0, 1, 2, 3, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, denote the Fourier coefficients of F given in (5.60)–(5.62), with F in place of F. Then,

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0. \tag{5.80}$$

We restate the conclusion of the Riemann–Lebesgue Lemma in the form that will be applied in the proof of Theorem 5.1.2.

$$\lim_{N\to\infty}\int_{-L}^{L} F(y)\cos\left(\frac{N\pi y}{L}\right)dy = 0. \tag{5.81}$$

and

$$\lim_{N\to\infty}\int_{-L}^{L} F(y)\sin\left(\frac{N\pi y}{L}\right)dy = 0. \tag{5.82}$$

A more general version of the Riemann–Lebesgue Lemma is the following.

Theorem 5.1.4. Assume that $F\colon [a, b] \to \mathbb{R}$ is absolutely integrable on $[a, b]$; so that,

$$\int_a^b |F(x)|\,dx < \infty.$$

Then,

$$\lim_{|\lambda|\to\infty} \int_a^b F(x)\cos(\lambda x)\ dx = 0, \tag{5.83}$$

and

$$\lim_{|\lambda|\to\infty} \int_a^b F(x)\sin(\lambda x)\ dx = 0. \tag{5.84}$$

For a proof of this result, see [Tol62]. We presents two examples that the the result in Theorem 5.1.4 plausible.

Example 5.1.5. Let $F(x) = 1$ for all $x \in \mathbb{R}$ and compute

$$\begin{aligned} \int_a^b F(x)\cos(\lambda x)\ dx &= \int_a^b \cos(\lambda x)\ dx \\ &= \left[\frac{1}{\lambda}\sin(\lambda x)\right]_a^b; \end{aligned}$$

so that

$$\int_a^b F(x)\sin(\lambda x)\ dx = \frac{1}{\lambda}[\sin(\lambda b) - \sin(\lambda a)], \quad \text{for } \lambda \neq 0. \tag{5.85}$$

Take absolute value of both sides of (5.85) and apply the triangle inequality to get

$$\left|\int_a^b F(x)\cos(\lambda x)\ dx\right| \leqslant \frac{2}{|\lambda|}, \quad \text{for } \lambda \neq 0,$$

from which (5.83) follows for the case $F(x) = 1$ for all x.

Similar calculations show that (5.84) is also true in this case.

We present a second example as a proposition that will used later in this section when we prove the uniform convergence of Fourier series of C^1 functions.

Proposition 5.1.6. Assume that $F\colon [a,b] \to \mathbb{R}$ is bounded on $[a,b]$ and differentiable on (a,b) and that

$$\int_a^b |F'(x)|\ dx < \infty; \tag{5.86}$$

that is, F' is absolutely integrable over $[a,b]$. Then, (5.83) and (5.84) hold true.

Proof: Use integration by parts to compute

$$\int_a^b F(x)\cos(\lambda x)\ dx = \left[\frac{1}{\lambda}F(x)\sin(\lambda x)\right]_a^b - \frac{1}{\lambda}\int_a^b F'(x)\sin(\lambda x)\ dx,$$

or

$$\begin{aligned}\int_a^b F(x)\cos(\lambda x)\ dx &= \frac{1}{\lambda}\left[F(b)\sin(\lambda b) - F(a)\sin(\lambda a)\right] \\ &\quad -\frac{1}{\lambda}\int_a^b F'(x)\sin(\lambda x)\ dx.\end{aligned} \tag{5.87}$$

Next, take absolute value of both sides of (5.87) and apply the triangle inequality to get

$$\left|\int_a^b F(x)\cos(\lambda x)\ dx\right| \leqslant \frac{2M}{|\lambda|} + \frac{1}{|\lambda|}\int_a^b |F'(x)|\ dx, \qquad \text{for } \lambda \neq 0, \tag{5.88}$$

where M is an upper bound of F on $[a, b]$. In view on (5.86), we can see that (5.83) follows from (5.88).

Similar calculations show that (5.84) is also true in this case. ■

Proof of Theorem 5.1.2: Using (5.76) we see that

$$f(x) = \frac{1}{L}\int_{-L}^{L} f(x)D_{N}(z)\ dz, \qquad \text{for } x \in [-L, L]. \tag{5.89}$$

Then, using the representation for S_N in (5.77) together with (5.89),

$$S_N(x) - f(x) = \frac{1}{L}\int_{-L}^{L}[f(x-z) - f(x)]D_N(z)\ dz, \qquad \text{for } x \in [-L, L], \tag{5.90}$$

Next, we use the formula for the Dirichlet kernel in (5.74) to write (5.90) as

$$S_N(x) - f(x) = \frac{1}{L}\int_{-L}^{L} \frac{f(x-z) - f(x)}{2\sin\left(\dfrac{\pi z}{2L}\right)} \sin\left[\left(N + \frac{1}{2}\right)\frac{\pi z}{L}\right] dz, \tag{5.91}$$

for $x \in [-L, L]$.

Write

$$G(x, z) = \frac{f(x-z) - f(x)}{2\sin\left(\dfrac{\pi z}{2L}\right)}, \qquad \text{for } z \neq 0 \text{ and } x, z \in [-L, L], \tag{5.92}$$

and observe that, using L'Hospital's Rule,

$$\lim_{z\to 0} \frac{f(x-z) - f(x)}{2\sin\left(\dfrac{\pi z}{2L}\right)} = -\frac{L}{\pi} f'(x),$$

where we have used the assumption that f is a C^1 function. Consequently, defining

$$G(x, 0) = -\frac{L}{\pi} f'(x), \qquad \text{for } x \in [-L, L], \tag{5.93}$$

We then see that the function G defined in (5.92) and (5.93), for $x \in [-L, L]$ and $z \in [-L, L]$, is continuous on $[-L, L] \times [-L, L]$. (Observe that denominator in (5.92) is zero only at $z = 0$ in $[-L, L]$.) Thus, the function G is bounded and continuous on $[-L, L] \times [-L, L]$, and therefore the map

$$z \mapsto G(x, z), \qquad \text{for } z \in [-L, L]$$

is absolutely integrable for each $x \in [-L, L]$. (We are aiming here at applying the Riemann–Lebesgue Lemma as stated in (5.81) and (5.82).)

Using $G(x, z)$ as defined in (5.92) and (5.93), we can rewrite (5.91) as

$$S_N(x) - f(x) = \frac{1}{L} \int_{-L}^{L} G(x, z) \sin\left[\left(N + \frac{1}{2}\right) \frac{\pi z}{L}\right] dz,$$

for $x \in [-L, L]$, which we can in turn write as

$$\begin{aligned} S_N(x) - f(x) \quad = \quad & \frac{1}{L} \int_{-L}^{L} G(x, z) \cos\left(\frac{\pi z}{2L}\right) \sin\left(\frac{N\pi z}{L}\right) dz \\ & + \frac{1}{L} \int_{-L}^{L} G(x, z) \sin\left(\frac{\pi z}{2L}\right) \cos\left(\frac{N\pi z}{L}\right) dz, \end{aligned} \tag{5.94}$$

for $x \in [-L, L]$.

Next, observe that, for each $x \in [-L, L]$,

$$\int_{-L}^{L} \left| G(x, z) \cos\left(\frac{\pi z}{2L}\right) \right| dz \leqslant \int_{-L}^{L} |G(x, z)|\, dz < \infty,$$

since $G(x, z)$ is absolutely integrable over $[-L, L]$ for each $x \in [-L, L]$. Thus, we can apply the result of the Riemann–Lebesgue Lemma in (5.82) to deduce that

$$\lim_{N \to \infty} \int_{-L}^{L} G(x, z) \cos\left(\frac{\pi z}{2L}\right) \sin\left(\frac{N\pi z}{L}\right) dz = 0, \qquad \text{for each } x \in [-L, L]. \tag{5.95}$$

Similar calculations can be used to show that

$$\lim_{N \to \infty} \int_{-L}^{L} G(x, z) \sin\left(\frac{\pi z}{2L}\right) \cos\left(\frac{N\pi z}{L}\right) dz = 0, \qquad \text{for each } x \in [-L, L]. \tag{5.96}$$

Combining (5.94), (5.95) and (5.96 we conclude that

$$\lim_{N \to \infty} (S_N(x) - f(x)) = 0, \qquad \text{for all } x \in [-L, L],$$

from which (5.79) follows. We have therefore completed the proof of Theorem 5.1.2. ■

We will next show that then the convergence of S_N to f in (5.79) is actually uniform convergence over $[-L, L]$. This means that, for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$, which depends only on ε, such that

$$N \geqslant N_\varepsilon \Rightarrow |S_N(x) - f(x)| < \varepsilon, \quad \text{for all } x \in [-L, L]. \tag{5.97}$$

We will prove this as a consequence of the fact that $f \in C^1(\mathbb{R}, \mathbb{R})$; so that, f' is bounded on $[-L, L]$; that is, there exists a constant $M > 0$ such that

$$|f'(x)| \leqslant M, \quad \text{for all } x \in [-L, L]. \tag{5.98}$$

Theorem 5.1.7 (Uniform Convergence 1)**.** Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a bounded, periodic function of period $2L$. Then, the sequence of functions (S_N) given in (5.78) converges uniformly to f over $[-L, L]$.

Proof: As in the proof of Theorem 5.1.2, we begin with

$$S_N(x) - f(x) = \frac{1}{L}\int_{-L}^{L} [f(x - z) - f(x)]D_N(z)\, dz, \quad \text{for } x \in [-L, L]. \tag{5.99}$$

Our goal is to show that, given any $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that the absolute value of the integral on the right–hand side of (5.99) is less than εL for all $N \geqslant N_\varepsilon$ and all $x \in [-L, L]$.

Before we proceed any further, let's make the change of variables $y = -z$ in the integral on the right–hand side of (5.99) and use the fact that the Dirichlet kernel D_N is an even function to write

$$S_N(x) - f(x) = \frac{1}{L}\int_{-L}^{L} [f(x + y) - f(x)]D_N(y)\, dy, \quad \text{for } x \in [-L, L]. \tag{5.100}$$

Let $\varepsilon > 0$ be given and let $\delta > 0$ (to be chosen later) be such that $\delta < L$.

Write the expression on the right–hand side of (5.100) as the sum of the three terms

$$I_1(x) = \frac{1}{L}\int_{-L}^{-\delta} [f(x + y) - f(x)]D_N(y)\, dy, \quad \text{for } x \in [-L, L], \tag{5.101}$$

$$I_2(x) = \frac{1}{L}\int_{-\delta}^{\delta} [f(x + y) - f(x)]D_N(y)\, dy, \quad \text{for } x \in [-L, L], \tag{5.102}$$

and

$$I_3(x) = \frac{1}{L}\int_{\delta}^{L} [f(x + y) - f(x)]D_N(y)\, dy, \quad \text{for } x \in [-L, L]; \tag{5.103}$$

so that,

$$S_N(x) - f(x) = I_1(x) + I_2(x) + I_3(x), \quad \text{for } x \in [-L, L]. \tag{5.104}$$

We estimate each of the quantities I_1, I_2 and I_3 in (5.101), (5.102) and (5.103), respectively, separately. We begin with I_2 in (5.102).

Use integration by parts to compute

$$\begin{aligned}\int_{-\delta}^{\delta}[f(x+y)-f(x)]D_N(y)\,dy &= [[f(x+y)-f(x)]V_N(y)]_{-\delta}^{\delta} \\ &\quad -\int_{-\delta}^{\delta} f'(x+y)V_N(y)\,dy,\end{aligned} \tag{5.105}$$

for $x \in [-L, L]$, where

$$V_N(y) = \int_0^y D_N(z)\,dz, \quad \text{for } y \in [-L, L]. \tag{5.106}$$

We derive a few properties of the function V_N defined in (5.106). In particular, we show that V_N is bounded independently of N; that is, there exists a constant $M_1 > 0$ such that

$$|V_N(y)| \leqslant M_1, \quad \text{for all } y \in [-L, L] \text{ and all } N \in \mathbb{N}. \tag{5.107}$$

We first note that, since D_N is an even function, then V_N is odd.

Compute, using (5.74),

$$\begin{aligned}V_N(y) &= \int_0^y \frac{\sin\left[\left(N+\frac{1}{2}\right)\frac{\pi z}{L}\right]}{2\sin\left(\frac{\pi z}{2L}\right)}\,dz \\ &= \frac{L}{\pi}\int_0^{\pi y/L} \frac{\sin\left[\left(N+\frac{1}{2}\right)r\right]}{2\sin(r/2)}\,dr,\end{aligned}$$

where we have made the change of variables: $r = \dfrac{\pi z}{L}$. We therefore have that

$$V_N(y) = \frac{L}{\pi}\int_0^{\pi y/L} \sin\left[\left(N+\frac{1}{2}\right)r\right]\,\frac{1}{2\sin(r/2)}\,dr,$$

which we can write as

$$\begin{aligned}V_N(y) &= \frac{L}{\pi}\int_0^{\pi y/L} \sin\left[\left(N+\frac{1}{2}\right)r\right]\left[\frac{1}{2\sin(r/2)}-\frac{1}{r}\right]dr \\ &\quad + \frac{L}{\pi}\int_0^{\pi y/L} \frac{\sin\left[\left(N+\frac{1}{2}\right)r\right]}{r}\,dr.\end{aligned} \tag{5.108}$$

Define

$$g(r) = \begin{cases} \dfrac{1}{2\sin(r/2)} - \dfrac{1}{r}, & \text{if } r \neq 0 \text{ and } r \in [-\pi, \pi]; \\ \\ 0, & \text{if } r = 0. \end{cases} \tag{5.109}$$

By virtue of L'Hospital's Rule, we can see that the function $g\colon [-\pi, \pi] \to \mathbb{R}$ defined in (5.109) is continuous and hence bounded on $[-\pi, \pi]$. We can use this function to rewrite the expression for $V_N(y)$ in (5.108) as

$$\begin{aligned} V_N(y) &= \frac{L}{\pi} \int_0^{\pi y/L} g(r) \sin\left[\left(N + \frac{1}{2}\right) r\right] dr \\ &\qquad + \frac{L}{\pi} \int_0^{(N+1/2)\pi y/L} \frac{\sin t}{t}\, dt, \end{aligned} \tag{5.110}$$

where we have used the change of variables $t = \left(N + \dfrac{1}{2}\right) r$ in the second integral on the right–hand side of (5.108).

Next, since

$$\lim_{N\to\infty} \int_0^{(N+1/2)\pi y/L} \frac{\sin t}{t}\, dt = \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2},$$

the second integral on the right–hand side of (5.110) is bounded.

We can bound the first integral on the right hand side of (5.110) as follows:

$$\left| \int_0^{\pi y/L} g(r) \sin\left[\left(N + \frac{1}{2}\right) r\right] dr \right| \leqslant \pi \max_{y\in[-\pi,\pi]} |g(r)|, \quad \text{for } y \in [-L, L].$$

Consequently, both integrals on the right–hand side of (5.110) are bounded independently of N over $[-L, L]$, which establishes the estimate in (5.107).

Next, rewrite (5.105) as

$$\begin{aligned} \int_{-\delta}^{\delta} [f(x+y) - f(x)] D_N(y)\, dy &= [f(x+\delta) - f(x)] V_N(\delta) \\ &\quad -[f(x-\delta) - f(x)] V_N(-\delta) \\ &\quad - \int_{-\delta}^{\delta} f'(x+y) V_N(y)\, dy, \end{aligned} \tag{5.111}$$

for $x \in [-L, L]$.

Using the fact that V_N is odd (since the Dirichlet kernel is an even function), we can rewrite the right–hand side of (5.111) as

$$[(f(x+\delta) - f(x)) + (f(x-\delta) - f(x))] V_N(\delta) - \int_{-\delta}^{\delta} f'(x+y) V_N(y)\, dy. \tag{5.112}$$

The first term in (5.112) can be estimated using the Mean Value Theorem and the estimates in (5.107) and (5.98) as follows:

$$|[(f(x+\delta)-f(x))+(f(x-\delta)-f(x))]V_{_N}(\delta)| \leqslant 2MM_1\delta, \tag{5.113}$$

for all $x \in [-L, L]$.

We can also estimate the second term in (5.112) as follows;

$$\left| -\int_{-\delta}^{\delta} f'(x+y)V_{_N}(y)\ dy \right| \leqslant 2MM_1\delta, \quad \text{ for all } x \in [-L, L]. \tag{5.114}$$

Thus, we can estimate the integral on the left–hand side of (5.111) as

$$\left| \int_{-\delta}^{\delta} [f(x+y)-f(x)]D_{_N}(y)\ dy \right| \leqslant 4MM_1\delta, \quad \text{ for all } x \in [-L, L], \tag{5.115}$$

where we have used (5.111), (5.112), (5.113), (5.114) and the triangle inequality. Consequently, the integral I_2 in (5.102) can be estimated using (5.115) as

$$|I_2(x)| \leqslant \frac{4MM_1\delta}{L}, \quad \text{ for all } x \in [-L, L]. \tag{5.116}$$

We choose $\delta > 0$ such that

$$\frac{4MM_1\delta}{L} < \frac{\varepsilon}{3},$$

or

$$\delta < \frac{\varepsilon L}{12MM_1}.$$

We then obtain from (5.116) that

$$|I_2(x)| < \frac{\varepsilon}{3}, \quad \text{ for all } x \in [-L, L]. \tag{5.117}$$

Next, we estimate I_3 in (5.103).

Use the formula for the Dirichlet kernel in (5.74) to write (5.103) as

$$I_3(x) = \frac{1}{L}\int_{\delta}^{L} \frac{f(x+y)-f(x)}{2\sin(\pi y/2L)} \sin\left[\left(N+\frac{1}{2}\right)\frac{\pi y}{L}\right]\ dy, \tag{5.118}$$

for $x \in [-L, L]$.

Define $F\colon [-L, L] \times [\delta, L] \to \mathbb{R}$ by

$$F(x,y) = \frac{f(x+y)-f(x)}{2\sin(\pi y/2L)}, \quad \text{ for } \delta \leqslant y \leqslant L \text{ and } -L \leqslant x \leqslant L. \tag{5.119}$$

Observe that the function F defined in (5.119) is a C^1 function over the rectangle $[-L, L] \times [\delta, L]$. Hence, there exist positive constants M_2 and M_3 such that

$$|F(x,y)| \leqslant M_2, \quad \text{ for all } \delta \leqslant y \leqslant L \text{ and } -L \leqslant x \leqslant L, \tag{5.120}$$

and

$$\left|\frac{\partial}{\partial y}[F(x,y)]\right| \leqslant M_3, \quad \text{for all } \delta \leqslant y \leqslant L \text{ and } -L \leqslant x \leqslant L. \tag{5.121}$$

With the definition of F given in (5.119), we can rewrite (5.118) as

$$I_3(x) = \frac{1}{L}\int_\delta^L F(x,y)\sin\left[\left(N+\frac{1}{2}\right)\frac{\pi y}{L}\right]\,dy, \quad \text{for } x \in [-L,L]. \tag{5.122}$$

Next, let

$$\left(N+\frac{1}{2}\right)\frac{\pi}{L} = \lambda \tag{5.123}$$

in (5.122) to get

$$I_3(x) = \frac{1}{L}\int_\delta^L F(x,y)\sin\lambda y\;dy, \quad \text{for } x \in [-L,L]. \tag{5.124}$$

and observe from (5.123 that $\lambda \to \infty$ as $N \to \infty$.

As in (5.88 in the proof of Proposition 5.1.6, we can use integration by parts to obtain the estimate

$$\left|\int_\delta^L F(x,y)\sin(\lambda x)\;dx\right| \leqslant \frac{2M_2}{|\lambda|} + \frac{M_3 L}{|\lambda|}, \quad \text{for all } x \in [-L,L]. \tag{5.125}$$

and for $\lambda \neq 0$, where we have used the estimates in (5.120) and (5.121).

Combining (5.124), (5.123) and the estimate in (5.125) we obtain

$$|I_3(x)| \leqslant \frac{2M_2 + M_3 L}{\pi(N+1/2)}, \quad \text{for all } x \in [-L,L]. \tag{5.126}$$

It follows from (5.126) that, if

$$N > \frac{6M_2 + 3M_3 L}{\pi\varepsilon},$$

then

$$|I_3(x)| < \frac{\varepsilon}{3}, \quad \text{for all } x \in [-L,L].$$

Thus, there exists $N_1 \in \mathbb{N}$ such that

$$N \geqslant N_1 \Rightarrow |I_3(x)| < \frac{\varepsilon}{3}, \quad \text{for all } x \in [-L,L]. \tag{5.127}$$

Similar calculations show that there exists $N_2 \in \mathbb{N}$ such that

$$N \geqslant N_2 \Rightarrow |I_1(x)| < \frac{\varepsilon}{3}, \quad \text{for all } x \in [-L,L]. \tag{5.128}$$

Letting $N_\varepsilon = \max\{N_1, N_2\}$ and combining (5.104), (5.117), (5.127) and (5.128), we obtain that

$$N \geqslant N_\varepsilon \Rightarrow |S_N(x) - f(x)| < \varepsilon, \quad \text{for all } x \in [-L,L],$$

which proves that (S_N) converges uniformly to f as $N \to \infty$. ■

5.1.3 Differentiability of Fourier Series Expansions

The uniform convergence of the Fourier series for $f \in C^1(\mathbb{R}, \mathbb{R})$ proved in the previous section justifies the term–by–term integration that was used in the derivation of the Fourier coefficients of f in (5.60), (5.61) and (5.62). That fact alone, however, is not enough to prove that the Fourier series expansion

$$u_{_N}(x,t) = \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right), \qquad \text{for } x \in [0,L],\ t \geqslant 0, \tag{5.129}$$

$N = 1, 2, 3, \ldots$, where b_n, for $n \in \mathbb{N}$, are given in (5.62), converges uniformly to a twice–differentiable function

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right), \qquad \text{for } x \in [0,L],\ t \geqslant 0. \tag{5.130}$$

The function $u\colon [0,L]\times[0,\infty) \to \mathbb{R}$ defined in (5.130) is a candidate for a solution of the vibrating string problem in (5.7), provided we can prove that it is well–defined and its partial derivatives u_t, u_x, u_{tt} and u_{xx} are well defined continuous functions. This will require term–by–term differentiation of the series in (5.130), which in turn requires proving that the partial derivatives of the trigonometric polynomials in (5.129) converge uniformly in $[-L,L]$ as $N \to \infty$. To achieve this, we will need to make further assumptions on $f\colon [0,L] \to \mathbb{R}$. Indeed, we will have to assume that f can be extended to a C^2 function over $\mathbb{R}$ that is odd and $2L$–periodic. We will also have to obtain more information on the rate of decay of $|a_n|$ and $|b_b|$, where a_n and b_n are the Fourier coefficients of f, as $n \to \infty$.

By virtue of the Riemann–Lebesgue Lemma (Theorem 5.1.4 on page 91 of these notes), we have that

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0. \tag{5.131}$$

In this section we will see an alternate proof of the limit facts in (5.131 based in the fact that

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty, \tag{5.132}$$

for the Fourier coefficients of a $2L$–periodic function $f \in C^1([-L,L])$. Observe that the convergence of the series in (5.132) implies the limit facts in (5.131).

We will prove (5.132) as a consequence of the following fact.

Proposition 5.1.8. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a $2L$–periodic function that is square–integrable over $[-L,L]$; that is,

$$\int_{-L}^{L} |f(x)|^2\, dx < \infty. \tag{5.133}$$

Let a_n and b_n, for $n \in \mathbb{N}$, denote the Fourier coefficients of f as given in (5.61) and (5.62), respectively. Then,

$$\sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right) < \infty. \tag{5.134}$$

To establish Proposition 5.1.8 we will first need to derive some estimates about sums of squares of Fourier coefficients of a $2L$–periodic function $f\colon \mathbb{R} \to \mathbb{R}$ that is also square integrable on $[-L, L]$.

For $N = 1, 2, 3, \ldots$, let

$$S_{N}(x) = a_o + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right], \tag{5.135}$$

for $x \in [-L, L]$, where a_n, for $n = 0, 1, 2, 3, \ldots$, and b_n, for $n \in \mathbb{N}$, are given in (5.60)–(5.62).

For $f\colon \mathbb{R} \to \mathbb{R}$, a $2L$–periodic function satisfying (5.133), and S_{N} given in (5.135), compute

$$\int_{-L}^{L} (S_{N}(x) - f(x))^2 dx = \int_{-L}^{L} (S_{N}(x))^2 dx - 2\int_{-L}^{L} f(x) S_{N}(x) dx + \int_{-L}^{L} (f(x))^2 dx. \tag{5.136}$$

We use the integration facts

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad \text{for all } n \in \mathbb{N}, \tag{5.137}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad \text{for all } n \in \mathbb{N}, \tag{5.138}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L, & \text{if } n = m; \\ 0, & \text{if } n \neq m, \end{cases} \tag{5.139}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L, & \text{if } n = m; \\ 0, & \text{if } n \neq m, \end{cases} \tag{5.140}$$

and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0, \quad \text{for all } m, n \in \mathbb{N}, \tag{5.141}$$

to compute the first two integrals on the right–hand side of (5.136).

We begin with

$$
\begin{aligned}
(S_N(x))^2 &= \left(a_o + \sum_{n=1}^{N}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]\right) \\
&\quad \times \left(a_o + \sum_{m=1}^{N}\left[a_m\cos\left(\frac{m\pi x}{L}\right) + b_m\sin\left(\frac{m\pi x}{L}\right)\right]\right),
\end{aligned}
$$

which we can rewrite as

$$
\begin{aligned}
(S_N(x))^2 &= a_o^2 + \sum_{m=1}^{N} a_o a_m \cos\left(\frac{m\pi x}{L}\right) \\
&\quad + \sum_{m=1}^{N} a_o b_m \sin\left(\frac{m\pi x}{L}\right) + \sum_{m=1}^{N} a_o a_n \cos\left(\frac{n\pi x}{L}\right) \\
&\quad + \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) \\
&\quad + \sum_{n=1}^{N}\sum_{m=1}^{N} a_n b_m \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \qquad (5.142) \\
&\quad + \sum_{m=1}^{N} a_o b_n \sin\left(\frac{n\pi x}{L}\right) \\
&\quad + \sum_{n=1}^{N}\sum_{m=1}^{N} a_m b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) \\
&\quad + \sum_{n=1}^{N}\sum_{m=1}^{N} b_n b_m \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right).
\end{aligned}
$$

Next, integrate on both sides of (5.142) from $-L$ to L and use the integration facts in (5.137)–(5.141) to obtain that

$$\int_{-L}^{L} (S_N(x))^2\, dx = 2La_o^2 + \sum_{n=1}^{N} La_n^2 + \sum_{n=1}^{N} Lb_n^2,$$

or

$$\int_{-L}^{L} (S_N(x))^2\, dx = 2La_o^2 + L\sum_{n=1}^{N}\left(a_n^2 + b_n^2\right), \quad \text{for } N = 1, 2, 3, \ldots \qquad (5.143)$$

Next, we compute

$$\int_{-L}^{L} f(x)S_{N}(x)\ dx = \int_{-L}^{L} f(x)\left(a_o + \sum_{n=1}^{N}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]\right) dx,$$

which we can write as

$$\begin{aligned}\int_{-L}^{L} f(x)S_{N}(x)\ dx &= a_o\int_{-L}^{L} f(x)\ dx \\ &\quad + \int_{-L}^{L} f(x)\left[\sum_{n=1}^{N} a_n\cos\left(\frac{n\pi x}{L}\right)\right] dx \\ &\quad + \int_{-L}^{L} f(x)\left[\sum_{n=1}^{N} b_n\sin\left(\frac{n\pi x}{L}\right)\right] dx,\end{aligned}$$

or

$$\begin{aligned}\int_{-L}^{L} f(x)S_{N}(x)\ dx &= a_o\int_{-L}^{L} f(x)\ dx \\ &\quad + \sum_{n=1}^{N} a_n\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad (5.144) \\ &\quad + \sum_{n=1}^{N} b_n\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx.\end{aligned}$$

Next, use the definitions of the Fourier coefficients of f in (5.60)–(5.62) to rewrite (5.144) as

$$\int_{-L}^{L} f(x)S_{N}(x)\ dx = 2La_o^2 + \sum_{n=1}^{N} La_n^2 + \sum_{n=1}^{N} Lb_n^2,$$

or

$$\int_{-L}^{L} f(x)S_{N}(x)\ dx = 2La_o^2 + L\sum_{n=1}^{N}\left(a_n^2 + b_n^2\right). \qquad (5.145)$$

to rewrite (5.136) as

$$\int_{-L}^{L}(S_{N}(x) - f(x))^2 dx = \int_{-L}^{L}(f(x))^2 dx - \left(2La_o^2 + L\sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right), \quad (5.146)$$

for $N = 1, 2, 3, \ldots$

Since the left–hand side of the equation in (5.146) is nonnegative, we obtain from (5.146) that

$$2a_o^2 + \sum_{n=1}^{N} \left(a_n^2 + b_n^2\right) \leqslant \frac{1}{L}\int_{-L}^{L} (f(x))^2 dx, \quad \text{for } N = 1, 2, 3, \ldots \tag{5.147}$$

Note that, if f is square–integrable over $[-L, L]$, the left–hand side of (5.147) is monotone, increasing sequence that is bounded above. Hence, it has a limit as $N \to \infty$. We therefore have that

$$2a_o^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \leqslant \frac{1}{L}\int_{-L}^{L} (f(x))^2 dx. \tag{5.148}$$

The inequality in (5.148) is an instance of **Bessel's Inequality**.

As a consequence of (5.148), we get that

$$\sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) < \infty,$$

for the case in which f is square integrable over $[-L, L]$. We have therefore proved Proposition 5.1.8.

Next, assume that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, $2L$–periodic and with square–integrable derivative over $[-L, L]$. Thus, letting a_n' and b_n', for $n \in \mathbb{N}$, denote the Fourier coefficients of f', we obtain from Proposition 5.1.8 applied to f' that

$$\sum_{n=1}^{\infty} \left((a_n')^2 + (b_n')^2\right) < \infty. \tag{5.149}$$

Next, use integration by parts to derive the identities

$$a_n' = \frac{n\pi}{L} b_n, \quad \text{for } n = 1, 2, 3, \ldots$$

and

$$b_n' = -\frac{n\pi}{L} a_n, \quad \text{for } n = 1, 2, 3, \ldots,$$

from which we get

$$a_n = -\frac{L}{\pi} \cdot \frac{1}{n} b_n', \quad \text{for } n = 1, 2, 3, \ldots, \tag{5.150}$$

and

$$b_n = \frac{L}{\pi} \cdot \frac{1}{n} a_n', \quad \text{for } n = 1, 2, 3, \ldots \tag{5.151}$$

Taking absolute values on both sides of (5.150) and (5.151), we obtain that

$$|a_n| = \frac{L}{\pi} \cdot \frac{1}{n} |b_n'|, \quad \text{for } n = 1, 2, 3, \ldots, \tag{5.152}$$

and

$$|b_n| = \frac{L}{\pi} \cdot \frac{1}{n} |a'_n|, \quad \text{for } n = 1, 2, 3, \ldots \tag{5.153}$$

Then, using the inequality

$$|x||y| \leqslant \frac{1}{2}x^2 + \frac{1}{2}y^2, \quad \text{for all } x, y \in \mathbb{R},$$

we obtain from (5.152) and (5.153) that

$$|a_n| \leqslant \frac{L}{2\pi} \left(\frac{1}{n^2} + (b'_n)^2 \right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{5.154}$$

and

$$|b_n| \leqslant \frac{L}{2\pi} \left(\frac{1}{n^2} + (a'_n)^2 \right), \quad \text{for } n = 1, 2, 3, \ldots \tag{5.155}$$

Adding the estimates in (5.154) and (5.155) we obtain

$$|a_n| + |b_n| \leqslant \frac{L}{2\pi} \left(\frac{2}{n^2} + (a'_n)^2 + (b'_n)^2 \right), \quad \text{for } n = 1, 2, 3, \ldots$$

Consequently,

$$\sum_{n=1}^{N} (|a_n| + |b_n|) \leqslant \frac{L}{2\pi} \left[2 \sum_{n=1}^{N} \frac{1}{n^2} + \sum_{n=1}^{N} \left((a'_n)^2 + (b'_n)^2 \right) \right], \tag{5.156}$$

for $N = 1, 2, 3, \ldots$

Now, since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows from Proposition 5.1.8 that, in the case that f' is square–integrable over $[-L, L]$, the sequences on the right–hand side of (5.156) converge and

$$\sum_{n=1}^{N} (|a_n| + |b_n|) \leqslant \frac{L}{2\pi} \left[2 \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \left((a'_n)^2 + (b'_n)^2 \right) \right], \tag{5.157}$$

for $N = 1, 2, 3, \ldots$; consequently, if f' is square–integrable over $[-L, L]$, the left–hand side of (5.157) is monotone, increasing sequence that is bounded above. Hence, it has a limit as $N \to \infty$. We therefore have that

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty,$$

which is the claim mane in (5.132). We summarize this result in the following proposition.

Proposition 5.1.9. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable, $2L$–periodic function whose derivative, f', is square–integrable over $[-L, L]$; that is,

$$\int_{-L}^{L} |f'(x)|^2 \; dx < \infty. \tag{5.158}$$

Let a_n and b_n, for $n \in \mathbb{N}$, denote the Fourier coefficients of f as given in (5.61) and (5.62), respectively. Then,

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty. \tag{5.159}$$

Remark 5.1.10. Note that in Proposition 5.1.9 we do not require that $f \in C^1(\mathbb{R}, \mathbb{R})$. The square–integrability condition in (5.158) can still hold true even if f' is not continuous; for instance, f' could have a finite number of jump discontinuities and the condition in (5.158) could still be true.

Next, we will show that the summability condition in (5.159) will imply that the sequence $(S_{_N})$ converges uniformly to f over $[-L, L]$ as $N \to \infty$. This result will be attained as a consequence of the Weierstrass Majorant Theorem or **Weierstrass M–Test**.

Theorem 5.1.11 (Weierstrass M–Test)**.** Let (g_n) denote a sequence of functions defined on $[a, b]$. Suppose that there exist positive numbers M_n, for $n = 1, 2, 3, \ldots$, for which

$$|g_n(x)| \leqslant M_n, \quad \text{ for all } x \in [a, b] \text{ and all } n \in \mathbb{N},$$

and

$$\sum_{n=1}^{\infty} M_n < \infty.$$

Then, the series

$$\sum_{n=1}^{\infty} g_n(x), \quad \text{ for } x \in [a, b],$$

converges absolutely and uniformly on $[a, b]$.

(See, for example, [Rud53, Theorem 7.10, pg. 119] for a proof of the Weierstrass M–Test).

We will also need the following lemma.

Lemma 5.1.12. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a continuous function that is also 2π–periodic. Suppose that all the Fourier coefficients of h are 0. Then, $h(x) = 0$ for all $x \in \mathbb{R}$.

Proof of Lemma 5.1.12: We are assuming that $h\colon \mathbb{R} \to \mathbb{R}$ continuous and 2π–periodic and that

$$\int_{-\pi}^{\pi} h(x)\ dx = 0, \tag{5.160}$$

$$\int_{-\pi}^{\pi} h(x)\cos(nx)\ dx = 0, \quad \text{for all } n = 1, 2, 3, \ldots, \tag{5.161}$$

and

$$\int_{-\pi}^{\pi} h(x)\sin(nx)\ dx = 0, \quad \text{for all } n = 1, 2, 3, \ldots. \tag{5.162}$$

We show that

$$h(x) = 0, \quad \text{for all } x \in [-\pi, \pi]. \tag{5.163}$$

Assume by way of contradiction that (5.163) is not true. Then, there exists $x_o \in (-\pi, \pi)$ such that $f(x_o) \neq 0$. Without loss of generality, we may assume that $f(x_o) > 0$. Then, since h is continuous at x_o, there exists $\delta > 0$ such that

$$h(x) > \frac{h(x_o)}{2}, \quad \text{for all } x \in (x_o - \delta, x_o + \delta), \tag{5.164}$$

where δ can be chosen small enough so that

$$[x_o - \delta, x_o + \delta] \subset (-\pi, \pi).$$

Note that the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = 1 + \cos(x_o - x) - \cos(\delta), \quad \text{for } x \in \mathbb{R}, \tag{5.165}$$

has the following properties:

$$g(x) > 1, \quad \text{for } x \in (x_o - \delta, x_o + \delta), \tag{5.166}$$

and

$$g(x) \leqslant 1, \quad \text{for } -\pi \leqslant x \leqslant x_o - \delta \ \text{ or } \ x_o + \delta \leqslant x \leqslant \pi. \tag{5.167}$$

Furthermore, $g(x)$ is a linear combination of the set $\{1, \cos x, \sin x\}$. To see this, use the trigonometric identity

$$\cos(A - B) = \cos A \cos A + \sin A \sin B, \quad \text{for } A, B \in \mathbb{R},$$

to write

$$g(x) = 1 - \cos(\delta) + \cos(\delta)\cos x + \sin(\delta)\sin x, \quad \text{for } x \in \mathbb{R};$$

so that, g is of the form

$$g(x) = a + b\cos x + c\sin x, \quad \text{for } x \in \mathbb{R}, \tag{5.168}$$

and real numbers a, b and c. Furthermore, using the trigonometric identities

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x), \quad \text{for all } x \in \mathbb{R},$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x), \qquad \text{for all } x \in \mathbb{R},$$

and

$$2\sin x \cos x = \sin(2x), \qquad \text{for all } x \in \mathbb{R},$$

we can see from (5.166) that $(g(x))^2$ is a linear combination of elements from the set

$$\{1, \cos x, \sin x, \cos(2x), \sin(2x)\}.$$

Similarly, noting that $(g(x))^2 = g(x)(g(x))^2$ and using the trigonometric identities

$$\cos x \cos(2x) = \frac{1}{2}\cos x + \frac{1}{2}\cos(3x), \qquad \text{for all } x \in \mathbb{R},$$

$$\sin x \sin(2x) = \frac{1}{2}\cos x 1 \frac{1}{2}\cos(3x), \qquad \text{for all } x \in \mathbb{R},$$

$$\cos x \sin(2x) = \frac{1}{2}\sin x + \frac{1}{2}\sin(3x), \qquad \text{for all } x \in \mathbb{R},$$

and

$$\sin x \cos(2x) = -\frac{1}{2}\sin x + \frac{1}{2}\sin(3x), \qquad \text{for all } x \in \mathbb{R},$$

we can see that $(g(x))^3$ is a linear combination of of elements from the set

$$\{1, \cos x, \sin x, \cos(2x), \sin(2x), \cos(3x), \sin(3x)\}.$$

Proceeding by induction, and using the appropriate trigonometric identities, we can show that $(g(x))^n$ is span of the set

$$\{1, \cos x, \sin x, \cos(2x), \sin(2x), \cos(3x), \sin(3x), \cos(4x), \sin(4x), \ldots\},$$

for all $n \in \mathbb{N}$. (Recall that the span of a set is the collection of all finite a linear combination of of elements from the set).

It then follows from the assumptions in (5.160), (5.161) and (5.162) that

$$\int_{-\pi}^{\pi} h(x)(g(x))^n \, dx = 0, \qquad \text{for all } n \in \mathbb{N}. \tag{5.169}$$

On the other hand, we can estimate the integral in (5.169) by writing

$$\begin{aligned} \int_{-\pi}^{\pi} h(x)(g(x))^n \, dx &= \int_{-\pi}^{x_o-\delta} h(x)(g(x))^n \, dx \\ &\quad + \int_{x_o-\delta}^{x_o+\delta} h(x)(g(x))^n \, dx \\ &\quad + \int_{x_o+\delta}^{\pi} h(x)(g(x))^n \, dx, \end{aligned} \tag{5.170}$$

for all $n \in \mathbb{N}$, and estimating the first and last integral in the right–hand side of (5.170) as follows.

Use (5.167) to estimate

$$\left|\int_{x_o+\delta}^{\pi} h(x)(g(x))^n\, dx\right| \leqslant \int_{x_o+\delta}^{\pi} |h(x)|\, dx, \quad \text{for all } n \in \mathbb{N};$$

so that

$$\left|\int_{x_o+\delta}^{\pi} h(x)(g(x))^n\, dx\right| \leqslant 2M\pi, \quad \text{for all } n \in \mathbb{N}, \tag{5.171}$$

where

$$M = \max_{x\in[-\pi,\pi]} |h(x)|\, dx. \tag{5.172}$$

Similarly,

$$\left|\int_{-\pi}^{x_o-\delta} h(x)(g(x))^n\, dx\right| \leqslant 2M\pi, \quad \text{for all } n \in \mathbb{N}, \tag{5.173}$$

where M is given in (5.172).

Combining (5.170), (5.171) and (5.173), we then get that

$$\int_{-\pi}^{\pi} h(x)(g(x))^n\, dx \geqslant \int_{x_o-\delta}^{x_o+\delta} h(x)(g(x))^n\, dx - 4M\pi, \quad \text{for all } n \in \mathbb{N}, \tag{5.174}$$

where M is given in (5.172).

Next, use the estimates in (5.164) and (5.166) to obtain

$$\int_{x_o-\delta}^{x_o+\delta} h(x)(g(x))^n\, dx > \frac{h(x_o)}{2}\int_{x_o-\delta}^{x_o+\delta} (g(x))^n\, dx;$$

so that, using (5.166) again,

$$\int_{x_o-\delta}^{x_o+\delta} h(x)(g(x))^n\, dx > \frac{h(x_o)}{2}\int_{x_o-\delta/2}^{x_o+\delta/2} (g(x))^n\, dx, \quad \text{for all } n \in \mathbb{N}. \tag{5.175}$$

Next, let

$$r = \min_{x_o-\delta/2\leqslant x\leqslant x_o+\delta/2} g(x). \tag{5.176}$$

Then, by virtue of (5.166),

$$r > 1. \tag{5.177}$$

Now, it follows from (5.176) and (5.175) that

$$\int_{x_o-\delta}^{x_o+\delta} h(x)(g(x))^n\, dx > \frac{h(x_o)}{2}\delta r^n, \quad \text{for all } n \in \mathbb{N}. \tag{5.178}$$

Thus, combining (5.174) and (5.178),

$$\int_{-\pi}^{\pi} h(x)(g(x))^n\, dx > \frac{h(x_o)\delta}{2} r^n - 4M\pi, \quad \text{for all } n \in \mathbb{N}. \tag{5.179}$$

Now, since $r^n \to \infty$ as $n \to \infty$, by virtue of (5.177), it follows from (5.179) that it is possible to find an $m \in \mathbb{N}$ sufficiently large so that

$$\int_{-\pi}^{\pi} h(x)(g(x))^m \, dx > 0. \tag{5.180}$$

We can now see that (5.180) is in direct contradiction with (5.169). Consequently, we conclude that $h(x) = 0$ for all $x \in [-\pi, \pi]$, if it is continuous and all its Fourier coefficients are 0. ■

Remark 5.1.13. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $2L$–periodic. Suppose that all the Fourier coefficients of f given in (5.60), (5.61) and (5.62) are zero; so that,

$$\int_{-L}^{L} f(y) \, dy = 0, \tag{5.181}$$

$$\int_{-L}^{L} f(y) \cos\left(\frac{n\pi y}{L}\right) dy = 0, \quad \text{for all } n \in \mathbb{N}, \tag{5.182}$$

and

$$\int_{-L}^{L} f(y) \sin\left(\frac{n\pi y}{L}\right) dx = 0, \quad \text{for all } n \in \mathbb{N}. \tag{5.183}$$

Define

$$h(x) = f\left(\frac{L}{\pi}x\right), \quad \text{for all } x \in \mathbb{R}. \tag{5.184}$$

Then, $h : \mathbb{R} \to \mathbb{R}$ is continuous and 2π–periodic. Furthermore, making the change of variables

$$y = \frac{L}{\pi}x \tag{5.185}$$

in the integral in (5.181),

$$\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}x\right) \frac{L}{\pi} \, dx = 0,$$

from which we get, using (5.184),

$$\int_{-\pi}^{\pi} h(x) \, dx = 0.$$

Similarly, making the change of variables give in (5.185) in the integrals in (5.182) and (5.183), we get that

$$\int_{-\pi}^{\pi} h(x) \cos(nx) \, dx = 0, \quad \text{for all } n \in \mathbb{N},$$

and

$$\int_{-\pi}^{\pi} h(x) \sin(nx) \, dx = 0, \quad \text{for all } n \in \mathbb{N},$$

respectively.

We have therefore shown that the hypotheses in Lemma 5.1.12 are satisfied for h given in (5.184), from which we obtain that $f(y) = 0$ for all $y \in \mathbb{R}$.

Theorem 5.1.14 (Uniform Convergence 2)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable, $2L$–periodic function whose derivative, f', is square–integrable over $[-L, L]$. Then, the sequence of functions $(S_{_N})$ given in (5.135) converges absolutely and uniformly to f over $[-L, L]$.

Proof: Write, using (5.135),

$$S_{_N}(x) = a_o + \sum_{n=1}^{N} g_n(x), \quad \text{for } x \in [-L, L], \tag{5.186}$$

where

$$g_n(x) = a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{for } x \in [-L, L] \text{ and } n \in \mathbb{N}. \tag{5.187}$$

Then, by the triangle inequality, we obtain from (5.187) that

$$|g_n(x)| \leqslant |a_n| + |b_n|, \quad \text{for } x \in [-L, L] \text{ and } n \in \mathbb{N}.$$

Thus, by virtue of (5.159) in Proposition 5.1.9, we can apply the Weierstrass M-Test to conclude that the series

$$\sum_{n=1}^{\infty} g_n(x), \quad \text{for } x \in [-L, L],$$

converges absolutely and uniformly to a continuous function, which we shall denote by g; so that,

$$g(x) = \sum_{n=1}^{\infty} g_n(x), \quad \text{for } x \in [-L, L]. \tag{5.188}$$

It then follows from (5.186) and (5.188) that

$$\lim_{N\to\infty} S_{_N}(x) = a_o + g(x), \quad \text{for all } x \in [-L, L] \text{ uniformly}, \tag{5.189}$$

where, according to (5.188) and (5.187),

$$g(x) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right], \quad \text{for } x \in [-L, L], \tag{5.190}$$

where the series on the right–hand side of (5.190) converges absolutely and uniformly on $[-L, L]$. Consequently, the Fourier coefficients of g are a_n and b_n, for $n \in \mathbb{N}$, which are the same as the Fourier coefficients of f for $n \in \mathbb{N}$. Observe also that

$$\int_{-L}^{L} g(x)\ dx = 0. \tag{5.191}$$

Consider the function

$$H(x) = f(x) - a_o - g(x), \quad \text{for } x \in \mathbb{R}. \tag{5.192}$$

It follows from (5.191) and (5.192) that

$$\int_{-L}^{L} H(x)\ dx = \int_{-L}^{L} f(x)\ dx - a_o 2L = 0, \tag{5.193}$$

where we have used the definition of a_o in (5.60).

Similarly, multiplying on both sides of (5.192) by $\cos\left(\frac{n\pi x}{L}\right)$, for $n \in \mathbb{N}$, and integrating from $-L$ to L,

$$\int_{-L}^{L} H(x)\cos\left(\frac{n\pi x}{L}\right)\ dx = \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx - \int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)\ dx,$$

where we have used the fact that $\cos\left(\frac{n\pi x}{L}\right)$ has mean–value 0, for $n \in \mathbb{N}$; consequently, since f and g have the same Fourier coefficients for $n \in \mathbb{N}$,

$$\int_{-L}^{L} H(x)\cos\left(\frac{n\pi x}{L}\right)\ dx = 0, \quad \text{for } n \in \mathbb{N}. \tag{5.194}$$

In the same way we get that

$$\int_{-L}^{L} H(x)\sin\left(\frac{n\pi x}{L}\right)\ dx = 0, \quad \text{for } n \in \mathbb{N}. \tag{5.195}$$

In view of (5.193), (5.194) and (5.195), we see that the function $H\colon \mathbb{R} \to \mathbb{R}$ is a continuous, $2L$–periodic function satisfying the conditions of Lemma 5.1.12 (see also Remark 5.1.13 following the proof of Lemma 5.1.12). Consequently,

$$f(x) - a_o - g(x) = 0, \quad \text{for } x \in \mathbb{R},$$

from which we get that

$$f(x) = a_o + g(x), \quad \text{for } x \in \mathbb{R}. \tag{5.196}$$

Combining (5.189) with (5.196) we get that

$$\lim_{N\to\infty} S_{N}(x) = f(x), \quad \text{for all } x \in [-L, L] \text{ uniformly},$$

which was to be shown. ■

Remark 5.1.15. For the case in which $f \in C^1(\mathbb{R}, \mathbb{R})$ is $2L$–periodic, we have that f' is bounded on $[-L, L]$. Consequently, f' is square–integrable on $[-L, L]$. Thus, Theorem 5.1.14 applies in this case, and we therefore recover Theorem 5.1.7.

Example 5.1.16 (Constructing a Solution of the Vibrating String Problem)**.** Assume that $f \colon [0, L] \to \mathbb{R}$ is a differentiable function satisfying $f(0) = f(L) = 0$. Extend f to an odd, $2L$–periodic function, and suppose the extension is differentiable with square–integrable derivative, f', on the interval $[-L, L]$. It then follows from Proposition 5.1.9 that

$$\sum_{n=1}^{\infty} |b_n| < \infty, \tag{5.197}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \qquad \text{for } n \in \mathbb{N}. \tag{5.198}$$

We can then use the Weierstrass M-Test (Theorem 5.1.11) to deduce that the series

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \tag{5.199}$$

where b_n, for $n \in \mathbb{N}$ are given in (5.198), converges absolutely and uniformly for $x \in [0, L]$ and $t \geqslant 0$. Indeed, setting

$$g_n(x, t) = b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \qquad \text{for } x \in [0, L], t \geqslant 0 \tag{5.200}$$

and $n \in \mathbb{N}$, we see that

$$|g_n(x, t)| \leqslant |b_n|, \qquad \text{for } x \in [0, L], t \geqslant 0 \text{ and } n \in \mathbb{N}.$$

Consequently, in view of (5.197), the Weierstrass M–Test applies. We therefore deduce that

$$\sum_{n=1}^{\infty} g_n(x, t) \tag{5.201}$$

converges absolutely and uniformly for $x \in [0, L]$ and $t \geqslant 0$. Thus, in view of (5.200) and (5.201), we see that the series in (5.199) converges absolutely and uniformly for $x \in [0, L]$ and $t \geqslant 0$.

Define

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \qquad \text{for } x \in [0, L],\ t \geqslant 0. \tag{5.202}$$

We have therefore shown that the function $u \colon [0, L] \times [0, \infty) \to \mathbb{R}$ given in (5.202) is well–defined. In particular, we get from (5.202) that

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \qquad \text{for } x \in [0, L]. \tag{5.203}$$

It follows from (5.198) and Theorem 5.1.14 that the series on the right–hand side of (5.203) converges to $f(x)$, for $x \in [0, L]$. Hence,

$$u(x, 0) = f(x), \qquad \text{for } x \in [0, L]. \tag{5.204}$$

This is the first of the initial conditions in the vibrating string problem in (5.7). We note also that the function u defined in (5.202) also satisfies the boundary conditions in (5.7):

$$u(0, t) = u(L, t) = 0, \qquad \text{for all } t \geqslant 0.$$

To see whether or not u in (5.202) satisfies the second initial condition in problem (5.7), we first need to see that u is differentiable with respect to t and that

$$\frac{\partial}{\partial t}[u(x, t)] = -\sum_{n=1}^{\infty} \frac{\pi c}{L} n b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right), \tag{5.205}$$

for $x \in [0, L]$, $t \geqslant 0$. This will require to determine conditions on f for which that series

$$\sum_{n=1}^{\infty} n b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right) \tag{5.206}$$

converges for $x \in [0, L]$ and $t \geqslant 0$. We will answer these questions in a subsequent example.

We now turn to the question of convergence of the series in (5.206) discussed in Example 5.1.16. We first note that, if

$$\sum_{n=1}^{\infty} n|b_n| < \infty, \tag{5.207}$$

the Weierstrass M-Test would imply the absolute and uniform convergence of the series in (5.206)

Consider the general situation of a $2L$–periodic function, $f : \mathbb{R} \to \mathbb{R}$. Assume now that f is twice differentiable, with second derivative, f'', that is square–integrable on $[-L, L]$; so, that

$$\int_{-L}^{L} |f''(x)|^2 \, dx < \infty. \tag{5.208}$$

Let a_n' and b_n', for $n \in \mathbb{N}$, denote the Fourier coefficients of f'. Applying the result of Proposition 5.1.9 to f', instead of f, we obtain that

$$\sum_{n=1}^{\infty} (|a_n'| + |b_n'|) < \infty, \tag{5.209}$$

in view of (5.208).

We have already seen the identities

$$a'_n = \frac{n\pi}{L} b_n, \quad \text{for } n = 1, 2, 3, \ldots \tag{5.210}$$

and

$$b'_n = -\frac{n\pi}{L} a_n, \quad \text{for } n = 1, 2, 3, \ldots, \tag{5.211}$$

from which we get that

$$n a_n = -\frac{L}{\pi} b'_n, \quad \text{for } n = 1, 2, 3, \ldots, \tag{5.212}$$

and

$$n b_n = \frac{L}{\pi} a'_n, \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.213}$$

It follows from (5.212) and (5.213) that

$$n(|a_n| + |b_n|) = \frac{L}{\pi}(|a'_n| + b'_n), \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.214}$$

Comparing (5.209) and (5.214) we then see that

$$\sum_{n=1}^{\infty} n(|a_n| + |b_n|) < \infty.$$

We have therefore established the following proposition.

Proposition 5.1.17. Assume that $f \colon \mathbb{R} \to \mathbb{R}$ is a twice–differentiable, $2L$–periodic function whose second derivative, f'', is square–integrable over $[-L, L]$; that is,

$$\int_{-L}^{L} |f''(x)|^2 \, dx < \infty.$$

Let a_n and b_n, for $n \in \mathbb{N}$, denote the Fourier coefficients of f as given in (5.61) and (5.62), respectively. Then,

$$\sum_{n=1}^{\infty} n\left(|a_n| + |b_n|\right) < \infty. \tag{5.215}$$

Example 5.1.18 (Constructing a Solution of the Vibrating String Problem (Part II))**.** In Example 5.1.16 we considered the function $u \colon [0, L] \times [0, \infty) \to \mathbb{R}$ defined by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right), \quad \text{for } x \in [0, L], \ t \geqslant 0, \tag{5.216}$$

where the coefficients b_n, for $n \in \mathbb{N}$ are given by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad \text{for } n \in \mathbb{N}, \tag{5.217}$$

the Fourier coefficients the odd, $2L$–periodic extension of $f\colon [0,L]\to\mathbb{R}$.

In Example 5.1.16 we showed that if the odd, $2L$–periodic extension, $f\colon \mathbb{R}\to\mathbb{R}$ is differentiable with derivative f' that is square–integrable over $[-L,L]$, then the series defining u in (5.216) converges absolutely and uniformly for $x\in[-L,L]$ and $t\geqslant 0$.

In this example, we assume further that $f\colon\mathbb{R}\to\mathbb{R}$ is twice–differentiable and that the second derivative, f'', is square integrable over $[-L,L]$. We show that the partial derivatives, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$, of u exist and are given by

$$\frac{\partial}{\partial t}[u(x,t)] = -\sum_{n=1}^{\infty}\frac{\pi c}{L} n b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right), \tag{5.218}$$

and

$$\frac{\partial}{\partial x}[u(x,t)] = \sum_{n=1}^{\infty}\frac{\pi}{L} n b_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \tag{5.219}$$

respectively.

The series in (5.218) and (5.219) converge absolutely and uniformly, by the Weierstrass M–Test, provided that we can show that

$$\sum_{n=1}^{\infty} n|b_n| < \infty. \tag{5.220}$$

However, (5.220) follows from (5.215) in Proposition 5.1.17 because we are assuming that f'' is square integrable over $[-L,L]$. We therefore conclude that the series in (5.218) and (5.219) converge absolutely and uniformly for $x\in[0,L]$ and $t\geqslant 0$. In particular, we obtain from (5.218) that

$$u_t(x,0) = 0, \qquad \text{for all } x\in[0,L;$$

so that the function u defined in (5.216) satisfies the second of the initial conditions in the vibrating string problem in (5.7).

5.1.4 Solution of the Vibrating String Problem

To complete the construction of a solution of the Vibrating String Problem (5.7) begun in Example 5.1.16 and Example 5.1.18, we need to see that the function $u\colon [0,L]\times[0,\infty)\to\mathbb{R}$ defined in (5.216) has second partial derivatives with respect to t and with respect to x, $\dfrac{\partial^2 u}{\partial t^2}$ and $\dfrac{\partial^2 u}{\partial x^2}$, for $x\in(0,L)$ and $t>0$.

We have already seen in Example 5.1.18 that if the odd, $2L$–periodic extension, of $f\colon [0,L]\to\mathbb{R}$ is twice–differentiable with second–derivative, f'', that is square–integrable on $[-L,L]$, then the partial derivatives, $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$, of u exist and are given by (5.218) and (5.219), respectively. In this section we will show that, if the odd, $2L$–periodic extension of $f\colon [0,L]\to\mathbb{R}$ is thrice–differentiable with third derivative, f''', that is square–integrable over $[-L,L]$, the the function u defined in (5.216) has second partial derivatives, u_{tt} and u_{xx}, given by

$$u_{tt}(x,t) = -\sum_{n=1}^{\infty} \frac{\pi^2 c^2}{L^2} n^2 b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right), \tag{5.221}$$

for $x \in (0,L)$, $t > 0$, and

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} \frac{\pi^2}{L^2} n^2 b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right), \tag{5.222}$$

for $x \in (0,L)$, $t > 0$, respectively.

We will be able to apply the Weierstrass M–Test to show that the series in (5.221) and (5.222) converge absolutely and uniformly, provided that we can show that

$$\sum_{n=1}^{\infty} n^2 |b_n| < \infty. \tag{5.223}$$

The assertion in (5.223) will follow from the following proposition.

Proposition 5.1.19. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a thrice–differentiable, $2L$–periodic function whose third derivative, f''', is square–integrable over $[-L, L]$; that is,

$$\int_{-L}^{L} |f'''(x)|^2 \, dx < \infty. \tag{5.224}$$

Let a_n and b_n, for $n \in \mathbb{N}$, denote the Fourier coefficients of f as given in (5.61) and (5.62), respectively. Then,

$$\sum_{n=1}^{\infty} n^2 \left(|a_n| + |b_n|\right) < \infty. \tag{5.225}$$

Proof: Let a_n, b_n, for $n \in \mathbb{N}$, denote the Fourier coefficients of f as given in (5.61) and (5.62. Let a_n', b_n', for $n \in \mathbb{N}$, denote the corresponding Fourier coefficients of f', and a_n'', b_n'', for $n \in \mathbb{N}$, be those corresponding to f''.

Applying the result of Proposition 5.1.9 to f'', instead of f, we obtain that

$$\sum_{n=1}^{\infty} \left(|a_n''| + |b_n''|\right) < \infty, \tag{5.226}$$

in view of (5.224).

Next, use the identities in (5.210) and (5.211), applied to a_n'' and b_n'', to get that

$$a_n'' = \frac{n\pi}{L} b_n', \quad \text{for } n = 1, 2, 3, \ldots \tag{5.227}$$

and

$$b_n'' = -\frac{n\pi}{L} a_n', \quad \text{for } n = 1, 2, 3, \ldots \tag{5.228}$$

Using the identities in (5.210) and (5.211) again, we obtain from (5.227) and (5.228) that

$$a_n'' = -\frac{\pi^2}{L^2} n^2 a_n, \quad \text{for } n = 1, 2, 3, \ldots$$

and

$$b_n'' = -\frac{\pi^2}{L^2} n^2 b_n, \quad \text{for } n = 1, 2, 3, \ldots,$$

from which we get

$$|a_n''| = \frac{\pi^2}{L^2} n^2 |a_n|, \quad \text{for } n = 1, 2, 3, \ldots \tag{5.229}$$

and

$$|b_n''| = \frac{\pi^2}{L^2} n^2 |b_n|, \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.230}$$

It follows from (5.229) and (5.230) that

$$n^2(|a_n| + |b_n|) = \frac{L^2}{\pi^2}\left(|a_n''| + |b_n''|\right), \quad \text{for } n = 1, 2, 3, \ldots \tag{5.231}$$

In view of (5.231) we see that the assertion in (5.225) follows from (5.226), and the proof of the proposition is now complete. ∎

Theorem 5.1.20 (Existence of Solution for the Vibrating String Problem)**.** Suppose that $f\colon [0, L] \to \mathbb{R}$ satisfies $f(0) = f(L) = 0$ and that it extends to an odd, $2L$–periodic function, that is thrice–differentiable, and whose third derivative, f''', is square–integrable over $[-L, L]$. Then, the initial–boundary value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{for } x \in (0, L) \text{ and } t > 0; \\ u(0, t) = u(L, t) = 0, & \text{for } t \geqslant 0; \\ u(x, 0) = f(x), & \text{for } x \in [0, L]; \\ u_t(x, 0) = 0, & \text{for } x \in [0, L], \end{cases} \tag{5.232}$$

has a solution.

Proof: Define $u\colon [0, L] \to \mathbb{R}$ by

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right), \quad \text{for } x \in [0, L],\ t \geqslant 0, \tag{5.233}$$

where the coefficients b_n, for $n \in \mathbb{N}$ are given by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad \text{for } n \in \mathbb{N}, \tag{5.234}$$

the Fourier coefficients the odd, $2L$–periodic extension of $f\colon [0, L] \to \mathbb{R}$.

It follows from Proposition 5.1.19 that

$$\sum_{n=1}^{\infty} n^2|b_n| < \infty,$$

where the Fourier coefficients of f, b_n, for $n \in \mathbb{N}$, are given in (5.234). Consequently, by the Weierstrass M–Test, the series on the right–hand side of (5.221) and (5.222) converge absolutely and uniformly for $x \in [0, L]$ and $t \geqslant 0$. Thus, the function $u\colon [0, L] \to \mathbb{R}$ defined in (5.233) has continuous second partial derivatives, u_{tt} and u_{xx}, given in (5.221) and (5.222), respectively. Observe that

$$u_{tt}(x, t) = c^2 u_{xx}(x, t), \qquad \text{for } x \in (0, L) \text{ and } t > 0.$$

Thus, the function u given in (5.233) solves the PDE in the Vibrating String Problem (5.232).

We have already seen in Example 5.1.16 and Example 5.1.18 that the function u given in (5.233) also satisfies the boundary conditions and the initial conditions in problem (5.232). Hence, the proof of existence of a solution of problem (5.232) is now complete. ■

5.2 Fundamental Solutions

We will illustrate the concept of a fundamental solution by first finding a special solution of the one–dimensional diffusion equation.

5.2.1 Fundamental Solution to the Diffusion Equation

We compute a very special solution to the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.235}$$

In Section 4.2.3 we derived the following dilation–invariant solution to the diffusion equation in (5.235):

$$u(x, t) = c_1 \int_0^{x/\sqrt{t}} e^{-z^2/4D}\ dz + c_2, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.236}$$

and constants c_1 and c_2. Observe that the function $u\colon \mathbb{R}\times(0, \infty) \to \mathbb{R}$ defined in (5.236) is a composition of C^∞ functions. It then follows by the Fundamental Theorem of Calculus and the Chain Rule that $u \in C^\infty(\mathbb{R} \times (0, \infty))$. Hence, we can differentiate on both sides of the PDE in (5.235) with respect to x, for example, and get the valid statement

$$u_{tx} = D\ u_{xxx};$$

thus, by the equality of the mixed partial derivatives,

$$(u_x)_t = D\ (u_x)_{xx},$$

which shows that u_x is also a solution of the one–dimensional diffusion equation in (5.235). Hence, by taking the partial derivative with respect to x in (5.236) we obtain another solution to the the one–dimensional diffusion equation in (5.235). Set $v(x,t) = u_x(x,t)$, for $x \in \mathbb{R}$ and $t > 0$, where u is given in (5.236). Then, using the Fundamental Theorem of Calculus and the Chain Rule, we obtain from (5.236) that

$$v(x,t) = \frac{c_1}{\sqrt{t}}\ e^{-x^2/4Dt}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.237}$$

and some constant c_1, is a solution to the one–dimensional diffusion equation in (5.235).

An interesting property of the function defined in (5.237) is that the integral $\displaystyle\int_{-\infty}^{\infty} v(x,t)\ dx$ is finite and is independent of $t > 0$. Indeed, using the fact that

$$\int_{-\infty}^{\infty} e^{-z^2}\ dz = \sqrt{\pi},$$

and making the change of variables

$$z = \frac{x}{\sqrt{4Dt}},$$

so that

$$dx = \sqrt{4Dt}\ dz,$$

we obtain, for $t > 0$,

$$\int_{-\infty}^{\infty} v(x,t)\ dx = \frac{c_1}{\sqrt{t}}\sqrt{4Dt}\int_{-\infty}^{\infty} e^{-z^2}\ dz,$$

or

$$\int_{-\infty}^{\infty} v(x,t)\ dx = c_1\sqrt{4D\pi}, \quad \text{for all } t > 0. \tag{5.238}$$

We chose the constant c_1 in (5.238) so that

$$\int_{-\infty}^{\infty} v(x,t)\ dx = 1, \quad \text{for all } t > 0;$$

that is,

$$c_1 = \frac{1}{\sqrt{4D\pi}}. \tag{5.239}$$

Substituting the value of c_1 in (5.239) into the definition of $v(x,t)$ in (5.237), we obtain

$$v(x,t) = \frac{1}{\sqrt{4\pi Dt}}\ e^{-x^2/4Dt}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.240}$$

We shall denote the expression for $v(x,t)$ defined in (5.240) by $p(x,t)$, so that

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.241}$$

It then follows from what we have shown thus far that the function p defined in (5.241) is a C^∞ function defined in $\mathbb{R} \times (0,\infty)$ that solves the one–dimensional diffusion equation in (5.235); that is,

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.242}$$

Also, it follows from (5.238) and (5.239) that

$$\int_{-\infty}^{\infty} p(x,t)\; dx = 1, \qquad \text{for all } t > 0. \tag{5.243}$$

In fact, using a change of variables we obtain from (5.243) that

$$\int_{-\infty}^{\infty} p(x-y,t)\; dy = 1, \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0. \tag{5.244}$$

In addition to (5.244), the function p defined in (5.241) has the following properties:

Proposition 5.2.1 (Properties of p)**.** Let $p(x,t)$ be as defined in (5.241) for $x \in \mathbb{R}$ and $t > 0$.

(i) $p(x-y,t) > 0$ for all $x, y \in \mathbb{R}$ and $t > 0$

(ii) If $x \neq y$, then $\lim\limits_{t\to 0^+} p(x-y,t) = 0$.

(iii) If $x = y$, then $\lim\limits_{t\to 0^+} p(x-y,t) = +\infty$.

See Problem 5 in Assignment #14.

In this section we will see how to use the properties in (5.244) and in Proposition 5.2.1 to obtain a solution to the initial value problem for the one–dimensional diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{5.245}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded function that is also **piecewise continuous**.

Definition 5.2.2 (Piecewise Continuous Functions)**.** A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to have a **jump discontinuity** at $x \in \mathbb{R}$ if the one–sided limits

$$\lim_{y\to x^+} f(y) \quad \text{and} \quad \lim_{y\to x^-} f(y)$$

exist and

$$\lim_{y\to x^+} f(y) \neq \lim_{y\to x^-} f(y).$$

We say that f is piecewise continuous if it is continuous except at an at most countable number of points at which f has jump discontinuities.

Figure 5.2.1 shows a portion of the sketch of a piecewise continuous function. We will show that the function $u\colon \mathbb{R}\times(0,\infty)\to\mathbb{R}$ given by

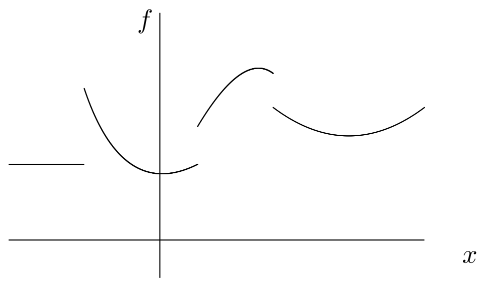

Figure 5.2.1: Sketch of a Piecewise Continuous Function

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t)f(y)\ dy, \qquad \text{for } x\in\mathbb{R} \text{ and } t>0, \tag{5.246}$$

is a candidate for a solution of the initial value problem in (5.245). We note that, since $p(x-y,t))$ is not defined at $t=0$, the initial condition in the IVP in (5.245) has to be understood as

$$\lim_{t\to 0^+} u(x,t) = f(x).$$

We will see in this section that (5.247) holds true foe values of x at which f is continuous. For values of x at which f has a jump discontinuity

$$\lim_{t\to 0^+} u(x,t) = \frac{f(x^+)+f(x^-)}{2},$$

where $f(x^+)$ and $f(x^-)$ are the one–sided limits

$$f(x^+) = \lim_{y\to x^+} f(y) \quad \text{and} \quad f(x^-) = \lim_{y\to x^-} f(y),$$

respectively.

We state the main result of this section as the following proposition:

Proposition 5.2.3. Let u be given by (5.246), where $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded, piecewise continuous function. Then, u is $C^{2,1}(\mathbb{R} \times (0, \infty))$[1] and

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t), \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.247}$$

Furthermore,

$$\lim_{t\to 0^+} u(x,t) = f(x). \tag{5.248}$$

if f is continuous at x, and

$$\lim_{t\to 0^+} u(x,t) = \frac{f(x^+) + f(x^-)}{2}, \tag{5.249}$$

if f has a jump discontinuity at x.

Once we have proved Proposition 5.2.3, we will have constructed a solution

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t)f(y)\ dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.250}$$

to the initial value problem of the initial value for the one–dimensional diffusion for the case of continuous initial data f, where p is defined in (5.241). Thus, a solution of the initial value problem in (5.245) is obtained by integrating $f(y)p(x-y,t)$ over y in the entire real line. The map

$$(x,y,t) \mapsto p(x-y,t), \quad \text{for all } x,y \in \mathbb{R} \text{ and } t > 0,$$

or

$$(x,y,t) \mapsto \frac{1}{\sqrt{4\pi Dt}}\ e^{-(x-y)^2/4Dt}, \quad \text{for all } x,y \in \mathbb{R} \text{ and } t > 0,$$

is usually called the **heat kernel**; we shall also call it the **fundamental solution** to the one–dimensional diffusion equation. We will denote it by $K(x,y,t)$, so that $K\colon \mathbb{R}^2 \times (0,\infty) \to \mathbb{R}$ and

$$K(x,y,t) = \frac{1}{\sqrt{4\pi Dt}}\ e^{-(x-y)^2/4Dt}, \quad \text{for all } x,y \in \mathbb{R} \text{ and } t > 0. \tag{5.251}$$

We shall reiterate the properties of the heat kernel that we have discussed for future reference in the following proposition, we will add the additional observation that K is symmetric in x and y; that is $K(x,y,t) = K(y,x,t)$ for all $x,y \in \mathbb{R}$ and $t > 0$.

Proposition 5.2.4 (Properties of the Heat Kernel)**.** Let $K(x,y,t)$ be as defined in (5.251) for $x,y \in \mathbb{R}$ and $t > 0$.

(i) $K(x,y,t) = K(y,x,t)$ for all $x,y \in \mathbb{R}$ and $t > 0$.

(ii) $K(x,y,t) > 0$ for all $x,y \in \mathbb{R}$ and $t > 0$.

[1]The function u is C^2 in the first variable, and C^1 in the second variable

(iii) $\displaystyle\int_{-\infty}^{\infty} K(y,x,t)\ dy = 1$ for all $x \in \mathbb{R}$ and $t > 0$.

(iv) If $x \neq y$, then $\displaystyle\lim_{t\to 0^+} K(x,y,t) = 0$.

(v) If $x = y$, then $\displaystyle\lim_{t\to 0^+} K(x,y,t) = +\infty$.

Before we prove Proposition 5.2.3, we will establish two Lemmas; the first one involves the error function,

$$\text{Erf}\colon \mathbb{R} \to \mathbb{R},$$

defined by

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-r^2}\ dr, \qquad \text{for } x \in \mathbb{R}, \tag{5.252}$$

and its properties:

Proposition 5.2.5. Let $\text{Erf}\colon \mathbb{R} \to \mathbb{R}$ be as given in (5.252). Then,

(i) $\text{Erf}(0) = 0$;

(ii) $\displaystyle\lim_{x\to\infty} \text{Erf}(x) = 1$;

(iii) $\displaystyle\lim_{x\to-\infty} \text{Erf}(x) = -1$;

See Problem 1 in Assignment #14.

A sketch of the graph of $y = \text{Erf}(x)$ is shown in Figure 5.2.2.

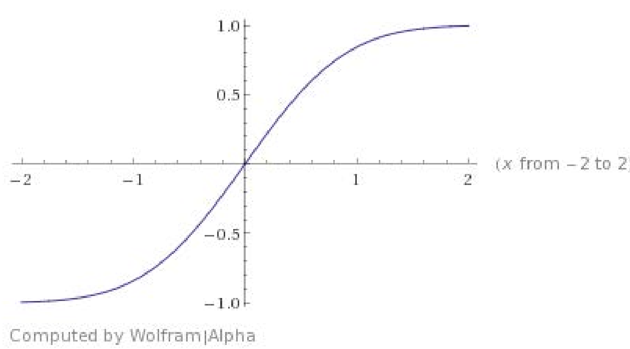

Figure 5.2.2: Sketch of Graph of Error Function

Lemma 5.2.6. Let $p(x,t)$ be as defined in (5.241) for $x \in \mathbb{R}$ and $t > 0$. For $\delta > 0$,

$$\lim_{t\to 0^+} \int_\delta^\infty p(x,t)\ dx = 0. \tag{5.253}$$

and

$$\lim_{t\to 0^+} \int_{-\infty}^{-\delta} p(x,t)\ dx = 0. \tag{5.254}$$

Proof: Make the change of variables $y = \dfrac{x}{\sqrt{4Dt}}$ to write

$$\begin{aligned} \int_{\delta}^{\infty} p(x,t)\ dx &= \int_{\delta}^{\infty} \frac{1}{\sqrt{4\pi Dt}} \cdot e^{-x^2/4Dt}\ dx \\ &= \frac{1}{\sqrt{\pi}} \int_{\delta/\sqrt{4Dt}}^{\infty} e^{-y^2}\ dy \\ &= \frac{1}{2}\left[1 - \text{Erf}\left(\frac{\delta}{\sqrt{4Dt}}\right)\right], \end{aligned}$$

where we have used the definition of the error function in (5.252) and the fact that

$$\int_0^{\infty} e^{-y^2}\ dy = \frac{\sqrt{\pi}}{2}.$$

We then have that

$$\int_{\delta}^{\infty} p(x,t)\ dx = \frac{1}{2}\left[1 - \text{Erf}\left(\frac{\delta}{\sqrt{4Dt}}\right)\right], \quad \text{for } t > 0. \tag{5.255}$$

Now, it follows from (5.255) and (ii) in Proposition 5.2.5 that

$$\lim_{t\to 0^+} \int_{\delta}^{\infty} p(x,t)\ dx = 0,$$

which is (5.253). Similar calculations can be used to derive (5.254). ■

Lemma 5.2.7. Let $p(x,t)$ be as defined in (5.241) for $x \in \mathbb{R}$ and $t > 0$. Then, we have the following estimates on integrals of the absolute values of the derivatives of p:

$$\int_{-\infty}^{\infty} \left|\frac{\partial p}{\partial t}(x-y,t)\right|\ dy \leqslant \frac{1}{t}, \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0, \tag{5.256}$$

and

$$\int_{-\infty}^{\infty} \left|\frac{\partial p}{\partial x}(x-y,t)\right|\ dy = \frac{1}{\sqrt{\pi Dt}}, \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0, \tag{5.257}$$

Proof: Compute the partial derivative of

$$p(x-y,t) = \frac{1}{\sqrt{4\pi Dt}}\ e^{-(x-y)^2/4Dt}, \quad \text{for all } x, y \in \mathbb{R} \text{ and } t > 0, \tag{5.258}$$

with respect to t to obtain

$$\frac{\partial}{\partial t}[p(x-y,t)] = -\frac{1}{2t}p(x-y,t) + \frac{(x-y)^2}{4Dt^2}p(x-y,t), \tag{5.259}$$

for all $x, y \in \mathbb{R}$ and $t > 0$. Next, take absolute value on both sides of (5.259), apply the triangle inequality, and use the positivity of the heat kernel (see (ii) in Proposition 5.2.4) to get

$$\left|\frac{\partial}{\partial t}[p(x-y,t)]\right| \leqslant \frac{1}{2t}p(x-y,t) + \frac{(x-y)^2}{4Dt^2}p(x-y,t), \tag{5.260}$$

for all $x, y \in \mathbb{R}$ and $t > 0$. Integrating on both sides of (5.260) and using (5.244) (see (iii) in Proposition 5.2.4) yields

$$\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial t}[p(x-y,t)]\right|\ dy \leqslant \frac{1}{2t} + \int_{-\infty}^{\infty}\frac{(x-y)^2}{4Dt^2}p(x-y,t)\ dy, \tag{5.261}$$

for all $x \in \mathbb{R}$ and $t > 0$.

Next, we evaluate the right–most integral in (5.261),

$$\int_{-\infty}^{\infty}\frac{(x-y)^2}{4Dt^2}p(x-y,t)\ dy = \int_{-\infty}^{\infty}\frac{(x-y)^2}{4Dt^2}\cdot\frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}}\ dy,$$

by making the change of variables

$$\xi = \frac{y-x}{\sqrt{4Dt}},$$

so that

$$\int_{-\infty}^{\infty}\frac{(x-y)^2}{4Dt^2}p(x-y,t)\ dy = \frac{1}{t\sqrt{\pi}}\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}\ d\xi, \tag{5.262}$$

for all $x \in \mathbb{R}$ and $t > 0$. The right–most integral in (5.262) can be evaluated using integration by parts to yield

$$\begin{aligned}\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}\ d\xi &= 2\int_0^{\infty}\xi^2 e^{-\xi^2}\ d\xi \\ &= -\xi e^{-\xi^2}\Big|_0^{\infty} + \int_0^{\infty}e^{-\xi^2}\ d\xi,\end{aligned}$$

so that

$$\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}\ d\xi = \frac{\sqrt{\pi}}{2}. \tag{5.263}$$

Combining (5.263), (5.262) and (5.261) yields the estimate

$$\int_{-\infty}^{\infty}\left|\frac{\partial}{\partial t}[p(x-y,t)]\right|\ dy \leqslant \frac{1}{t}, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

which is (5.256).

In order to establish (5.257), first take the partial derivative with respect to x on both side of (5.258) to get

$$\frac{\partial}{\partial x}[p(x-y,t)] = -\frac{x-y}{2Dt}p(x-y,t), \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.264}$$

so that, taking absolute value on both sides of (5.264) and integrating,

$$\int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x}[p(x-y,t)] \right| \, dy = \int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} p(x-y,t) \, dy, \tag{5.265}$$

Evaluate the right–most integral in (5.265),

$$\int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} p(x-y,t) \, dy = \int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} \cdot \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} \, dy, \tag{5.266}$$

by making the change of variables

$$\xi = \frac{y-x}{\sqrt{4Dt}},$$

to get

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} \cdot \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} \, dy &= \frac{1}{\sqrt{\pi Dt}} \int_{-\infty}^{\infty} |\xi| e^{-\xi^2} \, d\xi \\ &= \frac{2}{\sqrt{\pi Dt}} \int_{0}^{\infty} \xi e^{-\xi^2} \, d\xi, \end{aligned}$$

so that

$$\int_{-\infty}^{\infty} \frac{|x-y|}{2Dt} \cdot \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} \, dy = \frac{1}{\sqrt{\pi Dt}}. \tag{5.267}$$

The statement in (5.257) now follows by putting together the results in (5.267), (5.266) and (5.265). ■

Proof of Proposition 5.2.3: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function satisfying

$$|f(x)| \leqslant M, \quad \text{for all } x \in \mathbb{R}, \tag{5.268}$$

and some positive constant M, and define $u\colon \mathbb{R} \times (0,t) \to \mathbb{R}$ by

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t) f(y) \, dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.269}$$

where $p(x-y,t)$ denotes the heat kernel given in (5.258). We will show that u solves the one–dimensional diffusion equation in (5.247). Before we do that, though, we need to verify that the expression in (5.269) does indeed define a function $u\colon \mathbb{R}\times(0,\infty)\to\mathbb{R}$. In order to do this we need to make sure that the integral on the right–hand side of (5.269) is a real number. This will follow from the estimate

$$\int_{-\infty}^{\infty} |p(x-y,t)f(y)|\ dy < \infty \qquad \text{for } x\in\mathbb{R} \text{ and } t>0. \tag{5.270}$$

In order to derive the estimate in (5.270), use the positivity of the heat kernel (see (ii) in Proposition 5.2.4), (5.244) and (5.268) to compute

$$\int_{-\infty}^{\infty} |p(x-y,t)f(y)|\ dy \leqslant M\int_{-\infty}^{\infty} p(x-y,t)\ dy,$$

so that

$$\int_{-\infty}^{\infty} |p(x-y,t)f(y)|\ dy \leqslant M, \qquad \text{for } x\in\mathbb{R} \text{ and } t>0, \tag{5.271}$$

which implies (5.270). Observe that the estimate in (5.271) also implies that

$$|u(x,t)|\leqslant M, \qquad \text{for } x\in\mathbb{R} \text{ and } t>0,$$

by virtue of (5.269).

The fact that u defined in (5.269) solves the one–dimensional diffusion equation in (5.247) will follow from the fact that the heat kernel itself solves the one–dimensional heat equation,

$$\frac{\partial}{\partial t}[p(x-y,t)] = D\frac{\partial^2}{\partial x^2}[p(x-y,t)], \qquad \text{for } x,y\in\mathbb{R} \text{ and } t>0; \tag{5.272}$$

(see also (5.242). Indeed, suppose for the moment that we can interchange differentiation and integration in the definition of u in (5.269), so that

$$\frac{\partial u}{\partial t}(x,t) = \int_{-\infty}^{\infty} \frac{\partial p}{\partial t}(x-y,t)f(y)\ dy, \qquad \text{for } x\in\mathbb{R} \text{ and } t>0, \tag{5.273}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \int_{-\infty}^{\infty} \frac{\partial^2 p}{\partial x^2}(x-y,t)f(y)\ dy, \qquad \text{for } x\in\mathbb{R} \text{ and } t>0. \tag{5.274}$$

Thus, combining (5.273) and (5.274),

$$\frac{\partial u}{\partial t}(x,t) - D\frac{\partial^2 u}{\partial x^2}(x,t) = \int_{-\infty}^{\infty}\left[\frac{\partial p}{\partial t}(x-y,t) - D\frac{\partial^2 p}{\partial x^2}(x-y,t)\right] f(y)\ dy,$$

which shows that (5.247) holds true by virtue of (5.272)

The expressions in (5.273) and (5.274) are justified by the assumption that f is bounded (see (5.268) and the estimates (5.256) and (5.257) in Lemma 5.2.7; namely,

$$\int_{-\infty}^{\infty}\left|\frac{\partial p}{\partial t}(x-y,t)\right|\ dy \leqslant \frac{1}{t}, \qquad \text{for all } x\in\mathbb{R} \text{ and } t>0.$$

and

$$\int_{-\infty}^{\infty} \left| \frac{\partial p}{\partial x}(x-y,t) \right| \, dy = \frac{1}{\sqrt{\pi D t}}, \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

Observe that, (5.274) and (5.272) imply the estimate

$$\int_{-\infty}^{\infty} \left| \frac{\partial^2 p}{\partial x^2}(x-y,t) \right| \, dy \leqslant \frac{1}{Dt}, \qquad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

We have therefore established that the function $u\colon \mathbb{R} \times (0,1)$ defined in (5.269) is a C^2 function in the first variable, C^1 in the second variable, and is a solution to the one–dimensional diffusion equation.

Next, we will prove the second assertion in Proposition 5.2.3.
(i) Assume first that f is continuous at x and let $\varepsilon > 0$ be given. Then, there exists $\delta > 0$ such that

$$|y - x| < \delta \Rightarrow |f(y) - f(x)| < \frac{\varepsilon}{3}. \tag{5.275}$$

We consider

$$u(x,t) - f(x) = \int_{-\infty}^{\infty} p(x_o - y, t) f(y) \, dy - f(x) \int_{-\infty}^{\infty} p(x_o - y, t) \, dy,$$

where we have used the definition of $u(x,t)$ in (5.269) and (5.244) (see also the fact (iii) in Proposition 5.2.4). We then have that

$$u(x,t) - f(x) = \int_{-\infty}^{\infty} p(x-y,t)(f(y) - f(x)) \, dy,$$

so that

$$|u(x,t) - f(x)| \leqslant \int_{-\infty}^{\infty} p(x-y,t) |f(y) - f(x)| \, dy, \tag{5.276}$$

where we have used the fact that $p(x,t)$ is positive for all $x \in \mathbb{R}$ and all $t > 0$.

Next, re-write the integral on the right–hand side of (5.276) as a sum of three integrals,

$$\begin{aligned} \int_{-\infty}^{\infty} p(x-y, \quad t) |f(y) - f(x)| \, dy = & \\ & \int_{-\infty}^{x-\delta} p(x-y,t)|f(y) - f(x)| \, dy \\ & + \int_{x-\delta}^{x+\delta} p(x-y,t)|f(y) - f(x)| \, dy \\ & + \int_{x+\delta}^{\infty} p(x-y,t)|f(y) - f(x)| \, dy. \end{aligned} \tag{5.277}$$

We first estimate the middle integral on the right–hand side of (5.277), using (5.275) and (5.244) to get

$$\int_{x-\delta}^{x+\delta} p(x-y,t)|f(y)-f(x)|\ dy < \frac{\varepsilon}{3}. \tag{5.278}$$

Next, use (5.268) and the triangle inequality to obtain the following estimate for the last integral on the right–hand side of (5.277),

$$\int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x)|\ dy \leqslant 2M \int_{x+\delta}^{\infty} p(x-y,t)\ dy. \tag{5.279}$$

Make the change of variables $\xi = y - x$ in the integral on the right–hand side of (5.279) to obtain

$$\int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x_o)|\ dy \leqslant 2M \int_{\delta}^{\infty} p(\xi,t)\ d\xi, \tag{5.280}$$

where we have also used the symmetry of the heat kernel (see (i) in Proposition 5.2.4). It follows from (5.280) and (5.253) in Lemma 5.2.6 that

$$\lim_{t\to 0^+} \int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x)|\ dy = 0;$$

thus, there exists $\delta_1 > 0$ such that

$$0 < t < \delta_1 \Rightarrow \int_{x+\delta}^{\infty} p(x-y,t)|f(y)-f(x)|\ dy < \frac{\varepsilon}{3}. \tag{5.281}$$

Similar calculations to those leading to (5.281), using (5.254) in Lemma 5.2.6, can be used to show that there exists $\delta_2 > 0$ such that

$$0 < t < \delta_2 \Rightarrow \int_{-\infty}^{x-\delta} p(x-y,t)|f(y)-f(x)|\ dy < \frac{\varepsilon}{3}. \tag{5.282}$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. It then follows from (5.277), in conjunction with (5.278), (5.281) and (5.282), that

$$0 < t < \delta_3 \Rightarrow \int_{-\infty}^{\infty} p(x-y,t)|f(y)-f(x_o)|\ dy < \varepsilon.$$

We have therefore proved that

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} p(x-y,t)|f(y)-f(x)|\ dy = 0. \tag{5.283}$$

It follows from (5.283) and the estimate in (5.276) that

$$\lim_{t\to 0^+} |u(x,t) - f(x)| = 0,$$

which yields (5.248) and assertion (i) of Proposition 5.2.3 has been proved.

(ii) Assume that f has a jump discontinuity at x and put

$$f(x^+) = \lim_{y \to x^+} f(y) \quad \text{and} \quad f(x^-) = \lim_{y \to x^-} f(y). \tag{5.284}$$

Let $\varepsilon > 0$ be given. It follows from (5.284) that there exists $\delta > 0$ such that

$$x < y < x + \delta \Rightarrow |f(y) - f(x^+)| < \frac{\varepsilon}{3}, \tag{5.285}$$

and

$$x - \delta < y < x \Rightarrow |f(y) - f(x^-)| < \frac{\varepsilon}{3}. \tag{5.286}$$

Use the definition of $u(x,t)$ in (5.269) to write

$$u(x,t) - \frac{f(x^+) + f(x^-)}{2} = \int_{-\infty}^{\infty} p(x-y,t) f(y)\, dy - \frac{1}{2} f(x^+) - \frac{1}{2} f(x^-),$$

and note that

$$\frac{1}{2} = \int_{-\infty}^{x_o} p(x_o - y, t)\, dy = \int_{x_o}^{\infty} p(x_o - y, t)\, dy, \tag{5.287}$$

by virtue of (5.244), (5.243) and the symmetry of the heat kernel (see (i) in Proposition 5.2.4). We therefore have that

$$\begin{aligned} u(x,t) \quad - \quad & \frac{f(x^+) + f(x^-)}{2} \\ &= \int_{-\infty}^{x} p(x-y,t)(f(y) - f(x^-))\, dy \\ &\quad + \int_{x_o}^{\infty} p(x-y,t)(f(y) - f(x^+))\, dy, \end{aligned}$$

so that

$$\begin{aligned} \Bigg|\, u(x,t) \quad - \quad & \frac{f(x^+) + f(x^-)}{2} \Bigg| \\ &\leqslant \int_{-\infty}^{x} p(x-y,t)|f(y) - f(x^-)|\, dy \\ &\quad + \int_{x}^{\infty} p(x-y,t)|f(y) - f(x^+)|\, dy, \end{aligned} \tag{5.288}$$

We re-write the last integral on the right–hand side of (5.288) as a sum of two integrals,

$$\begin{aligned}\int_x^\infty p(x-y,\quad t)|f(y)-f(x^+)|\ dy \\ &= \int_x^{x+\delta} p(x-y,t)|f(y)-f(x^+)|\ dy \\ &\quad + \int_{x+\delta}^\infty p(x-y,t)|f(y)-f(x^+)|\ dy,\end{aligned} \tag{5.289}$$

where

$$\int_x^{x+\delta} p(x-y,t)|f(y)-f(x^+)|\ dy < \frac{\varepsilon}{3}\int_x^{x+\delta} p(x-y,t)\ dy < \frac{\varepsilon}{6}, \tag{5.290}$$

by virtue of (5.286) and (5.287).

Similar calculations to those leading to (5.281) can be used to show that there exists $\delta_1 > 0$ such that

$$0 < t < \delta_1 \Rightarrow \int_{x+\delta}^\infty p(x-y,t)|f(y)-f(x^+)|\ dy < \frac{\varepsilon}{3}. \tag{5.291}$$

Combining (5.290) and (5.291), we obtain from (5.289) that

$$0 < t < \delta_1 \Rightarrow \int_x^\infty p(x-y,t)|f(y)-f(x^+)|\ dy < \frac{\varepsilon}{2}. \tag{5.292}$$

Similarly, we can show that there exists $\delta_2 > 0$ such that

$$0 < t < \delta_2 \Rightarrow \int_{-\infty}^x p(x-y,t)|f(y)-f(x^-)|\ dy < \frac{\varepsilon}{2}. \tag{5.293}$$

Thus, letting $\delta_3 = \min\{\delta_1, \delta_2\}$ we see that the conjunction of (5.292) and (5.293), together with (5.288), implies that

$$0 < t < \delta_3 \Rightarrow \left| u(x,t) - \frac{f(x^+)+f(x^-)}{2} \right| < \varepsilon.$$

We have therefore established (5.249) and the proof of part (ii) of Proposition 5.2.3 is now complete. ■

Example 5.2.8. Solve the initial value problem for the diffusion equation in (5.245), where

$$f(x) = \begin{cases} 1, & \text{if } -1 < x \leqslant 1; \\ 0, & \text{elsewhere.} \end{cases} \tag{5.294}$$

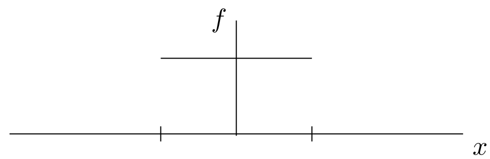

Figure 5.2.3: Initial Condition for Example 5.2.8

Solution: A sketch of the graph of the initial condition, f, is shown in Figure 5.2.3. Note that f has jump discontinuities at -1 and at 1.

Using the formula in (5.269) we get that a solution to the initial value problem (5.245) with initial condition given in (5.294) is given by

$$u(x,t) = \int_{-1}^{1} p(x-y,t)\ dy, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

or

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-1}^{1} e^{-(x-y)^2/4Dt}\ dy, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.295}$$

Make the change variables $r = \dfrac{x-y}{\sqrt{4Dt}}$ in (5.295) to obtain

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{\frac{x+1}{\sqrt{4Dt}}}^{\frac{x-1}{\sqrt{4Dt}}} e^{-r^2}\ dr, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

or

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+1}{\sqrt{4Dt}}} e^{-r^2}\ dr - \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-1}{\sqrt{4Dt}}} e^{-r^2}\ dr, \tag{5.296}$$

for $x \in \mathbb{R}$ and $t > 0$.

Making use of the error function defined in (5.252), we can rewrite (5.296) as

$$u(x,t) = \frac{1}{2}\left[\text{Erf}\left(\frac{x+1}{\sqrt{4Dt}}\right) - \text{Erf}\left(\frac{x-1}{\sqrt{4Dt}}\right)\right], \tag{5.297}$$

for $x \in \mathbb{R}$ and $t > 0$. Figure 5.2.4 shows plots of the graph of $y = u(x,t)$, where $u(x,t)$ is as given in (5.297), for various values of t in the case $4D = 1$. A few interesting properties of the function u given in (5.297) are apparent by examining the pictures in Figure 5.2.4. First, the graph of $y = u(x,t)$ is smooth for all $t > 0$. Even though the initial temperature distribution, f, in (5.294) is not even continuous, the solution to the initial value problem (5.245) given in (5.297) is in fact infinitely differentiable as soon as the process gets going for $t > 0$. Secondly, the values, $u(x,t)$, of the function u given in (5.297)) are positive at all values of $x \in \mathbb{R}$ and $t > 0$. In particular, for values of x with $|x| > 1$, where the initial temperature is zero, the temperature rises instantly

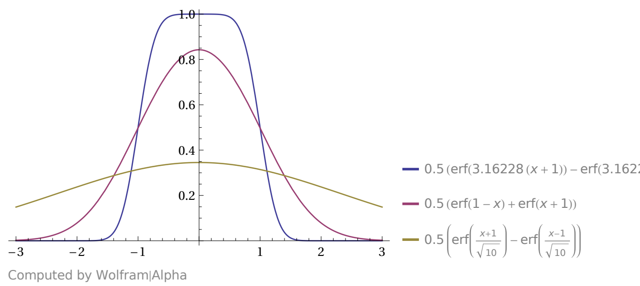

Figure 5.2.4: Sketch of Graph of $y = u(x,t)$ for $t = 0.1, 1, 10$

for $t > 0$. Thus, the diffusion model for heat propagation predicts that heat propagates with infinite speed. Thirdly, we see from the pictures in Figure 5.2.4 that

$$\lim_{t\to\infty} u(x,t) = 0, \qquad \text{for all } x \in \mathbb{R}. \tag{5.298}$$

□

5.2.2 Uniqueness for the Diffusion Equation

The observation (5.298) in Example 5.2.8 is true in general for solutions to the initial value problem in (5.245) for the case in which the initial condition, f, is square–integrable; that is,

$$\int_{-\infty}^{\infty} |f(x)|^2 \; dx < \infty \tag{5.299}$$

Observe that, for the function f in Example 5.2.8 satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 \; dx = 2,$$

so that the integrability condition in (5.299) holds true for the function in (5.294).

Before we establish that (5.298) is true for any solution of the initial value problem (5.245) in which the initial condition satisfies (5.299), we will first need to derive other properties of the function u given in (5.246).

Proposition 5.2.9. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and satisfying (5.299); that is,

$$\int_{-\infty}^{\infty} |f(x)|^2 \; dx < \infty.$$

Put

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t) f(y) \; dy, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.300}$$

Then,

$$\int_{-\infty}^{\infty} |u(x,t)|^2\ dx < \infty, \quad \text{for all } t > 0, \tag{5.301}$$

and

$$\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x}(x,t)\right|^2\ dx < \infty, \quad \text{for all } t > 0. \tag{5.302}$$

Proof: Let u be given by (5.300), where f satisfies the condition in (5.299). Apply the Cauchy–Schwarz inequality (or Jensen's Inequality) to get

$$|u(x,t)|^2 \leqslant \int_{-\infty}^{\infty} p(x-y,t)|f(y)|^2\ dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{5.303}$$

where we have also used (5.244) and the positivity of the heat kernel (see (ii) and (iii) in Proposition 5.2.4).

Integrate with respect to x on both sides of (5.303) to get

$$\int_{-\infty}^{\infty} |u(x,t)|^2\ dx \leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x-y,t)|f(y)|^2\ dydx, \tag{5.304}$$

for $t > 0$. Interchanging the order of integration in the integral on the right–hand side of (5.304) we obtain

$$\int_{-\infty}^{\infty} |u(x,t)|^2\ dx \leqslant \int_{-\infty}^{\infty} |f(y)|^2 \left\{\int_{-\infty}^{\infty} p(x-y,t)\ dx\right\} dy, \tag{5.305}$$

for $t > 0$. It follows from (5.305) and (5.244) that

$$\int_{-\infty}^{\infty} |u(x,t)|^2\ dx \leqslant \int_{-\infty}^{\infty} |f(y)|^2\ dy, \quad \text{for } t > 0, \tag{5.306}$$

Combining (5.306) and (5.299) then yields

$$\int_{-\infty}^{\infty} |u(x,t)|^2\ dx < \infty, \quad \text{for all } t > 0, \tag{5.307}$$

which is the condition in (5.301).

Next, differentiate u in (5.300) with respect to x to get

$$\frac{\partial u}{\partial x}(x,t) = -\int_{-\infty}^{\infty} \frac{(x-y)}{2Dt}\frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} f(y)\ dy,$$

so that

$$\frac{\partial u}{\partial x}(x,t) = -\int_{-\infty}^{\infty} p(x-y,t)\frac{(x-y)}{2Dt} f(y)\ dy, \tag{5.308}$$

for $x \in \mathbb{R}$ and $t > 0$.

Proceeding as in the first part of this proof, use the Cauchy–Schwarz inequality (or Jensen's inequality) to obtain from (5.308) that

$$\left|\frac{\partial u}{\partial x}(x,t)\right|^2 \leqslant \int_{-\infty}^{\infty} p(x-y,t)\frac{(x-y)^2}{4D^2t^2}|f(y)|^2\ dy, \tag{5.309}$$

for $x \in \mathbb{R}$ and $t > 0$.

Next, integrate on both sides of (5.309) with respect to x and interchange the order of integration to obtain

$$\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial x}(x,t)\right|^2\ dx \leqslant \frac{1}{4D^2t^2}\int_{-\infty}^{\infty}|f(y)|^2\int_{-\infty}^{\infty}(x-y)^2p(x-y,t)\ dx\ dy, \tag{5.310}$$

for $t > 0$.

Observe that the inner integral in the right–hand side of (5.310) is simply the variance, $2Dt$, of the probability density function $p(x,t)$, so that

$$\int_{-\infty}^{\infty}(x-y)^2p(x-y,t)\ dx = 2Dt, \qquad \text{for all } y \in \mathbb{R} \text{ and } t > 0. \tag{5.311}$$

Putting together (5.310) and (5.311)

$$\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial x}(x,t)\right|^2\ dx \leqslant \frac{1}{2Dt}\int_{-\infty}^{\infty}|f(y)|^2\ dy, \qquad \text{for } t > 0,$$

which implies (5.302) by virtue of (5.299). ■

We will next show that, if in addition to the integrability condition in (5.299) for the initial distribution, f, we also impose the conditions (5.301) and (5.302) on the initial value problem (5.245), then any solution must be of the form given in (5.246). This amounts to showing that the initial value problem (5.245) in which the initial condition satisfies (5.299), together with the integrability condition in (5.301) and (5.302), has a unique solution. We will need the estimate in the following lemma when we prove uniqueness.

Lemma 5.2.10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (5.299). Let v be any solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & \text{for } x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & \text{for } \ x \in \mathbb{R}; \\ \displaystyle\int_{-\infty}^{\infty}|u(x,t)|^2\ dx < \infty, & \text{for all } t > 0; \\ \displaystyle\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial x}(x,t)\right|^2\ dx < \infty, & \text{for all } t > 0. \end{cases} \tag{5.312}$$

Then,

$$\int_{-\infty}^{\infty} |v(x,t)|^2\ dx \leqslant \int_{-\infty}^{\infty} |f(x)|^2\ dx, \quad \text{for } t \geqslant 0. \tag{5.313}$$

Proof: Let v denote any solution to the problem (5.312), where f satisfies the integrability condition in (5.299).

In order to establish (5.313), set

$$E(t) = \int_{-\infty}^{\infty} |v(x,t)|^2\ dx, \quad \text{for all } t \geqslant 0. \tag{5.314}$$

It follows from the integrability condition in (5.312) that $E(t)$ in (5.314) is well defined for all $t \geqslant 0$ as a real valued function, $E\colon [0,\infty) \to \mathbb{R}$. Note also that

$$E(0) = \int_{-\infty}^{\infty} |f(x)|^2\ dx, \tag{5.315}$$

by virtue of the initial condition in problem (5.312).

Next, observe that, since v satisfies the diffusion equation in (5.312), that is

$$v_t = Dv_{xx},$$

then E is differentiable and

$$E'(t) = \int_{-\infty}^{\infty} 2v(x,t)v_t(x,t)\ dx = 2D\int_{-\infty}^{\infty} v(x,t)v_{xx}(x,t)\ dx, \tag{5.316}$$

for $t > 0$.

We note that the integrability conditions in (5.312) imply that

$$\lim_{x\to\infty} v(x,t) = 0 \text{ and } \lim_{x\to-\infty} v(x,t) = 0, \quad \text{for } t > 0, \tag{5.317}$$

and

$$\lim_{x\to\infty} v_x(x,t) = 0 \text{ and } \lim_{x\to-\infty} v_x(x,t) = 0. \quad \text{for } t > 0, \tag{5.318}$$

Integrate by parts the last integral in (5.316) to get

$$E'(t) = \lim_{R\to\infty}\left[v(R,t)v_x(R,t) - v(-R,t)v_x(-R,t) - \int_{-R}^{R}(v_x(x,t))^2\ dx\right],$$

so that

$$E'(t) = -\int_{-\infty}^{\infty}\left|\frac{\partial v}{\partial x}(x,t)\right|^2\ dx, \quad \text{for } t > 0, \tag{5.319}$$

by virtue of (5.317), (5.318) and the last integrability condition in (5.312).

Now, it follows from (5.319) that

$$E'(t) \leqslant 0, \quad \text{for all } t > 0,$$

so that E is nondecreasing in t and therefore

$$E(t) \leqslant E(0), \quad \text{for all } t > 0. \tag{5.320}$$

The estimate in (5.313) follows from (5.320) in view of (5.314) and (5.315). ■

Proposition 5.2.11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (5.299). The problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & \text{for } x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & \text{for } \ x \in \mathbb{R}; \\ \displaystyle\int_{-\infty}^{\infty} |u(x,t)|^2\ dx < \infty, & \text{for all } t > 0; \\ \displaystyle\int_{-\infty}^{\infty} \left|\frac{\partial u}{\partial x}(x,t)\right|^2\ dx < \infty, & \text{for all } t > 0, \end{cases} \tag{5.321}$$

has at most one solution.

Proof: Let v be any solution of the problem in (5.321) and let u be given by (5.300). It follows from Proposition 5.2.3 and Proposition 5.2.9 that u solves problem (5.321). Put

$$w(x,t) = v(x,t) - u(x,t), \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.322}$$

It follows from the linearity of the differential equation in (5.321) that w also solves the diffusion equation; indeed,

$$w_t = v_t - u_t = Dv_{xx} - Du_{xx} = D(v_{xx} - u_{xx}) = Dw_{xx}.$$

The function w defined in (5.322) also satisfies the integrability condition in problem (5.321) ; in fact, by the triangle inequality,

$$|w(x,t)| \leqslant |v(x,t)| + |u(x,t)|,$$

so that

$$|w(x,t)|^2 \leqslant |v(x,t)|^2 + 2|v(x,t)| \cdot |u(x,t)| + |u(x,t)|^2, \tag{5.323}$$

for all $x \in \mathbb{R}$ and all $t > 0$. Next, use the inequality

$$2ab \leqslant a^2 + b^2, \qquad \text{for } a, b \in \mathbb{R},$$

in (5.323) to get

$$|w(x,t)|^2 \leqslant 2\left[|v(x,t)|^2 + |u(x,t)|^2\right], \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{5.324}$$

Integrating on both sides of (5.324) with respect to x we then obtain that

$$\int_{-\infty}^{\infty} |w(x,t)|^2\ dx \leqslant 2\left[\int_{-\infty}^{\infty} |v(x,t)|^2\ dx + \int_{-\infty}^{\infty} |u(x,t)|^2\ dx\right], \qquad \text{for } t > 0,$$

so that

$$\int_{-\infty}^{\infty} |w(x,t)|^2\ dx < \infty, \qquad \text{for } t > 0,$$

since both u and v satisfy the integrability conditions in problem (5.321). Similarly, we can show that

$$\int_{-\infty}^{\infty} |w_x(x,t)|^2 \, dx < \infty, \quad \text{for } t > 0.$$

Now, observe that, since both v and u satisfy the initial condition in problem (5.321,

$$w(x,0) = v(x,0) - u(x,0) = f(x) - f(x) = 0, \quad \text{for all } x \in \mathbb{R},$$

so that w is a solution of problem (5.312) in which the initial condition is the constant function 0, it follows from the estimate (5.313) in Lemma 5.2.10 that

$$\int_{-\infty}^{\infty} |w(x,t)|^2 \, dx \leqslant 0, \quad \text{for } t \geqslant 0,$$

from which we get that

$$\int_{-\infty}^{\infty} |w(x,t)|^2 \, dx = 0, \quad \text{for } t \geqslant 0. \tag{5.325}$$

It follows from (5.325) and the continuity of w that

$$w(x,t) = 0, \quad \text{for all } x \in \mathbb{R} \text{ and } t \geqslant 0,$$

so that

$$v(x,t) = u(x,t), \quad \text{for all } x \in \mathbb{R} \text{ and } t \geqslant 0,$$

in view of the definition of w in (5.322). Hence, any solution to the problem in (5.321) must be that given by (5.300). ■

We will next show that, if u is any solution of problem (5.321), where f satisfies the integrability condition

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty, \tag{5.326}$$

then

$$\lim_{t\to\infty} u(x,t) = 0, \quad \text{for all } x \in \mathbb{R}. \tag{5.327}$$

To see why this is the case, apply Proposition 5.2.11 to write

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} f(y) \, dy,$$

for all $x \in \mathbb{R}$ and $t > 0$, from which we get that

$$|u(x,t)| \leqslant \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}} |f(y)| \, dy, \tag{5.328}$$

for all $x \in \mathbb{R}$ and $t > 0$. Next, square on both sides of (5.328) and apply the Cauchy–Schwarz inequality to get

$$|u(x,t)|^2 \leqslant \frac{1}{\sqrt{8\pi Dt}} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2Dt}}{\sqrt{2\pi Dt}}\, dy \int_{-\infty}^{\infty} |f(y)|^2\, dy, \tag{5.329}$$

where

$$\int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2Dt}}{\sqrt{2\pi Dt}}\, dy = 1. \tag{5.330}$$

Combining (5.329) and (5.330), we then get

$$|u(x,t)|^2 \leqslant \frac{1}{\sqrt{8\pi Dt}} \int_{-\infty}^{\infty} |f(y)|^2\, dy, \tag{5.331}$$

for $x \in \mathbb{R}$ and $t > 0$.

It follows from (5.326) and (5.331) that

$$\lim_{t\to\infty} |u(x,t)|^2 = 0, \qquad \text{for all } x \in \mathbb{R},$$

which implies (5.327).

5.3 Solving the Dirichlet Problem in the Unit Disc

The goal of this section is to construct a solution of the boundary value problem for the two–dimensional Laplacian

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } D_1; \\ u(x,y) = g(x,y), & \text{for } (x,y) \in \partial D_1, \end{cases} \tag{5.332}$$

where $D_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is the unit disk in $\mathbb{R}^2$, and g is a given function that is continuous in a neighborhood of the unit circle ∂D_1. A function $u \in C^2(D_1, \mathbb{R})$ that satisfies the PDE in Problem (5.332) is said to be **harmonic**. Thus, we would like to find a function, u, that is harmonic in D_1 and that takes on the values given by a continuous function, g, on the boundary of D_1.

To construct a solution of the Dirichlet problem in (5.3) we use the same procedure that we used to construct a solution of the Vibrating String Problem in Section 5.1. This will serve as another another illustration of the methods of separation of variables and eigenfunctions expansion. This approach has been particularly successful in the construction of solutions of boundary value problems for linear PDEs over domains with simple geometry.

The discussion here will parallel that of the Vibrating String Problem in Section 5.1.

- First, in view of the radial symmetry of the domain, we will express problem (5.3) in polar coordinate r and θ.

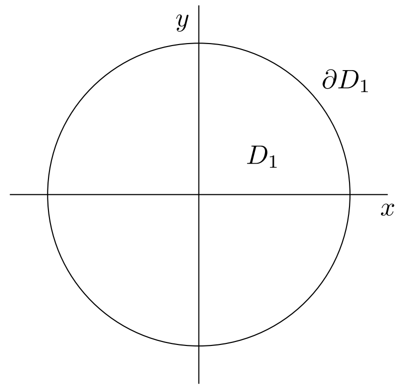

Figure 5.3.5: Unit Disk in $\mathbb{R}^2$

- Next, we look for a special type of solutions that are products of a function of r and a function of θ. In other words, we look for solutions in which the variables **separate**; this is where we use the method of **separation of variables**.

- When looking for solutions that are nonzero over the domain by means of separation of variables, we are invariable led to an **eigenvalue problem**. Solution of the eigenvalue problem leads to a family of solutions in one (or both of the variables), called **eigenfunctions**. These eigenfunctions generate a special family of solutions.

- We will then use the principle of superposition to construct linear combinations of the eigenfunction solutions. We hope that a sequence of these linear combinations will converge to a function that solves the PDE in (5.332) and satisfies the boundary condition in that problem; here is where we use the method of **eigenfunctions expansion**.

5.3.1 Separation of Variables

In view of the radial symmetry of the domain (see Figure 5.3.5), we will treat the problem in polar coordinates, (r, θ), where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

We will also exploit the linearity of the PDE and the boundary condition in (5.332) and use the principle of superposition to construct a solution of the problem by superposing simple solutions of the problem. The strategy then is to, first, find a special class of functions of r and θ that solve Laplace's equation, and then use sums of those solutions to construct a solution that also satisfies the boundary condition.

We begin by expressing the BVP (5.332) in polar coordinates:

$$\begin{cases} \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0, & 0 < r < 1, -\pi < \theta < \pi; \\ v(1,\theta) = g(\cos\theta, \sin\theta), & -\pi \leqslant \theta \leqslant \pi, \end{cases} \tag{5.333}$$

where we have set

$$v(r,\theta) = u(r\cos\theta, r\sin\theta).$$

We will denote $g(\cos\theta, \sin\theta)$ by $f(\theta)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous, periodic function of period 2π. We can then rewrite the BVP in (5.333) as

$$\begin{cases} \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0, & 0 < r < 1, -\pi < \theta < \pi; \\ v(1,\theta) = f(\theta), & -\pi \leqslant \theta \leqslant \pi. \end{cases} \tag{5.334}$$

We start out by looking for special solutions of the PDE in (5.334) of the form

$$v(r,\theta) = h(r)z(\theta), \quad \text{for } r \geqslant 0 \text{ and } -\pi < \theta < \pi, \tag{5.335}$$

where $h\colon [0,\infty) \to \mathbb{R}$ is a continuous functions that is C^2 in $(0,\infty)$, and $z\colon \mathbb{R} \to \mathbb{R}$ is a C^2, periodic function of period 2π. We can therefore compute the partial derivatives,

$$\frac{\partial v}{\partial r}(r,\theta) = h'(r)z(\theta), \quad r > 0, -\pi < \theta \leqslant \pi;$$

$$\frac{\partial^2 v}{\partial r^2}(r,\theta) = h''(r)z(\theta), \quad r > 0, -\pi < \theta \leqslant \pi;$$

$$\frac{\partial^2 v}{\partial \theta^2}(r,\theta) = h(r)z''(\theta), \quad r > 0, -\pi < \theta \leqslant \pi,$$

and substitute them into the PDE in (5.334) to obtain

$$h''(r)z(\theta) + \frac{1}{r}h'(r)z(\theta) + \frac{1}{r^2}h(r)z''(\theta) = 0, \quad \text{for } r > 0, -\pi < \theta \leqslant \pi. \tag{5.336}$$

Assuming that $v(r,\theta)$ is not zero for all values of r and θ, and dividing on both sides of (5.336) by $v(r,\theta)$ as given in (5.335), we obtain

$$\frac{h''(r)}{h(r)} + \frac{1}{r}\frac{h'(r)}{h(r)} + \frac{1}{r^2}\frac{z''(\theta)}{z(\theta)} = 0, \quad \text{for } 0 < r < 1, -\pi < \theta \leqslant \pi. \tag{5.337}$$

Multiplying on both sides of the equation in (5.337) by r^2, we notice that that equation can be written in such a way that the functions that depend only on r are on one side of the equation and those that depend only on θ are on the other side of the equation:

$$r^2\frac{h''(r)}{h(r)} + r\frac{h'(r)}{h(r)} = -\frac{z''(\theta)}{z(\theta)}, \quad \text{for } 0 < r < 1, -\pi < \theta \leqslant \pi. \tag{5.338}$$

Since (5.338) holds true for all values of r and θ in $(0,1)$ and $(-\pi,\pi]$, respectively, it follows from (5.338) that each side of the equation in (5.338) must be equal to a constant.[2] Call that constant λ so that

$$r^2\frac{h''(r)}{f(r)} + r\frac{h'(r)}{h(r)} = -\frac{z''(\theta)}{z(\theta)} = \lambda, \quad \text{for } 0 < r < 1, -\pi < \theta \leqslant \pi. \tag{5.339}$$

The expression in (5.339) leads to two ordinary differential equations

$$-z''(\theta) = \lambda z(\theta), \quad \text{for } -\pi < \theta \leqslant \pi, \tag{5.340}$$

and

$$r^2 h''(r) + r h'(r) = \lambda h(r), \quad \text{for } r > 0. \tag{5.341}$$

The requirement that the function g in (5.334) be periodic of period 2π yields the following conditions for z:

$$z(-\pi) = z(\pi) \quad \text{and} \quad z'(-\pi) = z'(\pi); \tag{5.342}$$

in other words, we will assume that z can be extended to a C^2 periodic function defined on $\mathbb{R}$ with period 2π. Putting together (5.340) and (5.342) yields the following **two–point boundary value problem**:

$$\begin{cases} -z''(\theta) &= \lambda z(\theta), \quad \text{for } -\pi < \theta < \pi; \\ z(-\pi) &= z(\pi); \\ z'(-\pi) &= z'(\pi). \end{cases} \tag{5.343}$$

5.3.2 An Eigenvalue Problem

Observe that the constant function $z(\theta) = 0$, for all values of θ, solves the two–point BVP in (5.343); we shall refer to this solution as the **trivial** solution. We are interested in **nontrivial** solutions of (5.343); otherwise, the special solutions in (5.335) of the BVP in (5.334) that we are seeking would all be the zero function. These solutions will not be helpful in the construction of a solution of the BVP in (5.334) for arbitrary (nonzero) boundary conditions. We will see shortly that the answer to the question of whether or not the two–point BVP in (5.343) has nontrivial solutions depends on the value of λ in the ODE in that problem. In fact, there is a certain set of values of λ for which (5.343) has nontrivial solutions; for the rest of the values of λ the two–point BVP (5.343) has only the trivial solution.

[2] To see why this assertion is true, pick θ_o in $(-\pi,\pi]$ such that $z(\theta_o) \neq 0$; then, by virtue of (5.338), $r^2\frac{h''(r)}{h(r)} + r\frac{h'(r)}{h(r)} = -\frac{z''(\theta_o)}{z(\theta_o)}$, for all $r > 0$; so that the left–hand side of (5.338) is constant. Similarly, for fixed r_o in $(0,1)$ with $f(r_o) \neq 0$, (5.338) implies that $\frac{z''(\theta)}{z(\theta)} = -r_o^2\frac{h''(r_o)}{h(r_o)} - r_o\frac{h'(r_o)}{h(r_o)}$, for all θ in $(-\pi,\pi]$, so that the right–hand side of (5.338) must also be constant.

Definition 5.3.1 (Eigenvalues and Eigenfunctions)**.** A value of λ in (5.343) for which the two–point BVP in (5.343) has a nontrivial is called an **eigenvalue** of the BVP; a corresponding nontrivial solution is called an **eigenfunction**.

We will next compute the eigenvalues and eigenfunctions of the two–point BVP in (5.343). Before we proceed with the calculations, it will be helpful to know that the eigenvalues of (5.343) must be nonnegative. We state that fact in the following proposition.

Proposition 5.3.2. Assume that the two–point BVP (5.343) has nontrivial solution. Then, $\lambda \geqslant 0$.

Proof: Let z be a nontrivial solution of (5.343). Multiply the ODE in (5.343) by z and integrate from $-\pi$ to π to get

$$-\int_{-\pi}^{\pi} z''(\theta)z(\theta)\ d\theta = \lambda \int_{-\pi}^{\pi} z(\theta)z(\theta)\ d\theta. \tag{5.344}$$

Use integration by parts to evaluate the left–most integral in (5.344) to get

$$\int_{-\pi}^{\pi} z''(\theta)z(\theta)\ d\theta = z(\theta)z'(\theta)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} z'(\theta)z'(\theta)\ d\theta,$$

so that, in view of the boundary conditions in (5.343),

$$\int_{-\pi}^{\pi} z''(\theta)z(\theta)\ d\theta = -\int_{-\pi}^{\pi} [z'(\theta)]^2\ d\theta. \tag{5.345}$$

Substituting the result in (5.345) into the left–hand side of (5.344) then yields

$$\int_{-\pi}^{\pi} [z'(\theta)]^2\ d\theta = \lambda \int_{-\pi}^{\pi} [z(\theta)]^2\ d\theta. \tag{5.346}$$

Since z is a nontrivial solution of the two–point BVP in (5.343), it follows that $\displaystyle\int_{-\pi}^{\pi} [z(\theta)]^2\ d\theta > 0$. We can therefore solve (5.346) for λ to obtain

$$\lambda = \frac{\displaystyle\int_{-\pi}^{\pi} [z'(\theta)]^2\ d\theta}{\displaystyle\int_{-\pi}^{\pi} [z(\theta)]^2\ d\theta},$$

which shows that λ is nonnegative. ■

In view of the result of Proposition 5.3.2, it suffices to look for nontrivial solutions of (5.343) for either $\lambda = 0$ or $\lambda > 0$.

For the case in which $\lambda = 0$ in (5.343), the ODE in (5.343) becomes

$$z''(\theta) = 0,$$

which has general solution

$$z(\theta) = c_1\theta + c_2, \tag{5.347}$$

for arbitrary constants c_1 and c_2.

Applying the first boundary condition to z given in (5.347) yields

$$-\pi c_1 + c_2 = \pi c_2 + c_2,$$

from which we get that $2\pi c_1 = 0$, so that $c_1 = 0$. It then follows from (5.347) any solution of the BVP in (5.343) with $\lambda = 0$ must be constant:

$$z(\theta) = c, \qquad \text{for all } \theta. \tag{5.348}$$

In particular, if $c \neq 0$ in (5.348), $z(\theta) = c$ for all θ is a nontrivial solution of the two–point BVP (5.343). Consequently, $\lambda = 0$ is an eigenvalue of (5.343). For future reference, we shall denote this eigenvalue by λ_o, so that

$$\lambda_o = 0, \tag{5.349}$$

and we shall pick the special eigenfunction

$$\varphi_o(\theta) = 1, \qquad \text{for all } \theta, \tag{5.350}$$

and note that any solution of the BVP in (5.343) for λ_o is a constant multiple of φ_o given in (5.350); so that

$$z_o(\theta) = a_o, \qquad \text{for all } \theta, \tag{5.351}$$

where a_o denotes a real constant, represents all solutions of the two–point BVP in (5.350) corresponding to the eigenvalue $\lambda_o = 0$.

Next, we look for positive eigenvalues of the BVP in (5.343). For the case in which $\lambda > 0$ in (5.343), the general solution of the ODE in (5.343) is

$$z(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta), \qquad \text{for all } \theta, \tag{5.352}$$

and arbitrary constants c_1 and c_2, so that

$$z'(\theta) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\theta) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\theta), \qquad \text{for all } \theta. \tag{5.353}$$

Imposing the the boundary conditions in (5.343) to the functions given in (5.352) and (5.353) yields the system of equations

$$\left\{\begin{array}{rcl} c_1\cos(-\sqrt{\lambda}\pi) + c_2\sin(-\sqrt{\lambda}\pi) & = & c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi); \\ -c_1\sqrt{\lambda}\sin(-\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(-\sqrt{\lambda}\pi) & = & -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi); \end{array}\right. \tag{5.354}$$

thus, dividing the second equation in (5.354) by $\sqrt{\lambda}$, since $\lambda > 0$, and using the fact that cos is even and sin is odd,

$$\left\{\begin{array}{rcl} c_1\cos(\sqrt{\lambda}\pi) - c_2\sin(-\sqrt{\lambda}\pi) & = & c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi); \\ c_1\sin(\sqrt{\lambda}\pi) + c_2\cos(\sqrt{\lambda}\pi) & = & -c_1\sin(\sqrt{\lambda}\pi) + c_2\cos(\sqrt{\lambda}\pi), \end{array}\right.$$

from which we get that

$$\begin{cases} 2c_2\sin(\sqrt{\lambda}\pi) & = & 0; \\ 2c_1\sin(\sqrt{\lambda}\pi) & = & 0. \end{cases} \tag{5.355}$$

Since we are looking for nontrivial solutions of (5.343), we require that c_1 and c_2 in (5.352) are not both zero. Consequently, we obtain from (5.355) that

$$\sin(\sqrt{\lambda}\pi) = 0. \tag{5.356}$$

Solutions of the trigonometric equation in (5.356) are given by

$$\sqrt{\lambda}\pi = n\pi \tag{5.357}$$

where n is an integer. It follows from (5.357) that the positive eigenvalues of the BVP in (5.343) are given by

$$\lambda = n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.358}$$

We will denote the positive eigenvalues of the BVP (5.343) in (5.358) by λ_n, for $n = 1, 2, 3, \ldots$, so that

$$\lambda_n = n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{5.359}$$

We will denote the corresponding eigenfunctions by z_n. These are linear combinations of $\cos(n\theta)$ and $\sin(n\theta)$, so that

$$z_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta) \quad \text{for } n = 1, 2, 3, \ldots, \text{ and } \theta \in \mathbb{R}, \tag{5.360}$$

where a_n and b_n, for $n = 1, 2, 3\ldots$, are real constants.

We shall put together the results in (5.349), (5.351), (5.359) and (5.360) in the following proposition:

Proposition 5.3.3 (Eigenvalues and eigenfunctions of BVP (5.343))**.** The eigenvalues of the two–point BVP (5.343) are given by

$$\lambda_n = n^2, \quad \text{for } n = 0, 1, 2, 3, \ldots, \tag{5.361}$$

with corresponding eigenfunctions of the form

$$\varphi_o(\theta) = 1, \quad \text{for all } \theta \in \mathbb{R},$$

corresponding to $\lambda_o = 0$, and

$$\varphi_{n1}(\theta) = \cos(n\theta) \;\text{ and }\; \varphi_{n2}(\theta) = \sin(n\theta) \quad \text{for } \theta \in \mathbb{R} \text{ and } n = 1, 2, 3, \ldots,$$

corresponding to $\lambda_n = n^2$, for $n = 1, 2, 3, \ldots$.

The corresponding eigenspaces are made up of the funnctions

$$z_o(\theta) = a_o, \quad \text{for all } \theta \in \mathbb{R},$$

and arbitrary real numbers a_o, and

$$z_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta) \quad \text{for } \theta \in \mathbb{R} \text{ and } n = 1, 2, 3, \ldots,$$

where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are arbitrary real constants.

With the values for λ given in (5.361), we now proceed to solve the ODE in (5.341) for the radial component of the special solutions of the BVP in (5.334) of the form given in (5.335); namely,

$$r^2 h''(r) + r h'(r) = n^2 h(r), \quad \text{for } r > 0 \text{ and } n = 0, 1, 2, \ldots \tag{5.362}$$

We shall first solve (5.362) for the case $n = 0$. In this case the equation becomes

$$r h''(r) + h'(r) = 0, \quad \text{for } r > 0, \tag{5.363}$$

where we have divided by $r > 0$. Observe that the equation in (5.363) can be written as

$$\frac{d}{dr}[r h'(r)] = 0, \quad \text{for } r > 0,$$

which can be integrated to yield

$$r h'(r) = c_1, \quad \text{for } r > 0,$$

and some constant c_1, or

$$h'(r) = \frac{c_1}{r}, \quad \text{for } r > 0, \tag{5.364}$$

and some constant c_1. Integrating the equation in (5.364) then yields

$$h(r) = c_1 \ln(r) + c_2, \quad \text{for } r > 0, \tag{5.365}$$

and some constants c_1 and c_2. Observe that, if $c_1 \neq 0$ in (5.365), the function h given a (5.365) is unbounded as $r \to 0^+$. Thus, since we are looking for C^2 functions defined in the unit disc, D_1, we must set c_1 equal to 0. This is equivalent to imposing the following boundary condition on h:

$$\lim_{r \to 0^+} h(r) \text{ exists.} \tag{5.366}$$

Hence, it follows from (5.365) and (5.366) that, for $n = 0$, a solution of (5.362) is given by

$$h(r) = c, \quad \text{for all } r, \tag{5.367}$$

is a solution, for some constant c. Taking $c = 1$ in (5.367) we get the solution of (5.362) corresponding to $n = 0$:

$$h_o(r) = 1, \quad \text{for all } r. \tag{5.368}$$

Next, consider the case $n \geqslant 1$ in (5.362). In this case the differential equation in (5.362) is an ODE of Euler type:

$$r^2 h''(r) + r h'(r) - n^2 h(r) = 0, \quad \text{for } r > 0. \tag{5.369}$$

The ODE in (5.369) can be solved by looking for solutions of the form

$$h(r) = r^q, \quad \text{for } r > 0, \tag{5.370}$$

and some real number q.

Taking derivatives of h in (5.370) and substituting into (5.370) yields

$$r^2 q(q-1)r^{q-2} + rqr^{q-1} - n^2 r^q = 0, \quad \text{for } r > 0,$$

or

$$q(q-1)r^q + qr^q - n^2 r^q = 0, \quad \text{for } r > 0,$$

or

$$[q(q-1) + q - n^2]r^q = 0, \quad \text{for } r > 0. \tag{5.371}$$

It follows from (5.371) that

$$q(q-1) + q - n^2 = 0,$$

or

$$q^2 - n^2 = 0,$$

or

$$(q+n)(q-n) = 0,$$

from which we get that

$$q = \pm n. \tag{5.372}$$

It follows from (5.371) and (5.372) that

$$h_{-n}(r) = r^{-n} \quad \text{and} \quad h_n(r) = r^n, \quad \text{for } r > 0. \tag{5.373}$$

In view of the boundary condition in (5.366), we take the second solution in (5.373),

$$h_n(r) = r^n, \quad \text{for all } r \text{ and } n = 1, 2, 3, \ldots. \tag{5.374}$$

Putting together (5.374), (5.368), (5.360), (5.351), and (5.335), we conclude that we have found an infinite collection of solutions of the PDE in (5.334); namely,

$$v_o(r,\theta) = a_o, \quad \text{for all } r \text{ and } \theta; \tag{5.375}$$

$$v_n(r,\theta) = r^n[a_n \cos(n\theta) + b_n \sin(n\theta)], \quad \text{for all } r \text{ and } \theta, \tag{5.376}$$

where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are real constants.

5.3.3 Expansion in Terms of Eigenfunctions

None of the functions in (5.375) and (5.376) by itself will satisfy the general boundary condition in (5.334). We can, however, attempt to construct a solution of (5.334) by adding all of them together; in other words, by applying the principle of superposition:

$$v(r,\theta) = a_o + \sum_{n=0}^{\infty} r^n[a_n \cos(n\theta) + b_n \sin(n\theta)], \; 0 \leqslant r < 1, \; -\pi < \theta \leqslant \pi. \tag{5.377}$$

provided the series in (5.377) converges to a C^2 function.

Let's assume for the moment that the series in (5.377) converges also for $r = 1$, so that we can apply the boundary condition in (5.334) to get

$$a_o + \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)] = f(\theta), \qquad \text{for } -\pi < \theta \leqslant \pi. \tag{5.378}$$

Assuming for the moment that the series on the left–hand side of (5.378) converges in such a way that it can be integrated term by term, we can compute the values of the coefficients a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, in terms of the function f by means of the following integration facts:

$$\int_{-\pi}^{\pi} \sin(n\theta)\cos(m\theta)\ d\theta = 0, \qquad \text{for all } m, n = 1, 2, 3, \ldots; \tag{5.379}$$

$$\int_{-\pi}^{\pi} \cos(n\theta)\cos(m\theta)\ d\theta = \begin{cases} 0, & \text{if } m \neq n; \\ \pi, & \text{if } m = n; \end{cases} \tag{5.380}$$

and

$$\int_{-\pi}^{\pi} \sin(n\theta)\sin(m\theta)\ d\theta = \begin{cases} 0, & \text{if } m \neq n; \\ \pi, & \text{if } m = n. \end{cases} \tag{5.381}$$

Indeed, integrating on both sides of (5.378) from $-\pi$ to π we get, assuming that the series in (5.378) can be integrated term by term,

$$2\pi a_o = \int_{-\pi}^{\pi} f(\theta)\ d\theta,$$

from which we get

$$a_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\ d\theta; \tag{5.382}$$

thus, a_o is the average value of f over the interval $(-\pi, \pi]$.

Next, multiply the equation in (5.378) on both sides by $\cos(m\theta)$ to obtain

$$a_o \cos m\theta + \sum_{n=0}^{\infty} [a_n \cos n\theta \cos m\theta + b_n \sin n\theta \cos m\theta] = f(\theta)\cos m\theta. \tag{5.383}$$

Then, integrate on both sides of (5.383) with respect to θ from $-\pi$ to π, and use the identities in (5.379) and (5.380) to get

$$\pi a_m = \int_{-\pi}^{\pi} f(\theta)\cos(m\theta)\ d\theta,$$

from which we get

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta)\cos(m\theta)\ d\theta, \qquad \text{for } m = 1, 2, 3, \ldots. \tag{5.384}$$

Similar calculations (this time multiplying the equation in (5.378) on both sides by $\sin(m\theta)$, integrating from $-\pi$ to π, and using the integral identities in (5.379) and (5.381)) lead to

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(m\theta)\ d\theta, \qquad \text{for } m = 1, 2, 3, \ldots. \tag{5.385}$$

The numbers defined in (5.382), (5.384) and (5.385) are the **Fourier coefficients** of the 2π–periodic function f. Note that the Fourier coefficients of f are defined whenever f is **absolutely integrable** over the interval $[-\pi, \pi]$.

Definition 5.3.4 (Absolute Integrability)**.** A function $f\colon [-\pi, \pi] \to \mathbb{R}$ is said to be absolutely integrable over $[-\pi, \pi]$ whenever

$$\int_{-\pi}^{\pi} |f(\theta)|\ d\theta < \infty. \tag{5.386}$$

Note that f doesn't have to be continuous for (5.386) to hold true. For instance, if f is bounded and piecewise continuous then (5.386) holds true; indeed, suppose that f piecewise continuous and

$$|f(\theta)| \leqslant M, \qquad \text{for } \theta \in [-\pi, \pi],$$

and some positive constant M; then

$$\int_{-\pi}^{\pi} |f(\theta)|\ d\theta \leqslant \int_{-\pi}^{\pi} M\ d\theta = 2\pi M < \infty.$$

Notation 5.3.5. We will denote the integral in (5.386) by $\|f\|_{L^1}$; so that

$$\|f\|_{L^1} = \int_{-\pi}^{\pi} |f(\theta)|\ d\theta. \tag{5.387}$$

If the integral in (5.387) ia understood as the Lebesgue integral, and $\|f\|_{L^1} < \infty$, we will say that f is an L^1 function and write $f \in L^1(-\pi, \pi)$. We shall refer to $\|f\|_{L^1}$ as the L^1 **norm of** $f \in L^1(-\pi, \pi)$.

The existence of the Fourier coefficients of f in (5.382), (5.384) and (5.385) is guaranteed for absolutely integrable 2π–periodic functions, f, or for $f \in L^1(-\pi, \pi)$. This is the content of the following proposition.

Proposition 5.3.6 (Existence of the Fourier Coefficients)**.** Let a_n, for $n = 0, 1, 2, \ldots$, be as given in (5.382) and (5.384), and b_n, for $n = 1, 2, 3, \ldots$, be as in (5.385), where $f \in L^1(-\pi, \pi)$. Then,

$$|a_n| \leqslant \frac{1}{\pi}\|f\|_{L^1}, \qquad \text{for } n = 0, 1, 2, 3, \ldots; \tag{5.388}$$

and

$$|b_n| \leqslant \frac{1}{\pi}\|f\|_{L^1}, \qquad \text{for } n = 1, 2, 3, \ldots. \tag{5.389}$$

Proof: The estimates in (5.388) and (5.389) follow from properties of the integral. For a_o, we get from (5.382) that

$$|a_o| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|\ d\theta,$$

so that, using the definition of the L^1 norm of f in (5.387),

$$|a_o| \leqslant \frac{1}{2\pi}\|f\|_{L^1} \leqslant \frac{1}{\pi}\|f\|_{L^1}.$$

For $n = 1, 2, 3, \ldots$ we obtain from (5.384) that

$$\begin{aligned} |a_n| &\leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |f(\theta)|\ |\cos(n\theta)|\ d\theta \\ &\leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |f(\theta)|\ d\theta, \end{aligned}$$

since $|\cos(n\theta)| \leqslant 1$ for all θ and all n, which yields (5.388). Similar calculations lead to (5.389). ■

It follows from Proposition 5.3.6 that the sequences of Fourier coefficients, (a_n) and (b_n), of f are bounded by a constant depending on the L^1 norm of f. In fact, it can be shown that the Fourier coefficients of an L^1, 2π–periodic functions tend to 0 as n goes to infinity; this is known as the Riemann–Lebesgue Lemma.

Proposition 5.3.7 (Riemann–Lebesgue Lemma)**.** Let a_n, for $n = 0, 1, 2, \ldots$, be as given in (5.382) and (5.384), and b_n, for $n = 1, 2, 3, \ldots$, be as in (5.385), where $f \in L^1(-\pi, \pi)$. Then,

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

For a proof of the Riemann-Lebesgue Lemma, see [Tol62].

We will next use the result of Proposition 5.3.6 to prove that the series defining the function v in (5.377) converges in D_1.

Proposition 5.3.8 (Point–wise Convergence of Series in (5.377))**.** Let a_n, for $n = 0, 1, 2, \ldots$, be as given in (5.382) and (5.384), and b_n, for $n = 1, 2, 3, \ldots$ be as in (5.385), where $f \in L^1(-\pi, \pi)$. Then, the series defining v in (5.377) converges absolutely in D_1.

Proof: The conclusion will follow by comparing with the geometric series since $0 \leqslant r < 1$ and

$$|r^n a_n \cos(n\theta)| \leqslant r^n |a_n| \leqslant \frac{\|f\|_{L^1}}{\pi} r^n, \quad \text{for all } n,$$

where we have used the estimate (5.388) in Proposition 5.3.6. Similarly, using (5.389) in Proposition 5.3.6,

$$|r^n b_n \sin(n\theta)| \leqslant \frac{\|f\|_{L^1}}{\pi} r^n,$$

for all n. ■

Proposition 5.3.8 allows us to conclude that the function v given in (5.377) is well defined. However, in order to prove that that function is harmonic in D_1, we have to be able to differentiate the series term by term. This would be possible, for instance, if we knew that the series on the right–hand–side of (5.377), and the series for the partial derivatives

$$\sum_{n=0}^{\infty} nr^n[-a_n \sin(n\theta) + b_n \cos(n\theta)],$$

$$-\sum_{n=0}^{\infty} n^2 r^n[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

$$\sum_{n=0}^{\infty} nr^{n-1}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

and

$$\sum_{n=0}^{\infty} n(n-1)r^{n-2}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

converge uniformly. However, we do not know that at this point. In order to answer these questions, though, we will have to make further assumptions on f. Before we deal with these questions, we will first answer the question of when the trigonometric series on the left–hand side of the equation in (5.378) converges uniformly. Uniform convergence will justify the term–by–term integration that was done in order to obtain the formulas in (5.382), (5.384) and (5.385). We will denote the trigonometric series on the left–hand side of the equation in (5.378) by $\widehat{f}(\theta)$, so that

$$\widehat{f}(\theta) = a_o + \sum_{n=0}^{\infty}[a_n \cos(n\theta) + b_n \sin(n\theta)], \quad \text{for } -\pi \leqslant \theta \leqslant \pi, \tag{5.390}$$

where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of f. Let $(\widehat{f}_n(\theta))$ denote the sequence of partial sums of the series in (5.390) so that

$$\widehat{f}_n(\theta) = a_o + \sum_{k=0}^{n}[a_k \cos(k\theta) + b_k \sin(k\theta)], \quad \text{for } -\pi \leqslant \theta \leqslant \pi. \tag{5.391}$$

Definition 5.3.9 (Uniform Convergence)**.** We say that sequence of functions, $(\widehat{f}_n)$, defined in (5.391) converges uniformly to f in $[-\pi, \pi]$ if

$$\lim_{n\to\infty} \max_{-\pi\leqslant\theta\leqslant\pi} |\widehat{f}_n(\theta) - f(\theta)| = 0.$$

The following proposition gives a sufficient condition for the trigonometric series in (5.390) to converge uniformly to f.

Theorem 5.3.10 (Uniform Convergence 3)**.** Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous, 2π–periodic function; assume also that f is piecewise differentiable with $f'\colon \mathbb{R} \to \mathbb{R}$ piecewise continuous. Let $(\widehat{f}_n)$ be the sequence of trigonometric functions defined in (5.390), where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of f. Then, $(\widehat{f}_n)$ converges uniformly to g in $[-\pi, \pi]$ as $n \to \infty$.

The proof of Theorem 5.3.10 follows from the Uniform Convergence Theorem 2 in Theorem 5.1.14, with $L = \pi$, on page 111 in these notes. Note that, if f is piecewise differentiable with piecewise continuous derivative, $f'\colon \mathbb{R} \to \mathbb{R}$, then f' is square–integrable on $[-\pi, \pi]$. (For another proof, see [Tol62, pp. 80-81]).

Let's assume for the moment that $f\colon \mathbb{R} \to \mathbb{R}$ is 2π–periodic, and satisfies the assumptions of 5.3.10. It then follows from Theorem 5.3.10 that the Fourier series on the left–hand side of (5.378), where a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of f, converges uniformly to the right–hand side of the equation. This justifies the term–by–term integration of the series that lead to the formulas for the Fourier coefficients in (5.382), (5.384) and (5.385) by virtue of the following theorem form Analysis:

Theorem 5.3.11 (Term–by–Term Integration)**.** Let (u_k) be a sequence of continuous functions over a closed and bounded interval, $[a, b]$. Assume that the series

$$\sum_{k=1}^{\infty} u_k$$

converges uniformly to f. Then, f is continuous on $[a, b]$, and

$$\int_a^b f(x)\ dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)\ dx,$$

or

$$\int_a^b \left(\sum_{k=1}^{\infty} u_k(x)\right)\ dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)\ dx,$$

For a proof of this theorem refer to Rudin [Rud53, pg. 121–122].

We saw in Proposition 5.3.8 that the trigonometric series defining $v(r, \theta)$ in (5.377),

$$v(r, \theta) = a_o + \sum_{n=0}^{\infty} r^n [a_n \cos n\theta + b_n \sin n\theta],\ 0 \leqslant r \leqslant 1,\ -\pi \leqslant \theta \leqslant \pi, \quad (5.392)$$

converges in D_1, provided that $f \in L^1(-\pi,\pi)$, or absolutely integrable on $[-\pi,\pi]$. In the next proposition we will use the Weierstrass M–Test for uniform convergence, (see [Rud53, Theorem 7.10, pg. 119]), in order to show that, for the case in which f is piecewise C^1 and 2π–periodic, then the series in (5.392), where the a_n, for $n = 0, 1, 2, \ldots$, and b_n, for $n = 1, 2, 3, \ldots$, are the Fourier coefficients of f, converges uniformly in $\overline{D}_1$, the closed unit disk in $\mathbb{R}^2$.

Proposition 5.3.12 (Uniform Convergence of Series in (5.377))**.** Let a_n, for $n = 0, 1, 2, \ldots$, be as given in (5.382) and (5.384), and b_n, for $n = 1, 2, 3, \ldots$ be as in (5.385), where f is a piecewise C^1, 2π–periodic function. Then, the series defining v in (5.392) converges uniformly in $\overline{D}_1$.

Proof: The assumptions that f is piecewise C^1 and 2π–periodic imply that the Fourier coefficients of f satisfy the estimate

$$\sum_{k=0}^{\infty}(|a_k| + |b_k|) < \infty. \tag{5.393}$$

See the calculations leading to the proof of Proposition 5.1.9, with $L = \pi$, on page 106 in these notes for a derivation of the estimate in (5.393).

Next, use the triangle inequality to estimate the absolute values of the terms of the series in (5.392) to get

$$|r^n[a_n\cos(n\theta) + b_n\sin(n\theta)]| \leqslant |a_n| + |b_n|, \quad \text{for all } n = 1, 2, 3, \ldots,$$

and all $r \in [0, 1]$ and $\theta \in [-\pi, \pi]$. Thus, the absolute values of the terms of the series in (5.392) are "majorized" by the terms of the convergent series in (5.393). It then follows by the the Weierstrass M–Test for uniform convergence ([Rud53, Theorem 7.10, pg. 119]) that the series in (5.392) converges uniformly for $r \in [0, 1]$ and $\theta \in [-\pi, \pi]$. ■

We will get a chance to use the Weierstrass M–Test for uniform convergence once again to justify the following calculations based on the trigonometric series representation for $v(r, \theta)$ in (5.392) and the assumption that f is a piecewise C^1, 2π–periodic function.

First, substitute the formulas defining the Fourier coefficients of f in (5.382), (5.384) and (5.385) into the right–hand side (5.392) to get

$$
\begin{aligned}
v(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\ d\xi + \sum_{n=0}^{\infty} r^n \left[\frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\ d\xi\right)\cos(n\theta)\right. \\
&\qquad \left. + \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\ d\xi\right)\sin(n\theta)\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\ d\xi + \frac{1}{\pi}\sum_{n=0}^{\infty} r^n \left[\int_{-\pi}^{\pi}\cos(n\theta)\cos(n\xi)f(\xi)\ d\xi\right. \\
&\qquad \left. + \int_{-\pi}^{\pi}\sin(n\theta)\sin(n\xi)f(\xi)\ d\xi\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\ d\xi \\
&\qquad + \frac{1}{\pi}\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} r^n[\cos(n\theta)\cos(n\xi)+\sin(n\theta)\sin(n\xi)]f(\xi)\ d\xi,
\end{aligned}
$$

which can be written as

$$
v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\ d\xi + \frac{1}{\pi}\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} r^n\cos[n(\theta-\xi)]f(\xi)\ d\xi, \tag{5.394}
$$

for $0 \leqslant r < 1$ and $\theta \in [-\pi,\pi]$, by virtue of the trigonometric identity

$$
\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.
$$

Next, we will interchange the order of integration and summation in (5.394). This is justified by the fact that the series

$$
\sum_{n=1}^{\infty} r^n\cos(n\xi)
$$

converges absolutely and uniformly in $\xi \in [-\pi,\pi]$ for $0 \leqslant r < 1$. To see why this is the case, note that

$$
|r^n\cos(n\xi)| \leqslant r^n, \qquad \text{for all } n = 1,2,3,\ldots
$$

Thus, the assertion follows by the Weierstrass M–Test for uniform convergence, for $0 \leqslant r < 1$.

Hence, interchanging the order of summation and integration in (5.394), we can write

$$
v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1+\sum_{n=1}^{\infty} 2r^n\cos[n(\theta-\xi)]\right] f(\xi)\ d\xi, \tag{5.395}
$$

for $0 \leqslant r < 1$ and $\theta \in [-\pi, \pi]$. Putting

$$P(r,\theta) = \frac{1}{2\pi}\left[1 + \sum_{n=1}^{\infty} 2r^n \cos(n\theta)\right], \quad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi], \quad (5.396)$$

we see that (5.397) can be written as

$$v(r,\theta) = \int_{-\pi}^{\pi} P(r, \theta - \xi) f(\xi)\, d\xi, \quad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi]. \quad (5.397)$$

5.3.4 The Poisson Kernel for the Unit Disc

The function P defined in (5.396) is called the **Poisson kernel** for the unit disc in $\mathbb{R}^2$, and the expression on the right–hand side of (5.397) is called the **Poisson integral representation** for v. In this section and the next, we will prove several important properties of the Poisson kernel and the Poisson integral in (5.397).

We will first show that the series defining the Poisson kernel in (5.396) converges uniformly over $\theta \in [-\pi, \pi]$ for each $0 \leqslant r < 1$. This will justify term–by–term integration of the series. We will use the Weierstrass M–Test for uniform convergence. Thus, we first estimate the absolute values of the of the terms of the series,

$$|2r^n \cos(n\theta)| \leqslant 2r^n, \quad \text{for all } \theta \in [-\pi, \pi], \quad (5.398)$$

and for $n \in \mathbb{N}$. It follows from (5.398) that the absolute values of the terms of the series in (5.396) are "majorized" by the terms of the convergent geometric series

$$\sum_{n=1}^{\infty} 2r^n,$$

for $0 \leqslant r < 1$. Hence, the Weierstrass M–Test applies, and we conclude that the series defining $P(r,\theta)$ in (5.396) converges uniformly in θ for $0 \leqslant r < 1$. This arguments can be carried out further to prove that, for any $0 < R < 1$, the series defining $P(r,\theta)$ in (5.396) converges uniformly for $\theta \in [-\pi, \pi]$ and $r \in [0, R]$. Hence, $P(r,\theta)$ defines a continuous function in the open unit disc, D_1, in $\mathbb{R}^2$. This follows from the following important consequence of the uniform convergence of a sequence of continuous functions:

Proposition 5.3.13 (Uniform Limit of Continuous Functions)**.** Let (f_n) be a sequence of continuous functions on $[a, b]$ that converges uniformly to a function $f\colon [a, b] \to \mathbb{R}$. Then, f is continuous.

For a proof of this proposition see Rudin [Rud53, Theorem 7.12, pg. 20].

We will next show that, in fact, the Poisson kernel is C^2 in D_1 and that it solves Laplace's equation in D_1. In order to show that the partial derivatives of

P exist, we need to show that the series

$$\sum_{n=1}^{\infty} 2nr^{n-1}\cos(n\theta) \quad \text{and} \quad \sum_{n=1}^{\infty} 2nr^{n}\sin(n\theta) \tag{5.399}$$

converge uniformly. This assertion will follow from the following proposition

Proposition 5.3.14 (Term–by–Term Differentiation)**.** Let (u_k) be a sequence of functions that are differentiable over a closed and bounded interval, $[a,b]$. Assume that the series

$$\sum_{k=1}^{\infty} u_k'$$

converges absolutely and uniformly over $[a,b]$. Assume also that the series

$$\sum_{k=1}^{\infty} u_k(x_o)$$

converges absolutely at some point x_o in $[a,b]$. Then, the series converges

$$\sum_{k=1}^{\infty} u_k$$

converges uniformly to a function f that is differentiable over $[a,b]$, and

$$f'(x) = \sum_{k=1}^{\infty} u_k'(x), \qquad \text{for all } x \in [a,b];$$

or

$$\frac{d}{dx}\left[\sum_{k=1}^{\infty} u_k(x)\right] = \sum_{k=1}^{\infty} u_k'(x)\ ,$$

for all $x \in [a,b]$.

This proposition can be proved by applying Theorem 7.17 in [Rud53, pg. 124].

In order to see that the series in (5.399) converge absolutely and uniformly, first note that

$$|2nr^{n-1}\cos(n\theta)| \leqslant 2nr^{n-1} \quad \text{and} \quad |2nr^{n}\sin(n\theta)| \leqslant 2nr^{2}$$

for all $\theta \in [-\pi,\pi]$; so that the series in (5.399) are "majorized" by the series

$$\sum_{n=1}^{\infty} 2nr^{n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} 2nr^{n}, \tag{5.400}$$

respectively; both of the series in (5.400) converge by the Ratio Test (or the Root Test), since $0 \leqslant r < 1$. It then follows by the Weierstrass M–Test for uniform convergence that the series in (5.400) converge uniformly in θ. The same argument applied to $r \in [0, R]$, where where $R < 1$, yields that the series in (5.399) are absolutely and uniformly convergent for $\theta \in [-\pi, \pi]$ and $r \in [0, R]$. This time the series in (5.399) are "majorized" by the convergent series

$$\sum_{n=1}^{\infty} 2nR^{n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} 2nR^{n},$$

respectively. It then follows from Proposition 5.3.14 that the partial derivatives of the Poisson kernel in (5.396) have partial derivatives in D_1 given by

$$\frac{\partial}{\partial r}[P(r,\theta)] = \frac{1}{2\pi}\sum_{n=1}^{\infty} 2nr^{n-1}\cos(n\theta), \quad 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi], \tag{5.401}$$

and

$$\frac{\partial}{\partial \theta}[P(r,\theta)] = -\frac{1}{2\pi}\sum_{n=1}^{\infty} 2nr^{n}\sin(n\theta), \quad 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi], \tag{5.402}$$

where we have differentiated the series in (5.396) term–by–term. A similar argument can be used to obtain the second partial derivatives of the Poisson kernel:

$$\frac{\partial^2}{\partial r^2}[P(r,\theta)] = \frac{1}{2\pi}\sum_{n=1}^{\infty} 2n(n-1)r^{n-2}\cos(n\theta), \quad 0 \leqslant r < 1,\ \theta \in [-\pi, \pi], \tag{5.403}$$

and

$$\frac{\partial^2}{\partial \theta^2}[P(r,\theta)] = -\frac{1}{2\pi}\sum_{n=1}^{\infty} 2n^2r^{n}\cos(n\theta), \quad 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi], \tag{5.404}$$

where the series in (5.401) and (5.402) have been differentiated term–by–term.

Next, substitute the partial derivatives in (5.401), (5.403) and (5.404) into the expression for the Laplacian of P in polar coordinates to get

$$\begin{aligned}
\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} &= \frac{1}{2\pi}\sum_{n=1}^{\infty} 2n(n-1)r^{n-2}\cos(n\theta) \\
&\quad + \frac{1}{2\pi r}\sum_{n=1}^{\infty} 2nr^{n-1}\cos(n\theta) \\
&\quad - \frac{1}{2\pi r^2}\sum_{n=1}^{\infty} 2n^2r^{n}\cos(n\theta) \qquad (5.405) \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}[n(n-1) + n - n^2]r^{n-2}\cos(n\theta) \\
&= 0,
\end{aligned}$$

for all $\theta \in [-\pi, \pi]$ and $0 \leqslant r < 1$. We have therefore shown that the Poisson kernel solves Laplace's equation in the open unit disc.

Next, integrate the series in (5.396) over the interval $[-\pi, \pi]$, which is justified by the uniform convergence of the series, to obtain

$$\int_{-\pi}^{\pi} P(r,\theta)\; d\theta = 1, \qquad \text{for all } 0 \leqslant r < 1. \tag{5.406}$$

The series defining the Poisson kernel in (5.396) can actually be evaluated by using the identity

$$2\cos(n\theta) = e^{in\theta} + e^{-in\theta},$$

and then adding geometric series. Indeed,

$$\begin{aligned}\sum_{n=1}^{\infty} 2r^n \cos(n\theta) &= \sum_{n=1}^{\infty} r^n [e^{in\theta} + e^{-in\theta}] \\ &= \sum_{n=1}^{\infty} r^n [e^{i\theta}]^n + \sum_{n=1}^{\infty} r^n [e^{-i\theta}]^n \\ &= \sum_{n=1}^{\infty} [re^{i\theta}]^n + \sum_{n=1}^{\infty} [re^{-i\theta}]^n,\end{aligned}$$

so that, since $|re^{\pm i\theta}| = r < 1$, for all θ,

$$\sum_{n=1}^{\infty} 2r^n \cos(n\theta) = \frac{re^{i\theta}}{1 - re^{i\theta}} + \frac{re^{-i\theta}}{1 - re^{-i\theta}},$$

which simplifies to

$$\sum_{n=1}^{\infty} 2r^n \cos(n\theta) = \frac{re^{i\theta} - r^2 + re^{-i\theta} - r^2}{1 - re^{i\theta} - re^{-i\theta} + r^2},$$

or

$$\sum_{n=1}^{\infty} 2r^n \cos(n\theta) = \frac{r[e^{i\theta} + e^{-i\theta}] - 2r^2}{1 - r[e^{i\theta} + e^{-i\theta}] + r^2},$$

or

$$\sum_{n=1}^{\infty} 2r^n \cos(n\theta) = \frac{2r\cos(\theta) - 2r^2}{1 - 2r\cos(\theta) + r^2}, \quad 0 \leqslant r < 1, \theta \in [-\pi, \pi]. \tag{5.407}$$

Substituting the value of the series in (5.407) into (5.396) then yields the formula

$$P(r,\theta) = \frac{1}{2\pi}\, \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi], \tag{5.408}$$

for the Poisson kernel.

We will next use the formula in (5.408) for the Poisson kernel for the unit disk to derive further properties of the Poisson kernel. We summarize these properties, as well as the ones we have already established using the representation in (5.396), in the following proposition.

Proposition 5.3.15 (Properties of the Poisson Kernel)**.** Let $P(r,\theta)$ be given by (5.408), or its equivalent representation as an infinite series in (5.396). Then, the function $P\colon [0,1)\times[-\pi,\pi]\to\mathbb{R}$ satisfies the following:

(i) $P(r,\theta) > 0$ for all $(r,\theta)\in[0,1)\times[-\pi,\pi]$;

(ii) $P\in C^\infty([0,1)\times[-\pi,\pi])$;

(iii) P is harmonic in D_1;

(iv) $\displaystyle\int_{-\pi}^{\pi} P(r,\theta-\xi)\;d\xi = 1,$ for all $\xi\in\mathbb{R}$ and all $0\leqslant r<1$.

(v) $\displaystyle\lim_{r\to 1^-} P(r,\theta-\xi) = 0,$ for $\xi\neq\theta$ and $|\xi-\theta|<\pi$;

(vi) $\displaystyle\lim_{r\to 1^-} P(r,\theta-\xi) = +\infty,$ for $\xi=\theta$.

Proof: In order to prove (i) and (ii), first note that, for all $\theta\in\mathbb{R}$ and $r\geqslant 0$,

$$2r\cos\theta\leqslant 2r,$$

so that

$$1-2r\cos\theta+r^2\geqslant 1-2r+r^2,$$

or

$$1-2r\cos(\theta)+r^2\geqslant(1-r)^2,\qquad\text{for } 0\leqslant r<1 \text{ and } \theta\in\mathbb{R}. \tag{5.409}$$

It follows from (5.409) and the formula for $P(r,\theta)$ in (5.408) that $P(r,\theta)$ is defined for all $r\in[0,r)$ and all $\theta\in\mathbb{R}$, and $P(r,\theta)>0$ for $r\in[0,1)$ and all $\theta\in\mathbb{R}$; we have therefore establlished (i).

From (5.409) we also obtain that

$$1-2r\cos(\theta)+r^2>0,\qquad\text{for } 0\leqslant r<1 \text{ and } \theta\in\mathbb{R}.$$

Thus, the denominator in the formula for $P(r,\theta)$ in (5.408) is not zero for $0\leqslant r<1$ and $\theta\in\mathbb{R}$; hence, since the numerator and denominator of the expression defining $P(r,\theta)$ in (5.408) are C^∞ functions, (ii) also follows.

We have already established that P satisfies Laplace's equation in D_1 (see the calculations leading up to (5.405) on page 158) using the definition of P in (5.396). Thus, P is harmonic in D_1 and so we have established (iii).

The integral identity in (iv) will follow from (5.406) and the 2π–periodicity of $P(r,\theta)$ in θ. Indeed, making the change of variables $\zeta=\theta-\xi$ in the integral in (iv) we have

$$\begin{aligned}
\int_{-\pi}^{\pi} P(r,\theta-\xi)\ d\xi &= -\int_{\theta+\pi}^{\theta-\pi} P(r,\zeta)\ d\zeta \\
&= \int_{\theta-\pi}^{\theta+\pi} P(r,\zeta)\ d\zeta \\
&= \int_{-\pi}^{\pi} P(r,\zeta)\ d\zeta \\
&= 1,
\end{aligned}$$

for all $\theta \in \mathbb{R}$.

Next, use the formula for $P(r,\theta)$ in (5.408) to obtain that

$$P(r,\theta-\xi) = \frac{1}{2\pi}\ \frac{1-r^2}{1-2r+r^2},$$

for $\theta = \xi$, from which we get that

$$P(r,\theta-\xi) = \frac{1}{2\pi}\ \frac{1+r}{1-r}, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta = \xi. \tag{5.410}$$

The assertion in (vi) follows from (5.410).

To prove (v), first note that

$$\begin{aligned}
\lim_{r\to 1^-} [1-2r\cos(\theta-\xi)+r^2] &= 2-2\cos(\theta-\xi) \\
&= \sin^2(\theta-\xi)
\end{aligned}$$

so that

$$\lim_{r\to 1^-} [1-2r\cos(\theta-\xi)+r^2] \neq 0, \qquad \text{for } \xi \neq 0 \text{ and } |\xi-\theta| < \pi \tag{5.411}$$

The assertion in (v) then follows from (5.411) and the expression for the Poisson kernel in (5.408). ■

5.3.5 The Poisson Integral Representation

Let $f\colon [-\pi,\pi] \to \mathbb{R}$ be a continuous function that can be extended to a continuous, 2π–periodic function in $\mathbb{R}$. We then have that

$$|f(\theta)| \leqslant M, \qquad \text{for all } \theta \in [-\pi,\pi],$$

and some positive constant M. The goal of this section is to use the properties of the Poisson kernel listed in Proposition 5.3.15 to prove that the function

$v\colon \overline{D}_1 \to \mathbb{R}$ defined by

$$v(r,\theta) = \begin{cases} \displaystyle\int_{-\pi}^{\pi} P(r,\theta-\xi)f(\xi)\, d\xi, & \text{for } 0 \leqslant r < 1,\ \theta \in [-\pi,\pi]; \\ \\ f(\theta), & \text{for } r = 1,\ \theta \in [-\pi,\pi], \end{cases} \tag{5.412}$$

where $P(r,\theta)$ denotes the Poisson kernel for the unit disc in $\mathbb{R}^2$ given in (5.396) or (5.408), solves the Dirichlet problem (5.334) for the unit disc in $\mathbb{R}^2$.

We first show that $v \in C^2(D_1,\mathbb{R})$ and that it solves Laplace's equation in D_1; in polar coordinates,

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \quad \text{for } 0 < r < 1,\ \theta \in [-\pi,\pi]. \tag{5.413}$$

This will follow from (iii) in Proposition 5.3.15, provided we can show that differentiation under the integral sign in the first part of the definition of v in (5.412) is valid. Indeed, property (iii) in Proposition 5.413 says that

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} = 0, \quad \text{for } 0 \leqslant r < 1,\ \theta \in [-\pi,\pi]. \tag{5.414}$$

Thus, assuming for the moment that differentiation under the integral sign in (5.412) is valid, we have that, for $0 \leqslant r < 1$ and $\theta \in [-\pi,\pi]$,

$$\begin{aligned} \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} &= \int_{-\pi}^{\pi} \frac{\partial^2}{\partial r^2}[P(r,\theta-\xi)]f(\xi)\, d\xi \\ &\quad + \int_{-\pi}^{\pi} \frac{1}{r}\frac{\partial}{\partial r}[P(r,\theta-\xi)]f(\xi)\, d\xi \\ &\quad + \int_{-\pi}^{\pi} \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[P(r,\theta-\xi)]f(\xi)\, d\xi, \end{aligned}$$

which can be written as

$$\Delta v = \int_{-\pi}^{\pi} \left[\frac{\partial^2}{\partial r^2}[P(r,\theta-\xi)] + \frac{1}{r}\frac{\partial}{\partial r}[P(r,\theta-\xi)] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[P(r,\theta-\xi)]\right] f(\xi)\, d\xi,$$

where we have used the short–hand notation, Δv, for the Laplacian of v. The fact that v is harmonic in D_1 then follows from the previous identity and (5.414).

We will next see that differentiation under the integral sign is justified. In order to do this, we first note that the continuity of f implies that there exists a positive constant, M, such that

$$|f(\theta)| \leqslant M, \quad \text{for all } \theta \in [-\pi,\pi]. \tag{5.415}$$

In view of (5.415) and (5.414), in order to justify the differentiation under the integral sign in the first part of the definition of u in (5.412), it suffices to prove that

$$\frac{\partial}{\partial \theta}[P(r,\theta-\xi)], \quad \frac{\partial}{\partial r}[P(r,\theta-\xi)] \ \text{ and } \ \frac{\partial^2}{\partial \theta^2}[P(r,\theta-\xi)]$$

are absolutely integrable over $[-\pi, \pi]$ for each $[-\pi, \pi]$.

Use (5.408) to compute

$$\frac{\partial}{\partial \theta}[P(r, \theta - \xi)] = -\frac{1}{2\pi} \frac{(1 - r^2)2r\sin(\theta - \xi)}{(1 - 2r\cos(\theta - \xi) + r^2)^2},$$

which can be written as

$$\frac{\partial}{\partial \theta}[P(r, \theta - \xi)] = -\frac{2r\sin(\theta - \xi)}{1 - 2r\cos(\theta - \xi) + r^2} P(r, \theta - \xi), \qquad (5.416)$$

by virtue of the expression for the Poisson kernel in (5.408). Next, take absolute values on both sides of (5.416) and use the estimate in (5.409) to get

$$\left| \frac{\partial}{\partial \theta}[P(r, \theta - \xi)] \right| \leqslant \frac{2r}{(1 - r)^2} \; P(r, \theta - \xi), \qquad (5.417)$$

where we have used the positivity of the Poisson kernel in (i) of Proposition 5.3.15. Integrating on both sides of the inequality in (5.417) form $-\pi$ to π and using property (iv) in Proposition 5.3.15 we obtain that

$$\int_{-\pi}^{\pi} \left| \frac{\partial}{\partial \theta}[P(r, \theta - \xi)] \right| \, d\xi \leqslant \frac{2r}{(1 - r)^2}, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi, \pi],$$

which shows that $\dfrac{\partial}{\partial \theta}[P(r, \theta - \xi)]$ is absolutely integrable over $[-\pi, \pi]$ for $0 \leqslant r < 1$ and $\theta \in [-\pi, \pi]$.

Next, take partial derivative with respect to θ on both sides of (5.416) to get

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2}[P(r, \theta - \xi)] \;\; = \;\; & -\frac{2r\cos(\theta - \xi)}{1 - 2r\cos(\theta - \xi) + r^2} P(r, \theta - \xi) \\ & + \frac{4r^2\sin^2(\theta - \xi)}{(1 - 2r\cos(\theta - \xi) + r^2)^2} P(r, \theta - \xi) \\ & - \frac{2r\sin(\theta - \xi)}{1 - 2r\cos(\theta - \xi) + r^2} \frac{\partial}{\partial \theta}[P(r, \theta - \xi)], \end{aligned}$$

so that, in view of (5.416),

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2}[P(r, \theta - \xi)] \;\; = \;\; & -\frac{2r\cos(\theta - \xi)}{1 - 2r\cos(\theta - \xi) + r^2} P(r, \theta - \xi) \\ & + \frac{4r^2\sin^2(\theta - \xi)}{(1 - 2r\cos(\theta - \xi) + r^2)^2} P(r, \theta - \xi) \\ & + \frac{4r^2\sin^2(\theta - \xi)}{(1 - 2r\cos(\theta - \xi) + r^2)^2} P(r, \theta - \xi), \end{aligned}$$

or

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)] &= -\frac{2r\cos(\theta-\xi)}{1-2r\cos(\theta-\xi)+r^2}P(r,\theta-\xi)\\ &\quad +\frac{8r^2\sin^2(\theta-\xi)}{(1-2r\cos(\theta-\xi)+r^2)^2}P(r,\theta-\xi).\end{aligned} \tag{5.418}$$

Taking absolute values on both sides of (5.418) and applying the triangle inequality, we obtain

$$\left|\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)]\right| \leqslant \frac{2r}{(1-r)^2}P(r,\theta-\xi)+\frac{8r^2}{(1-r)^4}P(r,\theta-\xi), \tag{5.419}$$

where we have also used the estimate in (5.409) and the positivity of the Poisson kernel (see property (i) in Proposition 5.3.15). Integrating from $-\pi$ to π on both sides of (5.419) then yields

$$\int_{-\pi}^{\pi}\left|\frac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)]\right|\, d\xi \leqslant \frac{2r}{(1-r)^2}+\frac{8r^2}{(1-r)^4}, \qquad \text{for } 0\leqslant r<1,\ \theta\in\mathbb{R},$$

where we have also used property (iv) in Proposition 5.3.15; thus, we have shown that $\dfrac{\partial^2}{\partial\theta^2}[P(r,\theta-\xi)]$ is absolutely integrable over $[-\pi,\pi]$ for $0\leqslant r<1$ and $\theta\in\mathbb{R}$.

Next, differentiate the Poisson kernel in (5.408) with respect to r, for $0\leqslant r<1$, to obtain

$$\frac{\partial}{\partial r}[P(r,\theta-\xi)] = \frac{1}{2\pi}\,\frac{-2r}{1-2r\cos(\theta-\xi)+r^2}-\frac{1}{2\pi}\,\frac{(1-r^2)(2r-2\cos(\theta-\xi))}{(1-2r\cos(\theta-\xi)+r^2)^2},$$

where we have applied the Product Rule; so that, in view of the expression for the Poison kernel in (5.408),

$$\frac{\partial}{\partial r}[P(r,\theta-\xi)] = -\frac{2r}{1-r^2}P(r,\theta-\xi)-\frac{2r-2\cos(\theta-\xi)}{1-2r\cos(\theta-\xi)+r^2}P(r,\theta-\xi),$$

or

$$\frac{\partial}{\partial r}[P(r,\theta-\xi)] = \left[\frac{-2r}{1-r^2}+\frac{2r-\cos(\theta-\xi)}{1-2r\cos(\theta-\xi)+r^2}\right]P(r,\theta-\xi), \tag{5.420}$$

for $0\leqslant r<1$ and all ξ and θ in $\mathbb{R}$.

Next, take absolute values on both sides of (5.420), applying the triangle inequality, and use the estimate in (5.409) to obtain

$$\left|\frac{\partial}{\partial r}[P(r,\theta-\xi)]\right| \leqslant \left[\frac{2r}{1-r^2}+\frac{2r+1}{(1-r)^2}\right]P(r,\theta-\xi), \tag{5.421}$$

for $0\leqslant r<1$ and $\theta,\xi\in\mathbb{R}$, where we have also used the positivity of the Poisson kernel (see property (i) in Proposition 5.3.15).

Integrating from $-\pi$ to π on both sides of (5.421) and using property (iv) in Proposition 5.3.15 then yields

$$\int_{-\pi}^{\pi} \left| \frac{\partial}{\partial r}[P(r,\theta-\xi)] \right| \, d\xi \leqslant \frac{2r}{1-r^2} + \frac{2r+1}{(1-r)^2}, \qquad \text{for } 0 \leqslant r < 0,\ \theta \in \mathbb{R},$$

which shows that $\dfrac{\partial}{\partial r}[P(r,\theta-\xi)]$ is absolutely integrable over $[-\pi,\pi]$ for $0 \leqslant r < 1$ and $\theta \in [-\pi,\pi]$.

Hence, differentiation under the integral sign in the first part of the definition of u in (5.412) is justified. (See the results in Appendix B.1). We have therefore established that the function v defined in (5.412) is in $C^2(D_1)$ and satisfies Laplace's equation. It remains to prove that $v \in C(\overline{D}_1)$ and that it satisfies the boundary conditions in problem (5.334). This will be accomplished once we prove the following lemma:

Lemma 5.3.16 (Boundary Limits of the Poisson Integral Representation)**.** Let v be as given in (5.412) where f is continuous on $[-\pi,\pi]$. Then, for every $\zeta \in [-\pi,\pi]$,

$$\lim_{(r,\theta)\to(1,\zeta)} |v(r,\theta) - f(\zeta)| = 0. \tag{5.422}$$

Proof: First consider the case in which $\zeta \in (-\pi,\pi)$.

Let $\varepsilon > 0$ be given. Since f is is continuous on $[-\pi,\pi]$, there exists $\delta_1 > 0$ such that $\delta_1 < \dfrac{\pi}{2}$, and

$$|\xi - \zeta| < \delta_1 \Rightarrow \xi \in (-\pi,\pi) \ \text{ and } \ |f(\xi) - f(\zeta)| < \frac{\varepsilon}{3}. \tag{5.423}$$

Next, use property (iii) of the Poisson kernel in Proposition 5.3.15 to write

$$\begin{aligned} v(r,\theta) - f(\zeta) &= \int_{-\pi}^{\pi} P(r,\theta-\xi) g(\xi)\, d\xi - f(\zeta) \int_{-\pi}^{\pi} P(r,\theta-\xi)\, d\xi \\ &= \int_{-\pi}^{\pi} P(r,\theta-\xi) f(\xi)\, d\xi - \int_{-\pi}^{\pi} P(r,\theta-\xi) f(\zeta)\, d\xi, \end{aligned}$$

so that

$$v(r,\theta) - f(\zeta) = \int_{-\pi}^{\pi} P(r,\theta-\xi)(f(\xi) - f(\zeta))\, d\xi. \tag{5.424}$$

Next, take absolute values on both sides of (5.424) and use the positivity of the Poisson kernel (see property (i) in Proposition 5.3.15) to obtain that

$$|v(r,\theta) - f(\zeta)| \leqslant \int_{-\pi}^{\pi} P(r,\theta-\xi) |f(\xi) - f(\zeta)|\, d\xi. \tag{5.425}$$

We'll next divide the integral on the right–hand side of (5.425) into three integrals over the domains $[-\pi,\zeta-\delta_1]$, $[\zeta-\delta_1,\zeta+\delta_1]$ and $[\zeta+\delta_1,\pi]$, respectively.

We first estimate the integral over $[\zeta-\delta_1,\zeta+\delta_1]$ using (5.424) to get

$$\int_{\zeta-\delta_1}^{\zeta+\delta_1} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < \frac{\varepsilon}{3}\int_{\zeta-\delta_1}^{\zeta+\delta_1} P(r,\theta-\xi)\ d\xi,$$

so that, by virtue of the positivity of the Poisson (property (i) in Proposition 5.3.15)

$$\int_{\zeta-\delta_1}^{\zeta+\delta_1} P(r,\theta-\xi)|g(\xi)-g(\zeta)|\ d\xi < \frac{\varepsilon}{3}\int_{-\pi}^{\pi} P(r,\theta-\xi)\ d\xi;$$

hence, by property (iv) in Proposition 5.3.15,

$$\int_{\zeta-\delta_1}^{\zeta+\delta_1} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < \frac{\varepsilon}{3}. \tag{5.426}$$

Next, we estimate the integral over $[\zeta+\delta_1,\pi]$. Using the estimate in (5.415) and the triangle inequality we obtain

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < 2M\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)\ d\xi, \tag{5.427}$$

for all $\theta\in(-\pi,\pi)$ and $0\leqslant r<1$. Then, for

$$|\theta-\zeta| < \frac{\delta_1}{2}, \tag{5.428}$$

we obtain from (5.427) and the positivity of the Poisson kernel that

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < 2M\int_{\theta+\delta_1/2}^{\pi} P(r,\theta-\xi)\ d\xi; \tag{5.429}$$

Thus, making the change of variables $\omega=\xi-\theta$ in (5.429) we get

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < 2M\int_{\delta_1/2}^{\pi-\theta} P(r,\omega)\ d\omega,$$

so that, by the positivity of the Poisson kernel,

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < 2M\int_{\delta_1/2}^{\pi} P(r,\omega)\ d\omega. \tag{5.430}$$

Now, it follows from the properties of the cosine function that

$$\frac{\delta_1}{2}<\omega<\pi \Rightarrow \cos(\omega)<\cos(\delta_1/2);$$

so that, for $0<r<1$,

$$\frac{\delta_1}{2}<\omega<\pi \Rightarrow 1-2r\cos(\omega)+r^2 > 1-2r\cos(\delta_1/2)+r^2;$$

so that, by the expression for the Poisson kernel in (5.408),

$$P(r,\omega) < P(r,\delta_1/2), \quad \text{for all } \omega \in [\delta_1/2, \pi]. \tag{5.431}$$

It then follows from (5.430) and (5.431) that

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < 2M \int_{\delta_1/2}^{\pi} P(r,\delta_1/2)\ d\omega,$$

for $0 < r < 1$; so that,

$$\int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < 2M\left(\pi - \frac{\delta_1}{2}\right) P(r,\delta_1/2), \tag{5.432}$$

for $0 < r < 1$.

Now, it follows property (v) in Proposition 5.3.15 that there exists $\delta_2 > 0$ such that

$$|r-1| < \delta_2 \Rightarrow P(r,\delta_1/2) < \frac{\varepsilon}{3M(2\pi-\delta_1)} \tag{5.433}$$

Thus, combining (5.432), (5.428) and (5.433) we see that

$$|r-1| < \delta_2 \text{ and } |\theta-\zeta| < \frac{\delta_1}{2} \Rightarrow \int_{\zeta+\delta_1}^{\pi} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < \frac{\varepsilon}{3}. \tag{5.434}$$

Calculations similar to those leading to (5.434) show that there exists $\delta_3 > 0$ such that

$$|r-1| < \delta_3 \text{ and } |\theta-\zeta| < \frac{\delta_1}{2} \Rightarrow \int_{-\pi}^{\zeta-\delta_1} P(r,\theta-\xi)|f(\xi)-f(\zeta)|\ d\xi < \frac{\varepsilon}{3}. \tag{5.435}$$

Letting $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2, \delta_3\right\}$, we see that, in view of (5.425), (5.426), (5.434) and (5.435),

$$|r-1| < \delta \text{ and } |\theta-\zeta| < \delta \Rightarrow |v(r,\theta)-f(\zeta)| < \varepsilon.$$

This completes the proof of the boundary limits lemma for the case $\zeta \in (-\pi,\pi)$. The case in which ζ is one of the end–points of the interval $[-\pi,\pi]$ can be treated in an analogous manner to the interior point case using one–sided limits at those points. ■

5.3.6 Existence for the Dirichlet Problem on the Unit Disc

In Section 5.3.5 we studied the Dirichlet problem of the two–dimensional Laplacian in the unit disc in polar coordinates,

$$\begin{cases} \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0, & 0 < r < 1, -\pi < \theta < \pi; \\ v(1,\theta) = f(\theta), & -\pi \leqslant \theta \leqslant \pi, \end{cases} \tag{5.436}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous, 2π–periodic function. We constructed a candidate for a solution given by (5.412); namely,

$$v(r,\theta) = \begin{cases} \displaystyle\int_{-\pi}^{\pi} P(r,\theta-\xi)f(\xi)\, d\xi, & \text{for } 0 \leqslant r < 1,\ \theta \in [-\pi,\pi]; \\ \\ f(\theta), & \text{for } r = 1,\ \theta \in [-\pi,\pi], \end{cases} \tag{5.437}$$

where $P(r,\theta)$ denotes the Poisson kernel for the unit disc in $\mathbb{R}^2$ given in (5.396) or (5.408); that is,

$$P(r,\theta) = \frac{1}{2\pi}\,\frac{1-r^2}{1-2r\cos(\theta)+r^2}, \qquad \text{for } 0 \leqslant r < 1 \text{ and } \theta \in [-\pi,\pi], \tag{5.438}$$

We showed in Section 5.3.5 that

$$v(r,\theta) = \int_{-\pi}^{\pi} P(r,\theta-\xi)f(\xi)\, d\xi, \qquad \text{for } 0 \leqslant r < 1,\ \theta \in [-\pi,\pi], \tag{5.439}$$

is a C^2 function that solves Laplace's equation in D_1. It remains to show that $v \in C(\overline{D}_1,\mathbb{R})$ and that $v(1,\theta) = f(\theta)$, for all $\theta \in [-\pi,\pi]$.

The fact that the function $v\colon \overline{D}_1 \to \mathbb{R}$ defined in (5.437) is continuous follows from Lemma 5.3.16. Indeed, for $v(r,\theta)$ as given in (5.439), we get from Lemma 5.3.16 that

$$\lim_{(r,\theta)\to(1,\zeta)} v(r,\theta) = f(\zeta), \qquad \text{for all } \zeta \in [-\pi,\pi];$$

so that, in view of the definition of v in (5.437),

$$\lim_{(r,\theta)\to(1,\zeta)} v(r,\theta) = v(1,\zeta), \qquad \text{for all } \zeta \in [-\pi,\pi],$$

which shows that v is continuous at every point $(1,\zeta) \in \partial D_1$. This completes the proof of the following result.

Theorem 5.3.17 (Existence for the Dirichlet Problem for the Disc, I)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous, 2π–periodic function. There exists a function $v \in C^2(D_1,\mathbb{R}) \cap C(\overline{D}_1,\mathbb{R})$ that solves the BVP in (5.436).

By converting back to Cartesian coordinates from polar coordinates, we obtain a solution of the Dirichlet problem in (5.332); namely,

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } D_1; \\ u(x,y) = g(x,y), & \text{for } (x,y) \in \partial D_1, \end{cases} \tag{5.440}$$

where $g \in C(\partial D_1,\mathbb{R})$. Indeed, for $(x,y) \in D_1$, write

$$x = r\cos\theta \ \text{ and } \ y = r\sin\theta, \qquad \text{for } 0 \leqslant r < 1 \text{ and } -\pi < \theta \leqslant \pi,$$

and, for $-\pi < \xi \leqslant \pi$, put

$$\omega_1(\xi) = \cos\xi \ \text{ and } \ \omega_2 = \sin\xi;$$

so that, $(\omega_1(\xi), \omega_2(\xi)) \in \partial D_1$ for all $\xi \in (-\pi, \pi]$. Consequently, denoting the Euclidean norm of vector in $\mathbb{R}^2$ by $|\cdot|$,

$$|(x, y) - (\omega_1, \omega_2)|^2 = r^2 - 2(x, y) \cdot (\omega_1, \omega_2) + 1, \tag{5.441}$$

where the middle–term on the right–hand side of (5.441) is the dot–product of the the vectors (x, y) and (ω_1, ω_2). Thus,

$$|(x, y) - (\omega_1, \omega_2)|^2 = r^2 - 2r\cos(\theta - \xi) + 1. \tag{5.442}$$

Hence, in view of (5.438), we see from (5.442) that

$$P(r, \theta - \xi) = \frac{1}{2\pi} \, \frac{1 - |(x, y)|^2}{|(x, y) - (\omega_1, \omega_2)|^2}, \tag{5.443}$$

for $(x, y) \in D_1$ and $(\omega_1, \omega_2) \in \partial D_1$.

The right–hand side of the equation in (5.443) gives the Poisson kernel in Cartesian coordinates. We write

$$P((x, y), (\omega_1, \omega_2)) = \frac{1}{2\pi} \, \frac{1 - |(x, y)|^2}{|(x, y) - (\omega_1, \omega_2)|^2}, \tag{5.444}$$

for $(x, y) \in D_1$ and $(\omega_1, \omega_2) \in \partial D_1$. Then, in view of (5.439) and (5.444), we define

$$u(x, y) = \oint_{\partial D_1} P((x, y), \omega) g(\omega) \; ds_\omega, \quad \text{for } (x, y) \in D_1, \tag{5.445}$$

where $\omega = (\omega_1, \omega_2) \in \partial D_1$ and ds_ω is the element of arc–length along the circle ∂D_1.

The boundary limit Lemma 5.3.16 implies that

$$\lim_{\substack{(x,y)\to(z_1,z_2) \\ (x,y)\in D_1}} \oint_{\partial D_1} P((x, y), \omega) g(\omega) \; ds_\omega = g(z_1, z_2), \tag{5.446}$$

for all $(z_1, z_2) \in \partial D_1$. Thus, in view of (5.437), (5.445) and (5.446), we see that the function $u\colon \overline{D}_1 \to \mathbb{R}$ given by

$$u(x, y) = \begin{cases} \displaystyle\oint_{\partial D_1} P((x, y), \omega) g(\omega) \; ds_\omega, & \text{for } (x, y) \in D_1; \\ \\ g(x, y), & \text{for } (x, y) \in \partial D_1, \end{cases} \tag{5.447}$$

where $P((x, y), \omega)$, for $(x, y) \in D_1$ and $\omega \in \partial D_1$, is the Poisson kernel for the unit disc, D_1, given in (5.444), gives a solution of the boundary value problem in (5.440). We state this fact as the following existence theorem, which is, essentially, the Cartesian coordinates version of Theorem 5.3.17.

Theorem 5.3.18 (Existence for the Dirichlet Problem for the Disc, II)**.** For any given $g \in C(\partial D_1, \mathbb{R})$, there exists a function $u \in C^2(D_1, \mathbb{R}) \cap C(\overline{D}_1, \mathbb{R})$ that solves the BVP in (5.440). Indeed, u is given by the Poisson integral representation in (5.447).

5.4 Green's Functions

In Section 5.3.6 we constructed a solution of the Dirichlet problem for Laplace's equation in the unit disc, D_1, in $\mathbb{R}^2$:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } D_1; \\ u(x,y) = g(x,y), & \text{for } (x,y) \in \partial D_1, \end{cases}$$

where $g \in C(\partial D_1, \mathbb{R})$ is given. The construction was based on the Poisson integral representation of a harmonic functions in terms of its boundary values. In this section, we turn to the Dirichlet problem for Laplace's equation in a general bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u(x,y) = g(x,y), & \text{for } (x,y) \in \partial\Omega, \end{cases} \tag{5.448}$$

where $g \in C(\partial\Omega, \mathbb{R})$ is given. We will be able to construct a solution of the Dirichlet problem in (5.448) by using the integral representation of a harmonic function in Ω in terms of its values on the boundary of Ω. This integral representation in turn relies on the existence of a special function tied to the domain Ω that is known as the **Green's function** of the domain.

We will also construct a solution of the Dirichlet problem for Poisson's equations:

$$\begin{cases} -\Delta u(x,y) = f(x,y), & \text{for } (x,y) \in \Omega; \\ u(x,y) = g(x,y), & \text{for } (x,y) \in \partial\Omega, \end{cases} \tag{5.449}$$

where $g \in C(\partial\Omega, \mathbb{R})$ and $f \colon \Omega \to \mathbb{R}$ is assumed to be Hölder continuous.

Definition 5.4.1 (Hölder Continuity)**.** Let Ω denote an open subset of $\mathbb{R}^2$ and $f \colon \Omega \to \mathbb{R}$ be a real valued function. The function f is said to be **Hölder continuous with Hölder exponent** α, where $0 < \alpha \leqslant 1$, if and only if there exists a positive constant M such that

$$|f(x,y) - f(\xi,\eta)| \leqslant M|(x,y) - (\xi,\eta)|^\alpha, \quad \text{for all } (x,y), (\xi,\eta) \in \Omega. \tag{5.450}$$

If $\alpha = 1$ in (5.450), we say that f is **Lipschitz continuous** in Ω, with Lipschitz constant M.

5.4.1 Green's Integral Representation Formula

In Section 4.2.1 we saw that the functions of the form

$$W(x,y) = C \ln \sqrt{x^2 + y^2}, \quad \text{for } (x,y) \neq (0,0), \tag{5.451}$$

where C is a constant, are harmonic in the punctured plane, $\mathbb{R}^2 \backslash \{(0,0)\}$; that is, W solves

$$u_{xx} + u_{yy} = 0 \quad \text{in } \mathbb{R}^2 \backslash \{(0,0)\}. \tag{5.452}$$

We also saw that the functions in (5.451) are radially symmetric solutions of the problem in (5.452).

Since Laplace's equation in (5.452) is also translation invariant, given any point $(x_o, y_o) \in \mathbb{R}^2$, it follows that the function

$$(x,y) \mapsto W(x-x_o, y-y_o), \quad \text{for } (x,y) \neq (x_o,y_o), \tag{5.453}$$

solves Laplace's equation in $\mathbb{R}^2 \backslash \{(x_o,y_o)\}$. Denoting the function defined in (5.453) by $W_{(x_o,y_o)}$, we see that

$$W_{(x_o,y_o)}(x,y) = C \ln \sqrt{(x-x_o)^2 + (y-y_o)^2}, \quad \text{for } (x,y) \neq (x_o,y_o), \tag{5.454}$$

solves

$$u_{xx} + u_{yy} = 0 \quad \text{in } \mathbb{R}^2 \backslash \{(x_o,y_o)\}. \tag{5.455}$$

Let $\mathcal{U}$ denote an open subset of $\mathbb{R}^2$ and let Ω be a connected, bounded and open subset of $\mathcal{U}$ such that $\overline{\Omega} \subset \mathcal{U}$. Assume that the boundary, $\partial\Omega$, of Ω is piecewise C^1. Let $u \in C^2(\Omega,\mathbb{R}) \cap C(\overline{\Omega},\mathbb{R})$ and $v \in C^2(\Omega,\mathbb{R}) \cap C(\overline{\Omega},\mathbb{R})$. Using the Divergence Theorem (or Green's Theorem), we can derive the following identity

$$\iint_\Omega (v\Delta u - u\Delta v)\ dxdy = \oint_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right)\ ds, \tag{5.456}$$

where Δu and Δv denote the Laplacian of u and v, respectively, and

$$\frac{\partial u}{\partial n} = \nabla u \cdot \widehat{n} \quad \text{and} \quad \frac{\partial v}{\partial n} = \nabla v \cdot \widehat{n} \tag{5.457}$$

denote the directional derivative of u and v, respectively, in the direction of the outward, unit normal vector, $\widehat{n}$, to the boundary, $\partial\Omega$, of Ω. The differential ds in the integral on the right–hand side of (5.456) denotes the element of arc–length along the curve $\partial\Omega$. The identity in (5.456) is called **Green's Identity II**.

Next, let $(x_o,y_o) \in \Omega$ and let $\varepsilon > 0$ be such that the closure of the disc of radius ε centered at (x_o,y_o) is contained entirely in Ω. Setting

$$D_\varepsilon(x_o,y_o) = \{(x,y) \in \mathbb{R}^2 \mid (x-x_o)^2 + (y-y_o)^2 < \varepsilon^2\},$$

we have that

$$\overline{D_\varepsilon(x_o,y_o)} \subset \Omega.$$

Define $\Omega_\varepsilon = \Omega \backslash \overline{D_\varepsilon(x_o,y_o)}$, the set Ω with $\overline{D_\varepsilon(x_o,y_o)}$ taken out. Observe that the function $W_{(x_o,y_o)}$ defined in (5.454) is harmonic in Ω_ε by virtue of (5.455).

Next, let $u \in C^2(\mathcal{U},\mathbb{R})$ and apply Green's Identity II in (5.456) with Ω_ε in place of Ω, and with $W_{(x_o,y_o)}$ in place of v to get

$$\iint_{\Omega_\varepsilon} W_{(x_o,y_o)}\Delta u\ dxdy = \oint_{\partial\Omega_\varepsilon} \left(W_{(x_o,y_o)}\frac{\partial u}{\partial n} - u\frac{\partial W_{(x_o,y_o)}}{\partial n} \right)\ ds, \tag{5.458}$$

where we have used the fact that $W_{(x_o,y_o)}$ is harmonic in Ω_ε.

The boundary of Ω_ε is the union of $\partial\Omega$ and $\partial D_\varepsilon(x_o,y_o)$. We can therefore rewrite the boundary integral on the right–hand side of (5.458) as

$$\oint_{\partial\Omega} \left(W\frac{\partial u}{\partial n} - u\frac{\partial W}{\partial n} \right)\ ds - \oint_{\partial D_\varepsilon} \left(W\frac{\partial u}{\partial n} - u\frac{\partial W}{\partial n} \right)\ ds, \tag{5.459}$$

where we have written W for $W_{(x_o,y_o)}$ and D_ε for $D_\varepsilon(x_o,y_o)$, and the minus sign before the second integral in (5.459) indicates the fact that the outward, unit normal of $\partial\Omega_\varepsilon$ at a point on $\partial D_\varepsilon(x_o,y_o)$, points toward the point (x_o,y_o). We can therefore rewrite (5.458) as

$$\iint_{\Omega_\varepsilon} W\Delta u\ dxdy = \oint_{\partial\Omega}\left(W\frac{\partial u}{\partial n} - u\frac{\partial W}{\partial n}\right)\ ds - \oint_{\partial D_\varepsilon}\left(W\frac{\partial u}{\partial n} - u\frac{\partial W}{\partial n}\right)\ ds,$$

which can be rewritten as

$$\begin{aligned}\oint_{\partial D_\varepsilon}\left(W\frac{\partial u}{\partial n} - u\frac{\partial W}{\partial n}\right)\ ds \quad = \quad & -\iint_{\Omega_\varepsilon} W\Delta u\ dxdy \\ & + \oint_{\partial\Omega}\left(W\frac{\partial u}{\partial n} - u\frac{\partial W}{\partial n}\right)\ ds.\end{aligned} \tag{5.460}$$

We proceed to compute the limit as $\varepsilon\to 0^+$ of the integrals in (5.460) that depend on ε. These calculations will depend on the definition of W in (5.454) and the assumption that $u\in C^2(\mathcal{U},\mathbb{R})$. We begin by showing that

$$\lim_{\varepsilon\to 0^+}\iint_{\Omega_\varepsilon} W\Delta u\ dxdy = \iint_{\Omega} W\Delta u\ dxdy. \tag{5.461}$$

To establish (5.461), first note that

$$\iint_{\Omega} W\Delta u\ dxdy - \iint_{\Omega_\varepsilon} W\Delta u\ dxdy = \iint_{D_\varepsilon} W\Delta u\ dxdy;$$

so that,

$$\left|\iint_{\Omega} W\Delta u\ dxdy - \iint_{\Omega_\varepsilon} W\Delta u\ dxdy\right| \leqslant \iint_{D_\varepsilon} |W||\Delta u|\ dxdy. \tag{5.462}$$

Now, since we are assuming that $u\in C^2(\mathcal{U},\mathbb{R})$ and Ω is bounded, $|\Delta u|$ is bounded on $\overline{\Omega}$. Putting

$$M = \max_{(x,y)\in\overline{\Omega}} |\Delta u(x,y)|,$$

we obtain from (5.462) that

$$\left|\iint_{\Omega} W\Delta u\ dxdy - \iint_{\Omega_\varepsilon} W\Delta u\ dxdy\right| \leqslant M\iint_{D_\varepsilon} |W(x,y)|\ dxdy, \tag{5.463}$$

where W is defined in (5.454). We can use polar coordinates to evaluate the integral on the right–hand side of (5.463) as follows

$$\iint_{D_\varepsilon} |W(x,y)|\ dxdy = |C|\int_0^{2\pi}\int_0^{\varepsilon} |\ln r| r\ drd\theta.$$

Thus, assuming that $\varepsilon < 1$,

$$\iint_{D_\varepsilon} |W(x,y)| \ dxdy = -2\pi |C| \int_0^\varepsilon r \ln r \ dr. \tag{5.464}$$

We can use integration by parts and L'Hospital's Rule to evaluate the integral on the right–hand side of (5.464) to get

$$\iint_{D_\varepsilon} |W(x,y)| \ dxdy = -2\pi |C| \left(\frac{\varepsilon^2}{2} \ln \varepsilon - \frac{\varepsilon^2}{4} \right). \tag{5.465}$$

Thus, combining the estimate in (5.463) with the result in (5.465),

$$\left| \iint_\Omega W \Delta u \ dxdy - \iint_{\Omega_\varepsilon} W \Delta u \ dxdy \right| \leqslant \pi M |C| \left(\frac{\varepsilon^2}{2} + \varepsilon |\varepsilon \ln \varepsilon| \right), \tag{5.466}$$

for $0 < \varepsilon < 1$. Letting $\varepsilon \to 0^+$ in (5.466) and using the fact that

$$\lim_{\varepsilon \to 0^+} \varepsilon \ln \varepsilon = 0, \tag{5.467}$$

which follows from L'Hospital's Rule, we obtain from (5.466) and (5.467) that

$$\lim_{\varepsilon \to 0^+} \left| \iint_{\Omega_\varepsilon} W \Delta u \ dxdy - \iint_\Omega W \Delta u \ dxdy \right| = 0,$$

which implies (5.461).

Next, we estimate the first boundary integral on the left–hand side of (5.460) to get

$$\left| \oint_{\partial D_\varepsilon} W \frac{\partial u}{\partial n} \ ds \right| \leqslant M_1 \oint_{\partial D_\varepsilon} |W| \ ds, \tag{5.468}$$

where M_1 is given by

$$M_1 = \max_{(x,y) \in \overline{\Omega}} |\nabla u(x,y)|,$$

according to the definition of the outward–normal derivative of u in (5.457).

Using the definition of W in (5.454) we can evaluate the integral on the right–hand side of (5.468) to obtain

$$\oint_{\partial D_\varepsilon} |W| \ ds = 2\pi \varepsilon |\ln \varepsilon|,$$

for $0 < \varepsilon < 1$; so that,

$$\left| \oint_{\partial D_\varepsilon} W \frac{\partial u}{\partial n} \ ds \right| \leqslant 2\pi M_1 |\varepsilon \ln \varepsilon|, \quad \text{for } 0 < \varepsilon < 1. \tag{5.469}$$

Consequently, using (5.467) again, we obtain from (5.469) that

$$\lim_{\varepsilon \to 0^+} \oint_{\partial D_\varepsilon} W \frac{\partial u}{\partial n} \ ds = 0. \tag{5.470}$$

It remains to evaluate

$$\lim_{\varepsilon\to 0^+} \oint_{\partial D_\varepsilon} u\frac{\partial W}{\partial n}\ ds,$$

where the outward, normal derivative, $\dfrac{\partial W}{\partial n}$, of W given in (5.454) is the derivative of W with respect to the outward, radial direction from (x_o, y_o) on the boundary of $D_\varepsilon(x_o, y_o)$. Wring

$$r = \sqrt{(x-x_o)^2 + (y-y_o)^2},$$

we then have that

$$\frac{\partial W}{\partial n}\Big|_{\partial D_\varepsilon(x_o,y_o)} = \frac{d}{dr}[C\ln r]\Big|_{r=\varepsilon} = \frac{C}{\varepsilon};$$

so that,

$$\oint_{\partial D_\varepsilon} u\frac{\partial W}{\partial n}\ ds = \frac{C}{\varepsilon}\oint_{\partial D_\varepsilon} u\ ds;$$

or

$$\oint_{\partial D_\varepsilon} u\frac{\partial W}{\partial n}\ ds = 2\pi C\frac{1}{2\pi\varepsilon}\oint_{\partial D_\varepsilon} u\ ds, \quad \text{for } 0<\varepsilon<1, \tag{5.471}$$

where $\dfrac{1}{2\pi\varepsilon}\oint_{\partial D_\varepsilon} u\ ds$ is the mean value (or average value) of u on $\partial D_\varepsilon(x_o, y_o)$.

Now, since u is continuous in $\mathcal{U}$, which contains Ω, it follows that

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\pi\varepsilon}\oint_{\partial D_\varepsilon} u\ ds = u(x_o, y_o). \tag{5.472}$$

Thus, combining (5.471) and (5.472), we get that

$$\lim_{\varepsilon\to 0^+} \oint_{\partial D_\varepsilon} u\frac{\partial W}{\partial n}\ ds = 2\pi C u(x_o, y_o). \tag{5.473}$$

Next, let $\varepsilon \to 0^+$ on both sides of (5.460), and use the limits in (5.461), (5.470) and (5.473), to get that

$$-2\pi C u(x_o, y_o) = -\iint_\Omega W\Delta u\ dxdy + \oint_{\partial\Omega}\left(W\frac{\partial u}{\partial n} - u\frac{\partial W}{\partial n}\right)\ ds. \tag{5.474}$$

From this point on we set

$$-2\pi C = 1,$$

or

$$C = -\frac{1}{2\pi}; \tag{5.475}$$

so that, the equation in (5.474) becomes

$$u(x_o, y_o) = -\iint_\Omega W\Delta u\ dxdy + \oint_{\partial\Omega}\left(W\frac{\partial u}{\partial n} - u\frac{\partial W}{\partial n}\right)\ ds, \tag{5.476}$$

where, according to (5.454) and (5.454), W represents the function

$$W_{(x_o,y_o)}\colon \mathbb{R}^2\backslash\{(x_o,y_o)\} \to \mathbb{R}$$

defined by

$$W_{(x_o,y_o)}(x,y) = -\frac{1}{2\pi}\ln\sqrt{(x-x_o)^2+(y-y_o)^2}, \text{ if } (x,y)\neq(x_o,y_o). \tag{5.477}$$

It what follows we will show how to use the formula in (5.476) to obtain a representation of harmonic function $u \in C^2(\Omega,\mathbb{R})\cap C(\overline{\Omega},\mathbb{R})$ in terms of its values on the boundary of Ω. If we succeed, we will be able to obtain a representation for a solution of the Dirichlet problem in (5.448); namely,

$$\begin{cases} u_{xx}+u_{yy} = 0 & \text{in } \Omega; \\ u(x,y) = g(x,y), & \text{for } (x,y)\in\partial\Omega, \end{cases} \tag{5.478}$$

where $g\colon \partial\Omega \to \mathbb{R}$ is a given continuous function on $\partial\Omega$.

To obtain an integral representation for $u \in C^2(\mathcal{U},\mathbb{R})$ in terms of its values on $\partial\Omega$, let $H \in C^2(\mathcal{U},\mathbb{R})$ be a function that is harmonic in Ω; so that,

$$\Delta H = 0, \quad \text{in } \Omega. \tag{5.479}$$

Next, apply Green's Identity II in (5.456), with $u \in C^2(\mathcal{U},\mathbb{R})$ and H in place of v, to get

$$\iint_\Omega H\Delta u\; dxdy = \oint_{\partial\Omega}\left(H\frac{\partial u}{\partial n} - u\frac{\partial H}{\partial n}\right)\, ds,$$

where we have used (5.479), which we can rewrite as

$$0 = -\iint_\Omega H\Delta u\; dxdy + \oint_{\partial\Omega}\left(H\frac{\partial u}{\partial n} - u\frac{\partial H}{\partial n}\right)\, ds. \tag{5.480}$$

Adding the expressions in (5.476) and (5.480) we obtain

$$\begin{aligned} u(x_o,y_o) &= -\iint_\Omega (W+H)\Delta u\; dxdy \\ &\quad + \oint_{\partial\Omega}(W+H)\frac{\partial u}{\partial n}\; ds - \oint_{\partial\Omega} u\frac{\partial}{\partial n}[W+H]\; ds. \end{aligned} \tag{5.481}$$

Suppose that we can find a C^2 function $H\colon \Omega \to \mathbb{R}$ that is harmonic in Ω and such that

$$W + H = 0 \quad \text{on } \partial\Omega,$$

or

$$H = -W \quad \text{on } \partial\Omega. \tag{5.482}$$

It then follows from (5.481) and (5.482) that

$$u(x_o,y_o) = -\iint_\Omega (W+H)\Delta u\; dxdy - \oint_{\partial\Omega} u\frac{\partial}{\partial n}[W+H]\; ds, \tag{5.483}$$

for any $(x_o, y_o) \in \Omega$, where W is given by (5.477), and $H \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ is a harmonic function in Ω satisfying the boundary condition in (5.482).

Since $(x_o, y_o) \in \Omega$ in (5.483) is arbitrary, by defining

$$W_{(x,y)}(\xi, \eta) = -\frac{1}{2\pi} \ln \sqrt{(\xi - x)^2 + (\eta - y)^2}, \text{ if } (\xi, \eta) \neq (x, y), \tag{5.484}$$

and $(x, y) \in \Omega$, we obtain from (5.483) the integral representation formula

$$\begin{aligned} u(x,y) &= -\iint_\Omega (W_{(x,y)}(\xi,\eta) + H(\xi,\eta))\Delta u(\xi,\eta)\ d\xi d\eta \\ &\quad - \oint_{\partial\Omega} u(\xi,\eta) \frac{\partial}{\partial n}[W_{(x,y)}(\xi,\eta) + H(\xi,\eta)]\ ds, \end{aligned} \tag{5.485}$$

for all $(x, y) \in \Omega$.

We shall refer to the expression in (5.485) as **Green's Integral Representation Formula** for a function $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$.

5.4.2 Definition of Green's Function

We will denote the expression

$$W_{(x,y)}(\xi, \eta) + H(\xi, \eta), \quad \text{for } (\xi, \eta) \neq (x, y), \tag{5.486}$$

in Green's Integral Representation Formula in (5.485), where $W_{(x,y)}(\xi, \eta)$ is as given in (5.484), by

$$G((x, y), (\xi, \eta)) \tag{5.487}$$

and call it the Green's function of the domain Ω. With this notation, the Green's Integral Representation Formula in (5.485) becomes

$$\begin{aligned} u(x,y) &= -\iint_\Omega G((x,y),(\xi,\eta))\Delta u(\xi,\eta)\ d\xi d\eta \\ &\quad - \oint_{\partial\Omega} u(\xi,\eta) \frac{\partial}{\partial n}[G((x,y),(\xi,\eta))]\ ds, \end{aligned} \tag{5.488}$$

for all $(x, y) \in \Omega$. Thus, according to (5.488), if we are able to find a Green's function for a domain $\Omega \subset \mathbb{R}^2$, we will be able to represent any function $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ in terms of its Laplacian, Δu, in Ω, and its boundary values on $\partial\Omega$. In other words, we will be able to construct a candidate for a solution of the Dirichlet problem for Poisson's equation in (5.449); namely,

$$\begin{aligned} u(x,y) &= \iint_\Omega G((x,y),(\xi,\eta)) f(\xi,\eta)\ d\xi d\eta \\ &\quad - \oint_{\partial\Omega} g(\xi,\eta) \frac{\partial}{\partial n}[G((x,y),(\xi,\eta))]\ ds, \end{aligned} \tag{5.489}$$

for all $(x, y) \in \Omega$, provided that we make additional assumptions on f. In particular, we will assume that f is Hölder continuous with Hölder exponent α, with $0 < \alpha \leqslant 1$ (see Definition 5.4.1 on page 170 in these notes).

For the special case in which $u \in C^2(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ is harmonic in Ω and u takes on values given by g on $\partial\Omega$, we obtain from (5.488) that

$$u(x,y) = -\oint_{\partial\Omega} g(\xi,\eta) \frac{\partial}{\partial n}[G((x,y),(\xi,\eta))]\; ds, \quad \text{for } (x,y) \in \Omega,$$

which we can rewrite as

$$u(x,y) = \oint_{\partial\Omega} g(\xi,\eta) \left(-\frac{\partial}{\partial n}[G((x,y),(\xi,\eta))] \right) \; ds, \quad \text{for } (x,y) \in \Omega. \tag{5.490}$$

Thus, setting

$$P((x,y),(\xi,\eta)) = -\frac{\partial}{\partial n}[G((x,y),(\xi,\eta)) \tag{5.491}$$

in (5.490), we recover the Poisson integral representation formula for a harmonic function in Ω in terms of of its boundary values,

$$u(x,y) = \oint_{\partial\Omega} g(\xi,\eta) P((x,y),(\xi,\eta))\; ds, \quad \text{for } (x,y) \in \Omega, \tag{5.492}$$

that we obtained in Section 5.3.6 for the unit disc, D_1, in $\mathbb{R}^2$.

In this section, we discuss some properties of the Green's function. We will then compute the Green's function for the disc D_R in $\mathbb{R}^2$, for $R > 0$. In subsequent sections, we will compute the Green's functions of other simple regions in the plane.

According to the definition of Green's function given in (5.487) and (5.486) at the start of this section, if a domain $\Omega \subset \mathbb{R}^2$ has a Green's function, $G((x, y), (\xi, \eta))$, for $(x, y), (\xi, \eta) \in \Omega$, with $(x, y) \neq (\xi, \eta)$, then the function

$$(\xi,\eta) \mapsto G((x,y),(\xi,\eta)) - W_{(x,y)}(\xi,\eta), \quad \text{for } (\xi,\eta) \in \Omega,$$

is harmonic in Ω and

$$G((x,y),(\xi,\eta)) = 0, \quad \text{for } (\xi,\eta) \in \partial\Omega.$$

We note that a given domain Ω can have at most one Green function. For suppose that there are two functions, $G_1((x, y), (\xi, \eta))$ and $G_2((x, y), (\xi, \eta))$, with the properties that the maps

$$(\xi,\eta) \mapsto G_i((x,y),(\xi,\eta)) - W_{(x,y)}(\xi,\eta), \quad \text{for } (\xi,\eta) \in \Omega, \text{ and } i = 1, 2,$$

are harmonic in Ω, and

$$G_i((x,y),(\xi,\eta)) = 0, \quad \text{for } (\xi,\eta) \in \partial\Omega, \text{ and } i = 1, 2.$$

Then, the functions

$$H_i(\xi,\eta) = G_i((x,y),(\xi,\eta)) - W_{(x,y)}(\xi,\eta), \quad \text{for } (\xi,\eta)\in\Omega, \text{ and } i=1,2,$$

solve the Dirichlet problem

$$\begin{cases} u_{xx}+u_{yy} &= 0 & \text{in } \Omega; \\ u(x,y) &= -W_{(x,y)}, & \text{on } \partial\Omega, \end{cases} \tag{5.493}$$

where $-W_{(x,y)}$ is continuous on $\partial\Omega$, since $(x,y)\notin\partial\Omega$ (see the definition of $W_{(x,y)}$ in (5.484)). Consequently, since the Dirichlet problem in (5.493) can have at most one solutions (see Problem 5 in Assignment #5), it follows that

$$H_1(\xi,\eta) = H_2(\xi,\eta), \quad \text{for all } (\xi,\eta)\in\overline{\Omega}.$$

Consequently,

$$G_1((x,y),(\xi,\eta)) = G_2((x,y),(\xi,\eta)), \quad \text{for all } (\xi,\eta)\in\Omega \text{ with } (\xi,\eta)\neq(x,y).$$

In the following example, we compute the Green's function for the disc of radius R around the origin, D_R, in $\mathbb{R}^2$.

Example 5.4.2 (Green's Function of D_R)**.** For any $(x,y)\in D_R$, we construct a function $H_{(x,y)}\colon D_R\to\mathbb{R}$ that is harmonic in Ω and such that

$$H_{(x,y)}(\xi,\eta) = -W_{(x,y)}(\xi,\eta), \quad \text{for all } (\xi,\eta)\in\partial D_R. \tag{5.494}$$

In the construction, we will use the fact that, if $(x^*,y^*)\notin\overline{D}_R$, then the function

$$(\xi,\eta)\mapsto \ln\sqrt{(\xi-x^*)^2+(\eta-y^*)^2}, \quad \text{for } (\xi,\eta)\in D_R,$$

or

$$(\xi,\eta)\mapsto \ln|(\xi,\eta)-(x^*,y^*)|, \quad \text{for } (\xi,\eta)\in D_R,$$

is harmonic in D_R.

We first consider the case in which $(x,y)\in D_R$ and $(x,y)\neq(0,0)$; the case $(x,y)=(0,0)$ with be dealt with separately.

Assume that $(x,y)\in D_R$ and $(x,y)\neq(0,0)$. We pick (x^*,y^*) to be a point on the ray emanating from the origin, $(0,0)$, and going through the point (x,y) at a distance $\lambda>R$ from the origin; so that,

$$(x^*,y^*) = \frac{\lambda}{|(x,y)|}(x,y), \tag{5.495}$$

where λ will be chosen shortly.

We consider the function $H_{(x,y)}\colon D_R\to\mathbb{R}$ given by

$$H_{(x,y)}(\xi,\eta) = \frac{1}{2\pi}\ln\left(Q(x,y)|(\xi,\eta)-(x^*,y^*)|\right), \quad \text{for } (\xi,\eta)\in\overline{D}_R, \tag{5.496}$$

Where $Q(x,y) > 0$, along with $\lambda > R$ in (5.495), be chosen so that the boundary condition in (5.494) is satisfied.

We note that $H_{(x,y)}$ is harmonic in D_R, since $|(x^*,y^*)| = \lambda > R$, according to (5.495).

Next, use the definition of (x^*,y^*) in (5.495) to compute, for $(\xi,\eta) \in \partial D_R$,

$$|(\xi,\eta) - (x^*,y^*)|^2 = R^2 - 2\frac{\lambda}{|(x,y)|}(x,y)\cdot(\xi,\eta) + \lambda^2,$$

or

$$|(\xi,\eta) - (x^*,y^*)|^2 = \frac{\lambda}{|(x,y)|}\left(\frac{R^2|(x,y)|}{\lambda} - 2(x,y)\cdot(\xi,\eta) + \lambda|(x,y)|\right). \quad (5.497)$$

Setting $\lambda|(x,y)| = R^2$ in (5.497); so that,

$$\lambda = \frac{R^2}{|(x,y)|}, \quad (5.498)$$

we obtain from (5.497) that

$$|(\xi,\eta) - (x^*,y^*)|^2 = \frac{R^2}{|(x,y)|^2}\left(|(x,y)|^2 - 2(x,y)\cdot(\xi,\eta) + R^2\right),$$

or

$$|(\xi,\eta) - (x^*,y^*)|^2 = \frac{R^2}{|(x,y)|^2}|(x,y) - (\xi,\eta)|^2,$$

from which we get that

$$\frac{|(x,y)|}{R}|(\xi,\eta) - (x^*,y^*)| = |(\xi,\eta) - (x,y)|, \quad \text{for } (\xi,\eta) \in \partial D_R. \quad (5.499)$$

It follows from (5.499) that

$$\frac{1}{2\pi}\ln\left(\frac{|(x,y)|}{R}|(\xi,\eta) - (x^*,y^*)|\right) = \frac{1}{2\pi}\ln\left(|(\xi,\eta) - (x,y)|\right), \quad (5.500)$$

for all $(\xi,\eta) \in \partial D_R$. Thus, comparing (5.496) with the left–hand side of (5.500) we see that, we see that we can take

$$Q(x,y) = \frac{|(x,y)|}{R}$$

in (5.496) to obtain

$$H_{(x,y)}(\xi,\eta) = \frac{1}{2\pi}\ln\left(\frac{|(x,y)|}{R}|(\xi,\eta) - (x^*,y^*)|\right), \quad \text{for } (\xi,\eta) \in \overline{D}_R, \quad (5.501)$$

where, according to (5.495) and (5.498),

$$(x^*,y^*) = \frac{R^2}{|(x,y)|^2}\,(x,y), \quad \text{for } (x,y) \in D_R \text{ with } (x,y) \neq (0,0). \quad (5.502)$$

The identity in (5.500) then shows that

$$H_{(x,y)}(\xi,\eta) = -W_{(x,y)}(\xi,\eta), \quad \text{for } (\xi,\eta)\in \overline{D}_R,$$

which is the assertion in (5.494). It then follows from the definition of the Green function in (5.486) and (5.487) that

$$G((x,y),(\xi,\eta)) = H_{(x,y)}(\xi,\eta) + W_{(x,y)}(\xi,\eta),$$

for $(x,y),(\xi,\eta)\in D_R$with $(x,y)\neq(\xi,\eta)$, and $(x,y)\neq(0,0)$, or

$$\begin{aligned} G((x,y),(\xi,\eta)) \quad &= \quad \frac{1}{2\pi}\ln\left(\frac{|(x,y)|}{R}|(\xi,\eta)-(x^*,y^*)|\right) \\ &\quad - \frac{1}{2\pi}\ln\left(|(\xi,\eta)-(x,y)|\right), \end{aligned} \tag{5.503}$$

for $(x,y),(\xi,\eta)\in D_R$with $(x,y)\neq(\xi,\eta)$, and $(x,y)\neq(0,0)$.

Next, for the case $(x,y)=(0,0)$, observe that the constant function

$$H(\xi,\eta) = \frac{1}{2\pi}\ln R, \quad \text{for } (\xi,\eta)\in\mathbb{R}^2,$$

is harmonic everywhere. Furthermore, setting

$$W(\xi,\eta) = -\frac{1}{2\pi}\ln|(\xi,\eta)|, \quad \text{for } (\xi,\eta)\neq(0,0),$$

we see that

$$H(\xi,\eta) = -W(\xi,\eta), \quad \text{for all } (\xi,\eta)\in\partial D_R.$$

Thus, we define

$$G((0,0),(\xi,\eta)) = \frac{1}{2\pi}\ln R - \frac{1}{2\pi}\ln|(\xi,\eta)|, \quad \text{for } (\xi,\eta)\neq(0,0). \tag{5.504}$$

The function, $G((x,y),(\xi,\eta))$, defined in (5.503) and (5.504) is the Green function of the disc of radius R around the origin in $\mathbb{R}^2$. We can use it to solve the following Dirichlet problem for Poisson's equation in D_R:

$$\begin{cases} -\Delta u(x,y) &= f(x,y), \quad \text{for } (x,y)\in D_R; \\ u(x,y) &= g(x,y), \quad \text{for } (x,y)\in\partial D_R, \end{cases} \tag{5.505}$$

where $g\in C(\partial D_R,\mathbb{R})$ and $f\colon D_R\to\mathbb{R}$ is assumed to be Hölder continuous. Indeed, according to (5.489), a solution of (5.505) should satisfy

$$\begin{aligned} u(x,y) \quad &= \quad \iint_{D_R} G((x,y),(\xi,\eta))f(\xi,\eta)\; d\xi d\eta \\ &\quad + \oint_{\partial D_R} g(\xi,\eta)\left(-\frac{\partial}{\partial n}[G((x,y),(\xi,\eta))]\right)\; ds, \end{aligned} \tag{5.506}$$

for all $(x, y) \in D_R$.

Next, we compute the normal derivative of $G((x, y), (\xi, \eta))$ in the last integral in (5.506), which is

$$\frac{\partial}{\partial n}[G((x, y), (\xi, \eta))] = \nabla_{(\xi,\eta)} G((x, y), (\xi, \eta)) \cdot \frac{1}{R}(\xi, \eta), \tag{5.507}$$

for $(\xi, \eta) \in \partial D_R$, where $\nabla_{(\xi,\eta)}$ denotes the gradient take with respect to the variables ξ and η.

For $(x, y) \in D_R$ and $(x, y) \neq (0, 0)$, use (5.503) to write

$$\begin{aligned} G((x, y), (\xi, \eta)) &= \frac{1}{2\pi} \ln \frac{|(x, y)|}{R} + \frac{1}{4\pi} \ln |(\xi, \eta) - (x^*, y^*)|^2 \\ &\quad - \frac{1}{4\pi} \ln |(\xi, \eta) - (x, y)|^2; \end{aligned} \tag{5.508}$$

so that, taking the partial derivative with respect to ξ,

$$\begin{aligned} \frac{\partial}{\partial \xi} G((x, y), (\xi, \eta)) &= \frac{1}{2\pi} \frac{\xi - x^*}{|(\xi, \eta) - (x^*, y^*)|^2} \\ &\quad - \frac{1}{2\pi} \frac{\xi - x}{|(\xi, \eta) - (x, y)|^2}, \end{aligned} \tag{5.509}$$

for $(\xi, \eta) \in \partial D_R$.

Similarly, taking the partial with respect to η on both sides of (5.508),

$$\begin{aligned} \frac{\partial}{\partial \eta} G((x, y), (\xi, \eta)) &= \frac{1}{2\pi} \frac{\eta - y^*}{|(\xi, \eta) - (x^*, y^*)|^2} \\ &\quad - \frac{1}{2\pi} \frac{\eta - y}{|(\xi, \eta) - (x, y)|^2}, \end{aligned} \tag{5.510}$$

for $(\xi, \eta) \in \partial D_R$.

We can now use (5.509) and (5.510) to compute (5.507) to get

$$\begin{aligned} \frac{\partial}{\partial n} G((x, y), (\xi, \eta)) &= \frac{1}{2\pi R} \frac{\xi^2 + \eta^2 - x^*\xi - y^*\eta}{|(\xi, \eta) - (x^*, y^*)|^2} \\ &\quad - \frac{1}{2\pi R} \frac{\xi^2 + \eta^2 - x\xi - y\eta}{|(\xi, \eta) - (x, y)|^2}, \end{aligned}$$

for $(\xi, \eta) \in \partial D_R$, or

$$\begin{aligned} \frac{\partial}{\partial n} G((x, y), (\xi, \eta)) &= \frac{1}{2\pi R} \frac{R^2 - (x^*, y^*) \cdot (\xi, \eta)}{|(\xi, \eta) - (x^*, y^*)|^2} \\ &\quad - \frac{1}{2\pi R} \frac{R^2 - (x, y) \cdot (\xi, \eta)}{|(\xi, \eta) - (x, y)|^2}, \end{aligned} \tag{5.511}$$

for $(\xi,\eta)\in\partial D_R$.

Next, use the definition of (x^*,y^*) in (5.502) and the identity in (5.499) to rewrite (5.511) as

$$\begin{aligned}\frac{\partial}{\partial n}G((x,y),(\xi,\eta)) &= \frac{1}{2\pi R}\frac{|(x,y)|^2-(x,y)\cdot(\xi,\eta)}{|(\xi,\eta)-(x,y)|^2}\\ &\quad -\frac{1}{2\pi R}\frac{R^2-(x,y)\cdot(\xi,\eta)}{|(\xi,\eta)-(x,y)|^2},\end{aligned}$$

for $(\xi,\eta)\in\partial D_R$, which can be rewritten as

$$\frac{\partial}{\partial n}G((x,y),(\xi,\eta)) = \frac{1}{2\pi R}\frac{|(x,y)|^2-R^2}{|(\xi,\eta)-(x,y)|^2},\qquad \text{for } (\xi,\eta)\in\partial D_R,$$

from which we get that

$$-\frac{\partial}{\partial n}G((x,y),(\xi,\eta)) = \frac{1}{2\pi R}\frac{R^2-|(x,y)|^2}{|(\xi,\eta)-(x,y)|^2},\qquad \text{for } (\xi,\eta)\in\partial D_R. \tag{5.512}$$

Observe that the expression in (5.512) also works for $(x,y)=(0,0)$, according to the definition of $G((0,0),(\xi,\eta))$ in (5.504).

The expression on the right–hand side of (5.512) is the Poisson kernel, $P((x,y),(\xi,\eta))$, for the disc D_R (compare with the result of Problem 2 in Assignment #6). We therefore get from (5.506) and (5.512) that the solution of the Dirichlet problem for Poisson's equation in (5.505) should have the representation

$$\begin{aligned}u(x,y) &= \iint_{D_R} G((x,y),(\xi,\eta))f(\xi,\eta)\; d\xi d\eta\\ &\quad + \oint_{\partial D_R} P((x,y),(\xi,\eta))g(\xi,\eta)\; ds,\end{aligned} \tag{5.513}$$

for all $(x,y)\in D_R$, where

$$P((x,y),(\xi,\eta)) = \frac{1}{2\pi R}\frac{R^2-|(x,y)|^2}{|(\xi,\eta)-(x,y)|^2},\qquad \text{for } (\xi,\eta)\in\partial D_R, \tag{5.514}$$

and $(x,y)\in D_R$, is the Poisson kernel of the disc D_R, and $G((x,y),(\xi,\eta))$ is its Green's function defined in (5.503) and (5.504).

Next, we consider two special cases of the representation formula in (5.513).

First, suppose that $f(x,y)=0$ for all $(x,y)\in D_R$ in (5.513) and (5.505); then,

$$u(x,y) = \oint_{\partial D_R} P((x,y),(\xi,\eta))g(\xi,\eta)\; ds, \tag{5.515}$$

for all $(x,y)\in D_R$, where $P((x,y),(\xi,\eta))$ is the Poisson kernel of D_R given in (5.514) solves the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D_R;\\ u(x,y) = g(x,y), & \text{for } (x,y)\in\partial D_R,\end{cases}$$

where $g \in C(\partial D_R, \mathbb{R})$ is a given continuous function. We have therefore recovered the Poisson integral representation of a harmonic function in (5.515) that we derived in Section 5.3 for the unit disc D_1 using separation of variables and Fourier series.

Second, suppose that $g(x,y) = 0$ for all $(x,y) \in \partial D_R$ in (5.513) and (5.505); then,

$$u(x,y) = \iint_{D_R} G((x,y),(\xi,\eta))f(\xi,\eta)\ d\xi d\eta, \quad \text{for } (x,y) \in D_R, \tag{5.516}$$

where $G((x,y),(\xi,\eta))$ is the Green's function of D_R defined in (5.503) and (5.504), is a candidate for a solution of the Dirichlet problem for Poisson's equation in D_R:

$$\begin{cases} -\Delta u(x,y) &= f(x,y), \quad \text{for } (x,y) \in D_R; \\ u(x,y) &= 0, \quad \text{for } (x,y) \in \partial D_R, \end{cases} \tag{5.517}$$

provided that we assume that $f\colon D_R \to \mathbb{R}$ is Hölder continuous in Ω. We will show this in the next section.

5.4.3 Solving Poisson's Equation

The goal of this section is to show that, if $f\colon D_R \to \mathbb{R}$ is Hölder continuous with Hölder exponent $\alpha \in (0,1]$ (see Definition 5.4.1), then then function u given in (5.516) solves the Dirichlet problem for Poisson's equation in (5.517).

We will frame the discussion in this section in the context of a general domain Ω, with piecewise C^1 boundary, $\partial\Omega$, and for which a Green's function, $G((x,y),(\xi,\eta))$, exists. An instance of such a domain is the disc of radius R around the origin in $\mathbb{R}^2$, D_R, discussed in the previous section. Thus, we will show that the function $u\colon \Omega \to \mathbb{R}$ given by

$$u(x,y) = \iint_{\Omega} G((x,y),(\xi,\eta))f(\xi,\eta)\ d\xi d\eta, \quad \text{for } (x,y) \in \Omega, \tag{5.518}$$

where $f\colon \Omega \to \mathbb{R}$ is a Hölder continuous function with Hölder exponent $\alpha \in (0,1]$, solves the Dirichlet problem

$$\begin{cases} -\Delta u(x,y) &= f(x,y), \quad \text{for } (x,y) \in \Omega; \\ u(x,y) &= 0, \quad \text{for } (x,y) \in \partial \Omega. \end{cases} \tag{5.519}$$

Before we prove this result, we will have to establish an additional property of the Green function.

Let Ω denote an open, bounded subset of $\mathbb{R}^2$ with piecewise C^1 boundary, $\partial\Omega$, and suppose that Ω has a Green function, $G((x,y),(\xi,\eta))$, for $(x,y),(\xi,\eta) \in \overline{\Omega}$ such that $(x,y) \neq (\xi,\eta)$. We have already seen that a Green function for Ω, if it exists, it is unique. We have also seen that

$$G((x,y),(\xi,\eta)) = H((x,y),(\xi,\eta)) + W((x,y),(\xi,\eta)), \tag{5.520}$$

where the map

$$(\xi,\eta) \mapsto H((x,y),(\xi,\eta)), \quad \text{for } (\xi,\eta)\in\Omega,$$

is harmonic if Ω; so that,

$$H_{\xi\xi} + H_{\eta\eta} = 0, \quad \text{in } \Omega, \tag{5.521}$$

and

$$W((x,y),(\xi,\eta)) = -\frac{1}{2\pi}\ln|(\xi,\eta)-(x,y)|, \quad \text{for } (x,y)\neq(\xi,\eta). \tag{5.522}$$

It is also the case that

$$G((x,y),(\xi,\eta)) = 0, \quad \text{for all } (x,y)\in\Omega \text{ and } (\xi,\eta)\in\partial\Omega. \tag{5.523}$$

We will next show that G is symmetric; that is,

$$G((x_1,y_1),(x_2,y_2)) = G((x_2,y_2),(x_1,y_1)), \quad \text{for } (x_1,y_1)\neq(x_2,y_2), \tag{5.524}$$

and $(x_1,y_1),(x_2,y_2)\in\Omega$.

To establish (5.524), we proceed as in the derivation of the Green Representation Formula in Section 5.4.1.

Assume that $(x_1,y_1),(x_2,y_2)\in\Omega$ are such that

$$(x_1,y_1)\neq(x_2,y_2). \tag{5.525}$$

Define a function $v_1\colon \Omega\backslash\{(x_1,y_1)\}\to\mathbb{R}$ by

$$v_1(\xi,\eta) = G((x_1,y_1),(\xi,\eta)), \quad \text{for } (\xi,\eta)\in\Omega\backslash\{(x_1,y_1)\}. \tag{5.526}$$

It then follows from the definition of the Green's function in (5.520), (5.521) and (5.522) that v_1 is harmonic in $\Omega\backslash\{(x_1,y_1)\}$; indeed, there exists a harmonic function $h_1\colon\Omega\to\mathbb{R}$ such that

$$v_1(\xi,\eta) = h_1(\xi,\eta) + W((x_1,y_1),(\xi,\eta)) \quad \text{for } (\xi,\eta)\in\Omega\backslash\{(x_1,y_1)\}. \tag{5.527}$$

Furthermore,

$$v_1(\xi,\eta) = 0, \quad \text{for } (\xi,\eta)\in\partial\Omega, \tag{5.528}$$

by virtue of (5.523).

Similarly, defining $v_2\colon\Omega\backslash\{(x_2,y_2)\}\to\mathbb{R}$ by

$$v_2(\xi,\eta) = G((x_2,y_2),(\xi,\eta)), \quad \text{for } (\xi,\eta)\in\Omega\backslash\{(x_2,y_2)\}, \tag{5.529}$$

there exists a harmonic function $h_2\colon\Omega\to\mathbb{R}$ such that

$$v_2(\xi,\eta) = h_2(\xi,\eta) + W((x_2,y_2),(\xi,\eta)), \quad \text{for } (\xi,\eta)\in\Omega\backslash\{(x_2,y_2)\}; \tag{5.530}$$

so that, v_2 is harmonic in $\Omega\backslash\{(x_2,y_2)\}$ and

$$v_2(\xi,\eta) = 0, \quad \text{for } (\xi,\eta)\in\partial\Omega. \tag{5.531}$$

We will show that

$$v_1(x_2, y_2) = v_2(x_1, y_1). \tag{5.532}$$

This will establish (5.524), in view of the definitions of v_1 and v_2 in (5.526) and (5.529), respectively.

Since Ω is open and (x_1, y_1) and (x_2, y_2) are distinct points in Ω (see (5.525), there exist $\varepsilon_o > 0$ such that

$$\overline{D}_{\varepsilon_o}(x_1, y_1) \subset \Omega, \quad \overline{D}_{\varepsilon_o}(x_2, y_2) \subset \Omega,$$

and

$$\overline{D}_{\varepsilon_o}(x_1, y_1) \cap \overline{D}_{\varepsilon_o}(x_2, y_2) = \emptyset.$$

We may assume that $\varepsilon_o < 1$.

Let $0 < \varepsilon < \varepsilon_o$. As in the derivation of the Green Representation Formula in Section 5.4.1, define

$$\Omega_\varepsilon = \Omega \backslash (\overline{D}_\varepsilon(x_1, y_1) \cup \overline{D}_\varepsilon(x_2, y_2)),$$

and observe that, by virtue of the definitions of v_1 and v_2 in (5.527) and (5.530), respectively, v_1 and v_2 are harmonic in Ω_ε. Thus, applying Green's Identity II in (5.456) with v_1 and v_2 in place of u and v, and Ω_ε in place of Ω,

$$\oint_{\partial\Omega_\varepsilon} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds = \iint_{\Omega_\varepsilon} (v_2 \Delta v_1 - v_1 \Delta v_2) \, d\xi d\eta,$$

from which we get that

$$\oint_{\partial\Omega_\varepsilon} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds = 0, \tag{5.533}$$

since v_1 and v_2 are harmonic in Ω_ε.

Observe that the boundary of Ω_ε is made up of the boundary of Ω together with the circles $\partial D_\varepsilon(x_1, y_1)$ and $\partial D_\varepsilon(x_2, y_2)$. We can therefore the integral on the left–hand side as

$$\begin{aligned} \oint_{\partial\Omega_\varepsilon} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds \quad &= \quad \oint_{\partial\Omega} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds \\ &\quad - \oint_{\partial D_1} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds \\ &\quad - \oint_{\partial D_2} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds, \end{aligned}$$

or, in view of (5.528) and (5.531),

$$\begin{aligned} \oint_{\partial\Omega_\varepsilon} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds \quad &= \quad - \oint_{\partial D_1} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds \\ &\quad - \oint_{\partial D_2} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds, \end{aligned} \tag{5.534}$$

where we have written D_1 for $D_\varepsilon(x_1, y_1)$ and D_2 for $D_\varepsilon(x_2, y_2)$. Combining (5.533) and (5.534) we then get that

$$\oint_{\partial D_2} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds \quad = \quad - \oint_{\partial D_1} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds,$$

or

$$\oint_{\partial D_2} \left(v_2 \frac{\partial v_1}{\partial n} - v_1 \frac{\partial v_2}{\partial n} \right) \, ds \quad = \quad \oint_{\partial D_1} \left(v_1 \frac{\partial v_2}{\partial n} - v_2 \frac{\partial v_1}{\partial n} \right) \, ds. \tag{5.535}$$

Next, we estimate the integral

$$\oint_{\partial D_2} v_2 \frac{\partial v_1}{\partial n} \, ds = \oint_{\partial D_2} \left(h_2(\xi, \eta) - \frac{1}{2\pi} \ln |(\xi, \eta) - (x_2, y_2)| \right) \frac{\partial v_1}{\partial n} \, ds, \tag{5.536}$$

where we have used the definition of v_2 in (5.530).

Observe that v_1 in C^2 on $\overline{D}_{\varepsilon_o}(x_2, y_2)$, since, in view of the definition of v_1 in (5.527), v_1 is harmonic on $\Omega \backslash \{(x_1, y_1)\}$. Consequently, there exists a constant $M_1' > 0$ such that

$$|\nabla v_1(\xi, \eta)| \leqslant M_1', \quad \text{for all } (\xi, \eta) \in \overline{D}_{\varepsilon_o}(x_2, y_2). \tag{5.537}$$

Similarly, since h_2 is harmonic in Ω, there exists a constant $M_2 > 0$ such that

$$|h_2(\xi, \eta)| \leqslant M_2, \quad \text{for all } (\xi, \eta) \in \overline{D}_{\varepsilon_o}(x_2, y_2). \tag{5.538}$$

Consequently, using the estimates in (5.537) and (5.538), we obtain the estimate

$$\left| \oint_{\partial D_2} h_2(\xi, \eta) \frac{\partial v_1}{\partial n} \, ds \right| \leqslant M_1' M_2 2\pi\varepsilon, \quad \text{for } 0 < \varepsilon < \varepsilon_o. \tag{5.539}$$

It follows from (5.539) that

$$\lim_{\varepsilon \to 0^+} \oint_{\partial D_\varepsilon(x_2, y_2)} h_2(\xi, \eta) \frac{\partial v_1}{\partial n} \, ds = 0. \tag{5.540}$$

Next, use the estimate in (5.537) again to estimate

$$\left| \oint_{\partial D_2} \frac{1}{2\pi} \ln |(\xi, \eta) - (x_2, y_2)| \frac{\partial v_1}{\partial n} \, ds \right| \leqslant M_1' |\varepsilon \ln \varepsilon|, \quad \text{for } 0 < \varepsilon < \varepsilon_o. \tag{5.541}$$

It follows from (5.541) that

$$\lim_{\varepsilon \to 0^+} \oint_{\partial D_\varepsilon(x_2, y_2)} \frac{1}{2\pi} \ln |(\xi, \eta) - (x_2, y_2)| \frac{\partial v_1}{\partial n} \, ds = 0. \tag{5.542}$$

Combining the results in (5.540) and (5.542) with (5.536) we conclude that

$$\lim_{\varepsilon \to 0^+} \oint_{\partial D_\varepsilon(x_2, y_2)} v_2 \frac{\partial v_1}{\partial n} \, ds = 0. \tag{5.543}$$

Similar arguments can be used to show that

$$\lim_{\varepsilon\to 0^+}\oint_{\partial D_\varepsilon(x_1,y_1)} v_1\frac{\partial v_2}{\partial n}\ ds = 0. \tag{5.544}$$

Next, we evaluate the limit

$$\lim_{\varepsilon\to 0^+}\oint_{\partial D_\varepsilon(x_2,y_2)} v_1\frac{\partial v_2}{\partial n}\ ds. \tag{5.545}$$

Using (5.530) we can rewrite the integral in (5.545) as

$$\oint_{\partial D_\varepsilon(x_2,y_2)} v_1\left(\frac{\partial h_2}{\partial n} - \frac{1}{2\pi}\frac{\partial}{\partial n}\ln|(\xi,\eta)-(x_2,y_2)|\right)\ ds;$$

so that,

$$\oint_{\partial D_2} v_1\frac{\partial v_2}{\partial n}\ ds = \oint_{\partial D_2} v_1\frac{\partial h_2}{\partial n}\ ds - \frac{1}{2\pi}\oint_{\partial D_2} v_1\frac{\partial}{\partial n}\ln|(\xi,\eta)-(x_2,y_2)|\ ds, \tag{5.546}$$

where we have written D_2 for $D_\varepsilon(x_2,y_2)$.

We estimate the first integral on the right–hand side of (5.546) as follows:

$$\left|\oint_{\partial D_2} v_1\frac{\partial h_2}{\partial n}\ ds\right| \leqslant M_1 M_2' 2\pi\varepsilon, \quad \text{for } 0<\varepsilon<\varepsilon_o, \tag{5.547}$$

where

$$M_1 = \max_{(\xi,\eta)\in\overline{D}_{\varepsilon_o}(x_2,y_2)} |v_1(\xi,\eta)|,$$

and

$$M_2' = \max_{(\xi,\eta)\in\overline{D}_{\varepsilon_o}(x_2,y_2)} |\nabla h_2(\xi,\eta)|.$$

It follows from (5.547) that

$$\lim_{\varepsilon\to 0^+}\oint_{\partial D_\varepsilon(x_2,y_2)} v_1\frac{\partial h_2}{\partial n}\ ds = 0. \tag{5.548}$$

Next, use the fact that

$$\frac{\partial}{\partial n}\ln|(\xi,\eta)-(x_2,y_2)| = \frac{d}{dr}\ln r\Big|_{r=\varepsilon} = \frac{1}{\varepsilon}, \quad \text{for all } (\xi,\eta)\in\partial D_\varepsilon(x_2,y_2),$$

to evaluate the last expression on the right–hand side of (5.546) as follows:

$$\frac{1}{2\pi}\oint_{\partial D_2} v_1\frac{\partial}{\partial n}\ln|(\xi,\eta)-(x_2,y_2)|\ ds = \frac{1}{2\pi\varepsilon}\oint_{\partial D_\varepsilon(x_2,y_2)} v_1\ ds, \tag{5.549}$$

the average value of v_1 over the circle $\partial D_\varepsilon(x_2,y_2)$. Thus, since v_1 is continuous on $\overline{D}_{\varepsilon_o}(x_2,y_2)$, it follows from (5.549) that

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\oint_{\partial D_2} v_1\frac{\partial}{\partial n}\ln|(\xi,\eta)-(x_2,y_2)|\ ds = v_1(x_2,y_2). \tag{5.550}$$

Hence, letting $\varepsilon \to 0^+$ in (5.546), and using the limits in (5.548) and (5.550), we get that

$$\lim_{\varepsilon\to 0^+} \oint_{\partial D_\varepsilon(x_2,y_2)} v_1 \frac{\partial v_2}{\partial n}\, ds = -v_1(x_2,y_2). \tag{5.551}$$

Similar calculations to those leading to (5.551) can be used to show that

$$\lim_{\varepsilon\to 0^+} \oint_{\partial D_\varepsilon(x_1,y_1)} v_2 \frac{\partial v_1}{\partial n}\, ds = -v_2(x_1,y_1). \tag{5.552}$$

Finally, letting $\varepsilon \to 0^+$ in (5.535), and using the limits in (5.543), (5.544), (5.551) and (5.552), we get that

$$v_1(x_2,y_2) = v_2(x_1,y_1),$$

which is the assertion in (5.532). We have therefore established the symmetry of the Green's function.

It follows from the definition of the Green's function in (5.520)–(5.523), and the fact that $G((x,y),(\xi,\eta))$, for $(x,y) \neq (\xi,\eta)$, is symmetric, that the map

$$(x,y) \mapsto G((x,y),(\xi,\eta)), \quad \text{for } (x,y) \neq (\xi,\eta),$$

is harmonic as a function of (x,y), if $(x,y) \neq (\xi,\eta)$. Using the definition of G in (5.520), we then get that

$$H((x,y),(\xi,\eta)) = G((x,y),(\xi,\eta)) - W((x,y),(\xi,\eta)),$$

for $(x,y) \neq (\xi,\eta)$, is also symmetric. Consequently, the map

$$(x,y) \mapsto H((x,y),(\xi,\eta)), \quad \text{for } (x,y) \in \Omega,$$

is harmonic as a function of (x,y) in Ω; so that,

$$H_{xx} + H_{yy} = 0, \quad \text{in } \Omega. \tag{5.553}$$

Using the symmetry of G, we may define

$$G((x,y),(\xi,\eta)) = 0, \quad \text{for } (x,y) \in \partial\Omega \text{ and } (\xi,\eta) \in \Omega. \tag{5.554}$$

Next, assume that $f\colon \Omega \to \mathbb{R}$ is Hölder continuous with Hölder α, with $0 < \alpha \leqslant 1$. Then, according to Definition 5.4.1, there exists a constant $M > 0$ such that

$$|f(x,y) - f(\xi,\eta)| \leqslant M|(x,y) - (\xi,\eta)|^\alpha, \quad \text{for all } (x,y),(\xi,\eta) \in \Omega. \tag{5.555}$$

It follows from (5.555) that f is continuous on Ω and that there exists a constant $M_o > 0$ such that

$$|f(\xi,\eta)| \leqslant M_o, \quad \text{for all } (\xi,\eta) \in \Omega. \tag{5.556}$$

(Recall that we are assuming that Ω is a bounded domain in $\mathbb{R}^2$).

Define $u\colon \Omega \to \mathbb{R}$ by

$$u(x,y) = \iint_\Omega G((x,y),(\xi,\eta))f(\xi,\eta)\ d\xi d\eta, \qquad \text{for } (x,y) \in \Omega. \tag{5.557}$$

We will show that u solves Poisson's equation

$$-\Delta u = f, \qquad \text{in } \Omega.$$

Furthermore, it follows from the definition of u in (5.557) and from (5.554) that

$$u(x,y) = 0, \qquad \text{for all } (x,y) \in \partial\Omega.$$

Thus, the function u given in (5.557) solves the Dirichlet problem for Poisson's equation given in (5.519); namely,

$$\begin{cases} -\Delta u(x,y) = f(x,y), & \text{for } (x,y) \in \Omega; \\ u(x,y) = 0, & \text{for } (x,y) \in \partial\Omega. \end{cases} \tag{5.558}$$

Appendix A

Facts from the Theory of Ordinary Differential Equations

In this appendix we present some of the facts from the theory of ordinary differential equations (ODEs) that are used in these notes. We begin with linear second order ODEs with constant coefficients.

A.1 Linear, Second Order ODEs with Constant Coefficients

We discuss here how to construct the general solution of the linear, homogeneous, second order ODE with constant coefficients

$$ay'' + by' + cy = 0, \tag{A.1}$$

where a, b and c are real constants, and y is assumed to be a twice differentiable function of $x \in \mathbb{R}$

We look for solutions of (A.1) of the form

$$y(x) = e^{mx}, \quad \text{for } x \in \mathbb{R}, \tag{A.2}$$

where m is a parameter to be determined shortly.

Differentiating the function y defined in (A.2) and substituting into the ODE in (A.1) yields the equation

$$am^2 + bm + c = 0. \tag{A.3}$$

The quadratic equation in (A.3) is called the **characteristic equation** of the ODE in (A.1). Its roots could be real and distinct, real and equal, or complex conjugates.

If the roots, r_1 and r_2, of the characteristic equation in (A.3) are real and distinct, the general solution of the ODE in (A.1) is given by

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad \text{for all } x \in \mathbb{R}, \tag{A.4}$$

for arbitrary constants c_1 and c_2.

If $r \in \mathbb{R}$ is the only root of the characteristic equation (A.3), the general solution of the ODE in (A.1) is given by

$$y(x) = c_1 e^{rx} + c_2 x e^{rx}, \quad \text{for all } x \in \mathbb{R}, \tag{A.5}$$

for arbitrary constants c_1 and c_2.

If the roots of the characteristic equation (A.3) are the complex conjugates $\alpha \pm i\beta$, the general solution of the ODE in (A.1) is given by

$$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x, \quad \text{for all } x \in \mathbb{R}, \tag{A.6}$$

for arbitrary constants c_1 and c_2.

Appendix B

Theorems About Integration

B.1 Differentiating Under the Integral Sign

Solutions of problems in the Calculus of Variations often require the differentiation of functions defined in terms of integrals of other functions. In many instance this involves differentiation under the integral sign. In this appendix we preset a few results that specify conditions under which differentiation under the integral sign is valid.

Proposition B.1.1 (Differentiation Under the Integral Sign)**.** Suppose that $H\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is a C^1 function. Define $h\colon \mathbb{R}\to\mathbb{R}$ by

$$h(t) = \int_a^b H(x,t)\ dx, \qquad \text{for all} \ \ t\in\mathbb{R}.$$

Assume that the functions H and $\dfrac{\partial H}{\partial t}$ are absolutely integrable over $[a,b]$. Then, h is C^1 and its derivative is given by

$$h'(t) = \int_a^b \frac{\partial}{\partial t}[H(x,t)]\ dx.$$

Proposition B.1.2 (Differentiation Under the Integral Sign and Fundamental Theorem of Calculus)**.** Suppose that $H\colon [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a C^1 function. Define

$$h(y,t) = \int_a^t H(x,y,t)\ dx, \qquad \text{for all } y\in\mathbb{R},\ t\in\mathbb{R}.$$

Assume that the functions H, $\dfrac{\partial}{\partial y}[H(x,y,t)]$ and $\dfrac{\partial}{\partial t}[H(x,y,t)]$ are absolutely integrable over $[a,b]$. Then, h is C^1 and its partial derivatives are given by

$$\frac{\partial}{\partial y}[h(y,t)] = \int_a^t \frac{\partial}{\partial y}[H(x,y,t)]\ dx$$

and

$$\frac{\partial}{\partial t}[h(x,t)] = H(t,y,t) + \int_a^t \frac{\partial}{\partial t}[H(x,y,t)]\ dx.$$

Proposition B.1.2 can be viewed as a generalization of the Fundamental Theorem of Calculus and is a special case of Leibnitz Rule.

B.2 The Divergence Theorem

We begin by stating the two–dimensional version of the divergence theorem. We then present some consequences of the result.

Let U denote an open subset of $\mathbb{R}^2$ and Ω a subset of U such that $\overline{\Omega} \subset U$. We assume that Ω is bounded with boundary, $\partial\Omega$, that can be parmetrized by $\sigma\colon [0,1] \to \mathbb{R}$, where $\sigma(t) = (x(t), y(t))$, for $t \in [0,1]$, with $x, y \in C^1([0,1],\mathbb{R})$ satisfying

$$(\dot{x}(t))^2 + (\dot{y}(t))^2 \neq 0, \quad \text{for all } t \in [0,1], \tag{B.1}$$

(where the dot on top of the variable indicates derivative with respect to t), and $\sigma(0) = \sigma(1)$. Implicit in the definition of a parametrization is the assumption that the map $\sigma\colon [0,1) \to \mathbb{R}^2$ is one–to–one on $[0,1)$. Thus, $\partial\Omega$ is a simple closed curve in U. Observe that the assumption in (B.1) implies that at every point $\sigma(t) \in \partial\Omega$, a tangent vector

$$\sigma'(t) = (\dot{x}(t), \dot{y}(t)), \quad \text{for } t \in [0,1]. \tag{B.2}$$

Let $\overrightarrow{F} : U \to \mathbb{R}^2$ denote a C^1 vector field in U; so that,

$$\overrightarrow{F}(x,y) = (P(x,y), Q(x,y)), \quad \text{for } (x,y) \in U, \tag{B.3}$$

where $P\colon U \to \mathbb{R}$ and $Q\colon U \to \mathbb{R}$ are C^1, real–valued functions defined on U.

The **divergence** of the vector field $\overrightarrow{F} \in C^1(U,\mathbb{R}^2)$ given in (B.3) is a scalar field $\operatorname{div}\overrightarrow{F} : U \to \mathbb{R}$ defined by

$$\operatorname{div}\overrightarrow{F}(x,y) = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y) \quad \text{for } (x,y) \in U. \tag{B.4}$$

Example B.2.1. Imagine a two–dimensional fluid moving through a region U in the xy–plane. Suppose the velocity of the fluid at a point $(x,y) \in \mathbb{R}^2$ is given by a C^1 vector field $\overrightarrow{V} : U \to \mathbb{R}^2$ in units of distance per time. Suppose that we also know the density of the fluid, $\rho(x,y)$ at any point $(x,y) \in U$ (in units of mass per area), and that $\rho\colon U \to \mathbb{R}$ is a C^1 scalar field. Define

$$\overrightarrow{F}(x,y) = \rho(x,y)\overrightarrow{V}(x,y), \quad \text{for } (x,y) \in U. \tag{B.5}$$

Then $\overrightarrow{F}$ has units of mass per unit length, per unit time. The vector field $\overrightarrow{F}$ in (B.5) is called the **flow field** and it measures the amount of fluid per unit time that goes through a cross section of unit length perpendicular to the direction of $\overrightarrow{V}$. Thus, to get a measure of the amount of fluid per unit time that crosses the boundary $\partial\Omega$ in direction away from the region Ω, we compute the line integral

$$\oint_{\partial\Omega} \overrightarrow{F} \cdot \widehat{n}\ ds, \tag{B.6}$$

where ds is the element of arc–length along $\partial\Omega$, and $\widehat{n}$ is unit vector that is perpendicular to the curve $\partial\Omega$ and points away from Ω. The, expression in (B.6) is called the **flux** of the flow field $\overrightarrow{F}$ across $\partial\Omega$ and it measures the amount of fluid per unit time that crosses the boundary $\partial\Omega$.

On the other hand, the divergence, $\operatorname{div}\overrightarrow{F}$, of the flow field $\overrightarrow{F}$ in (B.5) has units of mass/time $\times$ length^2, and it measures the amount of fluid that diverges from a point per unit time per unit area. Thus, the integral

$$\iint_\Omega \operatorname{div}\overrightarrow{F}\ dxdy \tag{B.7}$$

the total amount of fluid leaving the reagin Ω per unit time. In the case where there are not sinks or sources of fluid inside the region Ω, the integrals in (B.6) and (B.7) must be equal; so that,

$$\iint_\Omega \operatorname{div}\overrightarrow{F}\ dxdy = \oint_{\partial\Omega} \overrightarrow{F} \cdot \widehat{n}\ ds. \tag{B.8}$$

The expression in (B.8) is the Divergence Theorem.

Theorem B.2.2 (The Divergence Theorem in $\mathbb{R}^2$)**.** Let U be an open subset of $\mathbb{R}^2$ and Ω an open subset of U such that $\overline{\Omega} \subset U$. Suppose that Ω is bounded with boundary $\partial\Omega$. Assume that $\partial\Omega$ is a piece–wise C^1, simple, closed curve. Let $\overrightarrow{F} \in C^1(U, \mathbb{R}^2)$. Then,

$$\iint_\Omega \operatorname{div}\overrightarrow{F}\ dxdy = \oint_{\partial\Omega} \overrightarrow{F} \cdot \widehat{n}\ ds, \tag{B.9}$$

where $\widehat{n}$ is the outward, unit, normal vector to $\partial\Omega$ that exists everywhere on $\partial\Omega$, except possibly at finitely many points.

For the special case in which $\partial\Omega$ is parmatrized by $\sigma \in C^1([0,1], \mathbb{R}^2)$ satisfying (B.2), $\sigma(0) = \sigma(1)$, the map $\sigma\colon [0,1) \to \mathbb{R}^2$ is one–to–one, and σ is oriented in the counterclockwise sense, the outward unit normal to $\partial\Omega$ is given by

$$\widehat{n}(\sigma(t)) = \frac{1}{|\sigma'(t)|}(\dot{y}(t), -\dot{x}(t)), \qquad \text{for } t \in [0,1]. \tag{B.10}$$

Note that the vector $\widehat{n}$ in (B.10) is a unit vector that is perpendicular to the vector $\sigma'(t)$ in (B.2) that is tangent to the curve at $\sigma(t)$. In follows from (B.10) that, for the C^1 vector field $\overrightarrow{F}$ given in (B.3), the line integral on the right–hand side of (B.9) can be written as

$$\oint_{\partial\Omega} \overrightarrow{F} \cdot \widehat{n}\ ds = \int_0^1 (P(\sigma(t)), Q(\sigma(t))) \cdot \frac{1}{|\sigma'(t)|}(\dot{y}(t), -\dot{x}(t))\ |\sigma'(t)|\ dt,$$

or

$$\oint_{\partial\Omega} \overrightarrow{F} \cdot \widehat{n}\ ds = \int_0^1 [P(\sigma(t))\dot{y}(t) - Q(\sigma(t))\dot{x}(t)]\ dt,$$

which we can write, using differentials, as

$$\oint_{\partial\Omega} \overrightarrow{F} \cdot \widehat{n}\ ds = \oint_{\partial\Omega} (Pdy - Qdx). \tag{B.11}$$

Thus, using the definition of the divergence of $\overrightarrow{F}$ in (B.4) and (B.11), we can rewrite (B.9) as

$$\iint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\ dxdy = \oint_{\partial\Omega} (Pdy - Qdx), \tag{B.12}$$

which is another form of the Divergence Theorem in (B.9).

Applying the Divergence Theorem (B.9) to the vector field $\overrightarrow{F} = (Q, -P)$, where $P, Q \in C^1(U, \mathbb{R})$ yields from (B.12) that

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dxdy = \oint_{\partial\Omega} (Pdx + Qdy),$$

which is **Green's Theorem**.

As an application of the Divergence Theorem as stated in (B.12), consider the case of the vector field $(P, Q) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$. In this case (B.12) yields

$$\iint_\Omega 2\ dxdy = \oint_{\partial\Omega} (xdy - ydx),$$

or

$$2\ \text{area}(\Omega) = \oint_{\partial\Omega} (xdy - ydx),$$

from which we get the formula

$$\text{area}(\Omega) = \frac{1}{2}\oint_{\partial\Omega} (xdy - ydx), \tag{B.13}$$

for the area of the region Ω enclosed by a simple closed curve $\partial\Omega$.

Appendix C

Kernels

C.1 The Dirichlet Kernel

In this section we derive the formula for the Dirichlet kernel (5.74) used in connection with the proof of convergence of Fourier series.

The Dirichlet kernel was defined in (5.72) as

$$D_{_N}(\theta) = \frac{1}{2} + \sum_{n=1}^{N} \cos\left(\frac{n\pi\theta}{L}\right). \quad \text{for } \theta \in \mathbb{R}. \tag{C.1}$$

We will show that

$$D_{_N}(\theta) = \frac{\sin\left[\left(N + \frac{1}{2}\right)\frac{\pi\theta}{L}\right]}{2\sin\left(\frac{\pi\theta}{2L}\right)}, \quad \text{for } \theta \neq 0. \tag{C.2}$$

We present two derivations of (C.2). The first one involves the use of Euler's formula

$$e^{iy} = \cos y + i \sin y, \quad \text{for all } y \in \mathbb{R}. \tag{C.3}$$

The second involves the use of some trigonometric identities.

Denote $\dfrac{\pi\theta}{L}$ by x and compute

$$\sum_{n=0}^{N} e^{ix} = \frac{1 - e^{i(N+1)x}}{1 - e^{ix}}, \quad \text{for } x \neq 0. \tag{C.4}$$

Observe that the real part of the expression on the left–hand side of (C.4) is

$$\text{Re}\left[\sum_{n=0}^{N} e^{ix}\right] = 1 + \sum_{n=1}^{N} \cos(nx), \quad \text{for } x \in \mathbb{R}, \tag{C.5}$$

where we have used (C.3). We will compute the real part of

$$\frac{1-e^{i(N+1)x}}{1-e^{ix}} = \frac{(1-e^{i(N+1)x})(1-e^{-ix})}{(1-e^{ix})(1-e^{-ix})}, \quad \text{for } x \neq 0, \tag{C.6}$$

where

$$(1-e^{ix})(1-e^{-ix}) = 2 - e^{ix} - e^{-ix} = 2(1-\cos x), \tag{C.7}$$

be virtue of Euler's formula in (C.3).

Next, expand and simplify the numerator of the expression on the right–hand side of (C.6) to get

$$(1-\cos[(N+1)x] - i\sin[(N+1)x])(1-\cos x + i\sin x),$$

or

$$1-\cos x - \cos[(N+1)x](1-\cos x) + \sin[(N+1)x])\sin x$$
$$+\ i[(1-\cos[(N+1)x])\sin x - \sin[(N+1)x](1-\cos x)].$$

It then follows that the real part of the expression on the right–hand side of (C.4) is

$$\frac{1-\cos x - \cos[(N+1)x](1-\cos x) + \sin[(N+1)x])\sin x}{2(1-\cos x)},$$

or

$$\text{Re}\left[\frac{1-e^{i(N+1)x}}{1-e^{ix}}\right] = \frac{1}{2} - \frac{\cos[(N+1)x]}{2} + \frac{\sin[(N+1)x])\sin x}{2(1-\cos x)}, \tag{C.8}$$

for $x \neq 0$, where we have also used (C.7).

Next, use the trigonometric identities

$$1-\cos x = 2\sin^2\left(\frac{x}{2}\right)$$

and

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

to rewrite (C.8) as

$$\text{Re}\left[\frac{1-e^{i(N+1)x}}{1-e^{ix}}\right] = \frac{1}{2} - \frac{\cos[(N+1)x]}{2} + \frac{\sin[(N+1)x])\cos\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)},$$

which is turn can be rewritten as

$$\text{Re}\left[\frac{1-e^{i(N+1)x}}{1-e^{ix}}\right] = \frac{1}{2} + \frac{\sin[(N+1)x])\cos\left(\frac{x}{2}\right) - \cos[(N+1)x]\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)},$$

or

$$\mathrm{Re}\left[\frac{1-e^{i(N+1)x}}{1-e^{ix}}\right] = \frac{1}{2} + \frac{\sin\left[\left(N+\frac{1}{2}\right)x\right]}{2\sin\left(\frac{x}{2}\right)}, \tag{C.9}$$

where we have used the trigonometric identity

$$\sin(A-B) = \sin A\cos B - \cos A \sin B, \qquad \text{for } A, B \in \mathbb{R}.$$

In view of (C.4), (C.5) and (C.9), we can then write

$$1 + \sum_{n=1}^{N}\cos(nx) = \frac{1}{2} + \frac{\sin\left[\left(N+\frac{1}{2}\right)x\right]}{2\sin\left(\frac{x}{2}\right)}, \qquad \text{for } x \neq 0. \tag{C.10}$$

Finally, subtract $\frac{1}{2}$ from both sides of (C.10) to get

$$\frac{1}{2} + \sum_{n=1}^{N}\cos(nx) = \frac{\sin\left[\left(N+\frac{1}{2}\right)x\right]}{2\sin\left(\frac{x}{2}\right)}, \qquad \text{for } x \neq 0,$$

from which we derive (C.2), in view of (C.1).

Alternatively, use (C.1) to compute

$$\sin\left(\frac{\pi\theta}{2L}\right) D_{N}(\theta) = \frac{1}{2}\sin\left(\frac{\pi\theta}{2L}\right) + \sum_{n=1}^{N}\sin\left(\frac{\pi\theta}{2L}\right)\cos\left(\frac{n\pi\theta}{L}\right), \tag{C.11}$$

and use the trigonometric identity

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

to each of the terms in the sum in (C.11) to get

$$\sum_{n=1}^{N}\sin\left(\frac{\pi\theta}{2L}\right)\cos\left(\frac{n\pi\theta}{L}\right) = \sum_{n=1}^{N}\left[\frac{1}{2}\sin\left[\left(n+\frac{1}{2}\right)\frac{\pi\theta}{L}\right] + \frac{1}{2}\sin\left[\left(\frac{1}{2}-n\right)\frac{\pi\theta}{L}\right]\right],$$

or

$$\sum_{n=1}^{N}\sin\left(\frac{\pi\theta}{2L}\right)\cos\left(\frac{n\pi\theta}{L}\right) = \sum_{n=1}^{N}\left[\frac{1}{2}\sin\left[\left(n+\frac{1}{2}\right)\frac{\pi\theta}{L}\right] - \frac{1}{2}\sin\left[\left(n-\frac{1}{2}\right)\frac{\pi\theta}{L}\right]\right],$$

since sin is an odd function. Note that the sum above telescopes to

$$\sum_{n=1}^{N}\sin\left(\frac{\pi\theta}{2L}\right)\cos\left(\frac{n\pi\theta}{L}\right) = \frac{1}{2}\sin\left[\left(N+\frac{1}{2}\right)\frac{\pi\theta}{L}\right] - \frac{1}{2}\sin\left(\frac{\pi\theta}{2L}\right). \tag{C.12}$$

Combining (C.11) and (C.12) yields

$$\sin\left(\frac{\pi\theta}{2L}\right) D_{N}(\theta) = \frac{1}{2}\sin\left[\left(N+\frac{1}{2}\right)\frac{\pi\theta}{L}\right],$$

from which (C.2) follows.

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