Assignment #16

Due on Monday, April 20, 2020

Read Section 6.3 on the *Independent Random Variables* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 7.1 on *The Normal Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 5.6 on *The Normal Distributions* in DeGroot and Schervish.

Do the following problems

1. Suppose that $X \sim \text{normal}(\mu, \sigma^2)$ and define $Z = \dfrac{X - \mu}{\sigma}$.

   (a) Compute the mgf of Z.

   (b) Prove that $Z \sim \text{Normal}(0, 1)$ and give the pdf of Z.

2. (*The Chi–Square Distribution*) Let $X \sim \text{normal}(0, 1)$ and define $Y = X^2$.

   (a) Compute the pdf, f_Y, of Y.

   The distribution of Y is called the *Chi–Square distribution with one degree of freedom;* we write $Y \sim \chi^2(1)$.

   (b) Compute the mgf, ψ_Y, of Y by first computing $E(e^{tY}) = E(e^{tX^2})$, where $X \sim \text{Normal}(0, 1)$.

   (c) Use the mgf of Y to compute $E(Y)$ and $\text{Var}(Y)$ for $Y \sim \chi^2(1)$.

3. Let Y_1 and Y_2 denote two independent random variables such that $Y_1 \sim \chi^2(1)$ and $Y_2 \sim \chi^2(1)$. Define $W = Y_1 + Y_2$.

   (a) Use the mgf of the $\chi^2(1)$ distribution to compute the mgf of W. Give the distribution of W.

   (b) Let X and Y be independent $\text{Normal}(0, 1)$ random variables.

   Compute $\Pr(X^2 + Y^2 < 1)$.

4. Let $X_1, X_2, X_3, \ldots, X_n$ be independent identically distributed normal$(0,1)$ random variables. Define

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

   (a) Use moment generating functions to determine the distribution of $\overline{X}$.

   (b) Compute $E(\overline{X})$ and $\text{Var}(\overline{X})$.

5. Two instruments are used to measure the height, h, of a tower. The error made by the less accurate instrument is normally distributed with mean 0 and standard deviation $0.0056h$. The error made by the more accurate instrument is normally distributed with mean 0 and standard deviation $0.0044h$.

   Let X_1 denote the measurement made by the first instrument and X_2 the measurement made by the second instrument. Assume that X_1 and X_2 are independent random variables, and let $X = \dfrac{X_1 + X_2}{2}$, the average of the two instruments.

   (a) Determine the distribution of X.

   (b) Compute the probability that the average of the two measurements is within $0.005h$ of the height of the tower?