Solutions to Review Problems for Exam 2

1. A bowl contains 5 chips of the same size and shape. Two chips are red and the other three are blue. Draw three chips from the bowl at random, without replacement. Let X denote the number of blue chips in a drawing.

   (a) Give the pmf of X.

   Solution: Possible values of X are 1, 2 and 3.

   Compute, using equal likelihood assumption and the fact that the sampling is done without replacement,

$$\Pr(X = 1) = \frac{\binom{3}{1} \cdot \binom{2}{2}}{\binom{5}{3}} = \frac{3}{10}.$$

   Similarly

$$\Pr(X = 2) = \frac{\binom{3}{2} \cdot \binom{2}{1}}{\binom{5}{3}} = \frac{3}{5},$$

   and

$$\Pr(X = 3) = \frac{\binom{3}{3} \cdot \binom{2}{0}}{\binom{5}{3}} = \frac{1}{10}.$$

   We then have that the pmf of X is

$$p_X(k) = \begin{cases} \dfrac{3}{10}, & \text{if } k = 1; \\[2ex] \dfrac{3}{5}, & \text{if } k = 2; \\[2ex] \dfrac{1}{10}, & \text{if } k = 3; \\[2ex] 0, & \text{elsewhere.} \end{cases} \tag{1}$$

$\square$

(b) Compute $\Pr(X > 1)$.

***Solution**:* Use the definition of the pmf of X in (1) to get

$$\Pr(X > 1) = 1 - \Pr(X \leqslant 1) = 1 - p_{_X}(1) = \frac{7}{10},$$

or 70%. $\square$

(c) Compute $E(X)$.

***Solution**:* Using the definition of the pmf of X in (1), we compute

$$\begin{aligned} E(X) &= \sum_{k=1}^{3} k p_{_X}(k) \\ &= 1 \cdot \frac{3}{10} + 2 \cdot \frac{3}{5} + 3 \cdot \frac{1}{10} \\ &= \frac{18}{10}, \end{aligned}$$

or $E(X) = 1.8$. $\square$

2. Let X have pmf given by $p_{_X}(x) = \dfrac{1}{3}$ for $x = 1, 2, 3$ and $p(x) = 0$ elsewhere. Give the pmf of $Y = 2X + 1$.

***Solution**:* Note that the possible values for Y are 3, 5 and 7

Compute

$$\Pr(Y = 3) = \Pr(2X + 1 = 3) = \Pr(X = 1) = \frac{1}{3}.$$

Similarly, we get that

$$\Pr(Y = 5) = \Pr(X = 2) = \frac{1}{3},$$

and

$$\Pr(Y = 7) = \Pr(X = 3) = \frac{1}{3}.$$

Thus,

$$p_{_Y}(k) = \begin{cases} \dfrac{1}{3} & \text{for } k = 3, 5, 7; \\[2ex] 0 & \text{elsewhere.} \end{cases}$$

$\square$

3. A player simultaneously rolls a fair die and flips a fair coin. If the coin lands heads, she wins twice the value of the die roll (in dollars). If it lands tails, she wins half. Compute the expected earnings of the player.

   ***Solution**:* Let Y denote the outcome of the die toss. Then, the pmf of Y is

$$p_Y(k) = \begin{cases} \dfrac{1}{6}, & \text{if } k = 1, 2, 3, 4, 5, 6; \\ \\ 0, & \text{elsewhere.} \end{cases}$$

   Thus, the expected value of Y is

$$E(Y) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6},$$

   or

$$\begin{aligned} E(Y) &= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) \\ &= \frac{1}{6} \cdot \frac{6 \cdot 7}{2}; \end{aligned}$$

   so that, $E(Y) = \dfrac{7}{2}$, or $E(Y) = 3.5$.

   To model the flip of a fair coin, let $X \sim \text{Bernoulli}(p)$, where $p = \dfrac{1}{2}$; so that, the pmf of X is

$$p_X(k) = \begin{cases} \dfrac{1}{2}, & \text{if } k = 0, 1; \\ \\ 0, & \text{elsewhere.} \end{cases}$$

   The event $(X = 0)$ corresponds to the coin coming up tails; while the event $(X = 1)$ corresponds to the coin coming up heads.

   The expected value of X is

$$E(X) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

   We note that X and Y are independent random variables.

   Let Z denote the earnings of the player. Then,

$$Z = \begin{cases} 2Y, & \text{if coin flip yields heads;} \\ \dfrac{1}{2}Y, & \text{if coin flip yields tails.} \end{cases} \tag{2}$$

Letting

$$m(X) = \frac{1}{2} + \frac{3}{2}X,$$

we see that the earnings, X, given in (2) can be written as

$$Z = m(X)Y,$$

since $m(X) = \frac{1}{2}$ when $X = 0$ (tails) and $m(X) = 2$ when $X = 1$ (heads). Then,

$$Z = \left(\frac{1}{2} + \frac{3}{2}X\right)Y,$$

or

$$Z = \frac{1}{2}Y + \frac{3}{2}XY.$$

Thus, using the linearity property of expectation, the expected earnings are

$$E(Z) = \frac{1}{2}E(Y) + \frac{3}{2}E(XY)$$

Then, since X and Y are independent random variables,

$$E(Z) = \frac{1}{2}E(Y) + \frac{3}{2}E(X)E(Y). \tag{3}$$

See the solution of Problem 8 in this review sheet for a proof of the fact that $E(XY) = E(X)E(Y)$, if X and Y are independent and discrete.

We get from (3) that the expected earnings are

$$E(Z) = \frac{1}{2}\cdot\frac{7}{2} + \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{7}{2},$$

or

$$E(Z) = \frac{7}{4} + \frac{21}{8} = \frac{35}{8}.$$

Hence, the expected earnings are $E(Z) = 4.375$, or about \$4.38. □

4. A *mode* of a distribution of a random variable X is a value of x that maximizes the pdf or the pmf. If there is only one such value, it is called *the mode of the distribution.* Find the mode for each of the following distributions:

   (a) $p(x) = \left(\frac{1}{2}\right)^x$ for $x = 1, 2, 3, \ldots$, and $p(x) = 0$ elsewhere.

   ***Solution**:* Note that $p(x)$ is decreasing; so, $p(x)$ is maximized when $x = 1$. Thus, 1 is the mode of the distribution of X. □

(b) $f(x) = \begin{cases} 12x^2(1-x), & \text{if } 0 < x < 1; \\ \\ 0 & \text{elsewhere.} \end{cases}$

Solution: Maximize the function f over $[0,1]$.
Compute
$$f'(x) = 24x(1-x) - 12x^2 = 12x(2-3x),$$
so that f has a critical points at $x = 0$ and $x = \dfrac{2}{3}$.
Since $f(0) = f(1) = 0$ and $f(2/3) > 0$, it follows that f takes on its maximum value on $[0,1]$ at $x = \dfrac{2}{3}$. Thus, the mode of the distribution of X is $x = \dfrac{2}{3}$. □

5. Let X have pdf
$$f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

Compute the probability that X is at least 3/4, given that X is at least 1/2.
Solution: We are asked to compute
$$\Pr(X \geqslant 3/4 \mid X \geqslant 1/2) = \frac{\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)]}{\Pr(X \geqslant 1/2)}, \tag{4}$$
where
$$\begin{aligned} \Pr(X \geqslant 1/2) &= \int_{1/2}^{1} 2x\,dx \\ &= x^2\Big|_{1/2}^{1} \\ &= 1 - \frac{1}{4}, \end{aligned}$$
so that
$$\Pr(X \geqslant 1/2) = \frac{3}{4}; \tag{5}$$

and

$$\begin{aligned}
\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)] &= \Pr(X \geqslant 3/4) \\
&= \int_{3/4}^{1} 2x \; dx \\
&= x^2 \Big|_{3/4}^{1} \\
&= 1 - \frac{9}{16},
\end{aligned}$$

so that

$$\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)] = \frac{7}{16}. \tag{6}$$

Substituting (6) and (5) into (4) then yields

$$\Pr(X \geqslant 3/4 \mid X \geqslant 1/2) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{12}.$$

$\square$

6. Let X have pdf

$$f_X(x) = \begin{cases} x^2/9, & \text{if } 0 < x < 3; \\ 0, & \text{elsewhere.} \end{cases}$$

Find the pdf of $Y = X^3$.

***Solution**:* First, compute the cdf of Y,

$$F_Y(y) = \Pr(Y \leqslant y). \tag{7}$$

Observe that, since $Y = X^3$ and the possible values of X range from 0 to 3, the possible values of Y will range from 0 to 27. Thus, in the calculations that follow, we will assume that $0 < y < 27$.

From (7) we get that

$$\begin{aligned}
F_Y(y) &= \Pr(X^3 \leqslant y) \\
&= \Pr(X \leqslant y^{1/3}) \\
&= F_X(y^{1/3})
\end{aligned}$$

Thus, for $0 < y < 27$, we have that

$$f_Y(y) = f_X(y^{1/3}) \cdot \frac{1}{3} y^{-3/2}, \tag{8}$$

where we have applied the Chain Rule.

It follows from (8) and the definition of f_X that

$$f_Y(y) = \frac{1}{9}\left[y^{1/3}\right]^2 \cdot \frac{1}{3} y^{-3/2} = \frac{1}{27}, \qquad \text{for } 0 < y < 27. \tag{9}$$

Combining (9) and the definition of f_X we obtain the pdf for Y:

$$f_Y(y) = \begin{cases} \dfrac{1}{27}, & \text{for } 0 < y < 27; \\ & \\ 0 & \text{elsewhere;} \end{cases}$$

in other words, $Y \sim \text{Uniform}(0, 27)$. □

7. Divide a segment at random into two parts. Find the probability that the largest segment is at least three times the shorter.

   ***Solution**:* Assume the segment is the interval $(0, 1)$ and let $X \sim \text{Uniform}(0, 1)$. Then X models a random point in $(0, 1)$. We have two possibilities: Either $X \leqslant 1 - X$ or $X > 1 - X$; or, equivalently, $X \leqslant \frac{1}{2}$ or $X > \frac{1}{2}$.

   Define the events

$$E_1 = \left(X \leqslant \frac{1}{2}\right) \quad \text{and } E_2 = \left(X > \frac{1}{2}\right).$$

   Observe that $\Pr(E_1) = \frac{1}{2}$ and $\Pr(E_2) = \frac{1}{2}$.

   The probability that the largest segment is at least three times the shorter is given by

$$\Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2),$$

   by the Law of Total Probability, where

$$\Pr(1 - X > 3X \mid E_1) = \frac{\Pr[(X < 1/4) \cap E_1]}{\Pr(E_1)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Similarly,

$$\Pr(X > 3(1-X) \mid E_2) = \frac{\Pr[(X > 3/4) \cap E_1]}{\Pr(E_2)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Thus, the probability that the largest segment is at least three times the shorter is

$$\Pr(E_1)\Pr(1-X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1-X) \mid E_2) = \frac{1}{2}.$$

□

8. Assume that X and Y are independent, discrete random variables.

   Show that $E(XY) = E(X)E(Y)$.

   ***Solution**:* Assume that

$$\Pr(X = x, Y = y) = \Pr(X = x) \cdot \Pr(Y = y), \quad \text{for all values of } x \text{ and } y, \tag{10}$$

and compute

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy\Pr(X = x, Y = y) \\ &= \sum_x \sum_y xy\Pr(X = x) \cdot \Pr(Y = y), \end{aligned}$$

where we have used (10). We then have that

$$\begin{aligned} E(XY) &= \sum_x \sum_y xp_{_X}(x)yp_{_Y}(y) \\ &= \sum_x xp_{_X}(x) \sum_y yp_{_Y}(y), \end{aligned}$$

where we have used the distributive property. Then, using the definition of $E(Y)$,

$$\begin{aligned} E(XY) &= \sum_x xp_{_X}(x)E(Y) \\ &= \left(\sum_x xp_{_X}(x)\right) E(Y), \end{aligned}$$

where we have used the distributive property again. Consequently, using the definition of $E(X)$,

$$E(XY) = E(X)E(Y),$$

which was to be shown. $\square$

9. Assume that $X \sim \text{Uniform}(0,1)$ and define $Y = -\ln X$.

   (a) Compute the cdf of Y.

   Solution: Assume that $X \sim \text{Uniform}(0,1)$; so that, the pdf of X is

$$f_X(x) = \begin{cases} 1, & \text{if } 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Let $Y = -\ln X$; so that, the possible values of Y range from 0 to $+\infty$. Thus, for $y > 0$,

$$\begin{aligned} F_Y(y) &= \Pr(Y \leqslant y) \\ &= \Pr(-\ln(X) \leqslant y) \\ &= \Pr(\ln(X) \geqslant -y); \end{aligned}$$

so that,

$$F_Y(y) = \Pr(X \geqslant e^{-y}), \qquad \text{for } y > 0.$$

Thus, using the fact that X is a continuous random variable,

$$F_Y(y) = \Pr(X > e^{-y}), \qquad \text{for } y > 0,$$

which can be rewritten as

$$F_Y(y) = 1 - \Pr(X \leqslant e^{-y}), \qquad \text{for } y > 0,$$

from which we get that

$$F_Y(y) = 1 - F_X(e^{-y}), \qquad \text{for } y > 0. \tag{12}$$

Since Y has no negative possible values, we obtain from (12) that

$$F_Y(y) = \begin{cases} 1 - F_X(e^{-y}), & \text{for } y > 0; \\ 0, & \text{for } y \leqslant 0. \end{cases} \tag{13}$$

Now, the cdf of X can be computed from (11) to be

$$F_X(x) = \begin{cases} 0, & \text{if } x \leqslant 0; \\ x, & \text{if } 0 < x < 1; \\ 1, & \text{if } x \geqslant 1. \end{cases} \tag{14}$$

We therefore obtain from (13) and (14) that

$$F_Y(y) = \begin{cases} 1 - e^{-y}, & \text{for } y > 0; \\ 0, & \text{for } y \leqslant 0, \end{cases} \tag{15}$$

since $0 < e^{-y} < 1$ for $y > 0$.

□

(b) Compute the pdf of Y

Solution: Differentiating the expression for F_Y in (15) for $y \neq 0$, and setting $f_Y(0) = 0$, we obtain

$$f_Y(y) = \begin{cases} e^{-y}, & \text{for } y > 0; \\ 0, & \text{for } y \leqslant 0. \end{cases} \tag{16}$$

Observe that (16) implies that $Y \sim \text{Exponential}(1)$. □

(c) Compute $\Pr(Y > 1)$.

Solution: Compute

$$\Pr(Y > 1) = 1 - \Pr(Y \leqslant 1) = 1 - F_Y(1);$$

so that, according to (15), $\Pr(Y > 1) = 1 - (1 - e^{-1}) = e^{-1}$. □

(d) Compute $E(Y)$ and $\text{Var}(Y)$.

Solution: Use the pdf of Y in (16) to compute

$$\begin{aligned} E(Y) &= \int_0^\infty ye^{-y}\, dy \\ &= \lim_{b\to\infty} \int_0^b ye^{-y}\, dy \\ &= \lim_{b\to\infty} \left[-ye^{-y} - e^{-y}\right]_0^b \\ &= \lim_{b\to\infty} \left[1 - be^{-b} - e^{-b}\right] \\ &= 1, \end{aligned}$$

where we have used integration by parts and L'Hospital's rule.

Similarly, to compute the second moment of Y,

$$\begin{aligned} E(Y^2) &= \int_0^\infty y^2 e^{-y}\, dy \\ &= \lim_{b\to\infty} \int_0^b y^2 e^{-y}\, dy \\ &= \lim_{b\to\infty} \left[-y^2e^{-y} - 2ye^{-y} - 2e^{-y}\right]_0^b \\ &= \lim_{b\to\infty} \left[2 - b^2e^{-b} - 2be^{-b} - 2e^{-b}\right] \\ &= 2, \end{aligned}$$

where we have integrated by parts twice and used L'Hospital's rule twice. We have therefore shown that $E(Y) = 1$ and $E(Y^2) = 2$. Consequently, the variance of Y is

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = 2 - (1)^2 = 1.$$

$\square$

10. A box contains a certain number of balls of various colors. Assume that 10% of the balls are red. If 20 balls are selected from the box at random, with replacement, what is the probability that more than 3 red balls will be obtained in the sample?

***Solution**:* Let X denote the number of red balls in the sample of 20. Then, X has a binomial distribution with parameters $n = 20$ and $p = 0.10$. Thus, the pmf of X is

$$p_X(k) = \begin{cases} \displaystyle\binom{20}{k}(0.1)^k(0.9)^{20-k}, & \text{if } k = 0, 1, 2, \ldots, 20; \\ \\ 0, & \text{otherwise.} \end{cases}$$

Compute

$$\begin{aligned} \Pr(X > 3) &= 1 - \Pr(X \leqslant 3) \\ &= 1 - p_X(0) - p_X(1) - p_X(2) - p_X(3) \\ &\approx 0.133. \end{aligned}$$

or about 13.3%. □

11. Let X denote a continuous random variable with pdf

$$f_X(x) = \begin{cases} \dfrac{x}{8}, & \text{if } 0 < x < 4; \\ \\ 0, & \text{otherwise.} \end{cases}$$

Define Y to be the integer that is closest to X.

(a) Explain why Y is a discrete random variable and give possible values for Y.

***Solution**:* Possible values for Y are 0, 1, 2, 3 and 4. Hence, Y is discrete. □

(b) Compute the pmf of Y.

***Solution**:* Compute

$$\begin{aligned} \Pr(Y = 0) &= \Pr(0 \leqslant X < 0.5) \\ &= \int_0^{0.5} f_X(x)\, dx \\ &= \left[\frac{x^2}{16}\right]_0^{0.5} \\ &= \frac{1}{64}; \end{aligned}$$

$$\begin{aligned}
\Pr(Y=1) &= \Pr(0.5 \leqslant X < 1.5) \\
&= \int_{0.5}^{1.5} f_X(x)\, dx \\
&= \left[\frac{x^2}{16}\right]_{0.5}^{1.5} \\
&= \frac{1}{8}; \\
\Pr(Y=2) &= \Pr(1.5 \leqslant X < 2.5) \\
&= \int_{1.5}^{2.5} f_X(x)\, dx \\
&= \left[\frac{x^2}{16}\right]_{1.5}^{2.5} \\
&= \frac{1}{4}; \\
\Pr(Y=3) &= \Pr(2.5 \leqslant X < 3.5) \\
&= \int_{2.5}^{3.5} f_X(x)\, dx \\
&= \left[\frac{x^2}{16}\right]_{2.5}^{3.5} \\
&= \frac{3}{8};
\end{aligned}$$

and

$$
\begin{aligned}
\Pr(Y = 4) &= \Pr(3.5 \leqslant X < 4) \\
&= \int_{3.5}^{4} f_X(x)\, dx \\
&= \left[\frac{x^2}{16}\right]_{3.5}^{4} \\
&= \frac{15}{64}.
\end{aligned}
$$

We therefore have that the pmf of Y is

$$
p_Y(k) = \begin{cases} 1/64, & \text{if } k = 0; \\ 1/8, & \text{if } k = 1; \\ 1/4, & \text{if } k = 2; \\ 3/8, & \text{if } k = 3; \\ 15/64, & \text{if } k = 4; \\ 0, & \text{elsewhere.} \end{cases} \tag{17}
$$

□

(c) Compute $E(Y)$ and $\text{Var}(Y)$.

Solution: Using the pmf in (17) we compute

$$
\begin{aligned}
E(Y) &= \sum_{k=0}^{4} k p_Y(k) \\
&= \sum_{k=1}^{4} k p_Y(k),
\end{aligned}
$$

or

$$
E(Y) = 1 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{3}{8} + 4 \cdot \frac{15}{64},
$$

or

$$
E(Y) = \frac{43}{16}. \tag{18}
$$

To compute the variance of Y, we first compute the second moment

$$\begin{aligned} E(Y^2) &= \sum_{k=0}^{4} k^2 p_{_Y}(k) \\ &= \sum_{k=1}^{4} k^2 p_{_Y}(k), \end{aligned}$$

or

$$\begin{aligned} E(Y^2) &= 1^2 \cdot \frac{1}{8} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{3}{8} + 4^2 \cdot \frac{15}{64} \\ &= \frac{1}{8} + 1 + \frac{27}{8} + \frac{15}{4} \\ &= 1 + \frac{28}{8} + \frac{15}{4}; \end{aligned}$$

so that,

$$E(Y^2) = \frac{33}{4}. \tag{19}$$

The variance of Y is given by

$$\mathrm{Var}(Y^2) = E(Y^2) - [E(Y)]^2;$$

so that, using the results in (18) and (19),

$$\mathrm{Var}(Y^2) = \frac{263}{256}.$$

□

12. Assume that X has a uniform distribution on the subset of the integers given by

$$\{1, 2, 3, \ldots, 47\}$$

(a) Compute the probability of the event that X is even.

***Solution**:* The pmf of X is

$$p_{_X}(k) = \begin{cases} \dfrac{1}{47}, & \text{if } k = 1, 2, 3, \ldots, 47; \\ \\ 0, & \text{otherwise.} \end{cases}$$

The probability of the event (X is even) is

$$\Pr(X \text{ is even}) = \frac{23}{47},$$

since there are 23 even numbers between 1 and 47. $\square$

(b) Compute the expected value of X.

Solution: Compute

$$\begin{aligned} E(X) &= \sum_{k=1}^{47} k p_{_X}(k) \\ &= \frac{1}{47}\sum_{k=1}^{47} k \\ &= \frac{1}{47}\cdot\frac{47\cdot 48}{2} \\ &= 24. \end{aligned}$$

$\square$