Review Problems for Exam 3

1. Let X have mgf given by $\psi_X(t) = \dfrac{1}{3}e^t + \dfrac{2}{3}e^{2t}$, for $t \in \mathbb{R}$.

   (a) Give the distribution of X.

   (b) Compute the expected value and variance of X.

2. Let X have mgf given by $\psi_X(t) = \begin{cases} \dfrac{e^t - e^{-t}}{2t}, & \text{if } t \neq 0; \\ 1, & \text{if } t = 0. \end{cases}$

   (a) Give the distribution of X.

   (b) Compute the expected value and variance of X.

3. A random point (X, Y) is distributed uniformly on the square with vertices $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

   (a) Give the joint pdf for X and Y.

   (b) Compute the following probabilities:

      (i) $\Pr(X^2 + Y^2 < 1)$,

      (ii) $\Pr(2X - Y > 0)$,

      (iii) $\Pr(|X + Y| < 2)$.

4. A random vector (X, Y) has the joint distribution shown in the table below

$X\backslash Y$	2	3	4
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0

   (a) Show that X and Y are not independent.

   (b) Give a probability table for random variables U and V that have the same marginal distributions as X and Y, respectively, but are independent.

5. An experiment consists of independent tosses of a fair coin. Let X denote the number of trials needed to obtain the first head, and let Y be the number of trials needed to get two heads in repeated tosses. Are X and Y independent random variables? Explain the reasoning leading to your answer.

6. Let $g(t)$ denote a non–negative, integrable function of a single variable with the property that
$$\int_0^\infty g(t) \ \mathrm{d}t = 1.$$
Define
$$f(x,y) = \begin{cases} \dfrac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}}, & \text{for } 0 < x < \infty,\ 0 < y < \infty, \\ \\ 0, & \text{otherwise.} \end{cases}$$
Show that $f(x,y)$ is a joint pdf for two random variables X and Y.

7. Suppose that two persons make an appointment to meet between 5 PM and 6 PM at a certain location and they agree that neither person will wait more than 10 minutes for each person. If they arrive independently at random times between 5 PM and 6 PM, what is the probability that they will meet?

8. Assume that the number of calls coming per minute into a hotel's reservation center follows a Poisson distribution with mean 3.
   (a) Find the probability that no calls come in a given 1 minute period.
   (b) Assume that the number of calls arriving in two different minutes are independent. Find the probability that at least two calls will arrive in a given two minute period.

9. Let $Y \sim \text{Binomial}(100, 1/2)$. Use the Central Limit Theorem to estimate the value of $\Pr(Y = 50)$.

   Suggestion: Observe that $\Pr(Y = 50) = \Pr(49.5 < Y \leq 50.5)$, since Y is discrete.

10. Roll a balanced die 36 times. Let Y denote the sum of the outcomes in each of the 36 rolls. Estimate the probability that $108 \leq Y \leq 144$.

    Suggestion: Since the event of interest is $(Y \in \{108, 109, \ldots, 144\})$, rewrite $\Pr(108 \leq Y \leq 144)$ as $\Pr(107.5 < Y \leqslant 144.5)$.

11. The standard voltage in residences in the United States of America is 120 volts. Assume that this voltage can be modeled by a random variable with mean 120 and variance 25. Suppose that some sensitive electrical appliances can be damaged if the voltage is not between 110 and 130. Use Chebyshev's inequality to find an upper bound for the probability that damage will occur to a sensitive electrical appliance.

12. Many random number generators, like the `RAND()` function in MS Excel, are pseudo–random number generators. These are algorithms that provide a (real) random number in the interval $(0, 1)$. Many of these pseudo–random numbers can be modeled by a uniform$(0, 1)$ random variable.

    Suppose a pseudo–random number generator is used to generate 400 random numbers from the interval $[0, 1]$.

    (a) Use Chebyshevs inequality to find a lower bound for the probability that the sum of the numbers lies between 190 and 210.

    (b) Use the central limit theorem to estimate the probability that the sum of the numbers lies between 190 and 210.