Upper and lower bounds on speed function of an excited random walk

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August 2, 2016
Simple Random Walk
Simple Random Walk

1 - \( p \)  \quad p

-2 -1 0 1 2
Simple Random Walk

\[ 1 - p \quad p \]

1 - \( p \) and \( p \) represent the probabilities of moving left and right, respectively, in the random walk.
Simple Random Walk

- Markovian

\[
\lim_{n \to \infty} X_n = \frac{1}{n} \sum_{i=1}^{\infty} \xi_i = E_0 [X_1] = 2p - 1 \text{ by SLLN}
\]
Simple Random Walk

- Markovian
- Recurrence ($p = \frac{1}{2}$): hits every state infinitely often
- Transience ($p \neq \frac{1}{2}$): hits any given state finitely often
Simple Random Walk

- Markovian
- Recurrence \((p = \frac{1}{2})\): hits every state infinitely often
- Transience \((p \neq \frac{1}{2})\): hits any given state finitely often
- Speed

\[
\lim_{n \to \infty} \frac{X_n}{n} = \frac{1}{n} \sum_{i=1}^{\infty} \xi_i = \mathbb{E}_0[X_1] = 2p - 1 \text{ by SLLN}
\]
Excited Random Walks

Questions
1. Transient?
2. Recurrent?
3. Speed?
Excited Random Walks

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Questions

1. Transcient?
2. Recurrent?
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Prior Results That Structure Our Bounds

Drift defined as $\delta = M(2p - 1)$
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**Theorem (Zerner ’05)**

$transcience \iff \delta > 1$
Prior Results That Structure Our Bounds

Drift defined as $\delta = M(2p - 1)$

**Theorem (Zerner ‘05)**

- \textit{transcience} $\iff \delta > 1$

**Theorem (Basdevant and Singh ‘08)**

- \textit{positive speed} $v \iff \delta > 2$
Simulation of the Speed of an ERW with 3 Equal Cookies
Branching Process- This is what we study!

Random Walk Path

Backward Branching Process

bijection
Branching Process- This is what we study!

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Random Walk Path

Backward Branching Process

bijection
Alternate form of Speed

\[ V = \lim_{n \to \infty} \frac{X_n}{n} \]
Alternate form of Speed

- \( V = \lim_{n \to \infty} \frac{X_n}{n} \)
- \( \frac{1}{V} = \lim_{n \to \infty} \frac{T_n}{n} \)
- \( T_n = \inf\{k \geq 0 : X_k = n\} \)
Hitting Time Fun!

\[ T_n = n + 2(\text{number of total offspring of tree}) \]
Hitting Time Fun!

- \( T_n = n + 2 \) (number of total offspring of tree)
- \( T_n = n + 2 \sum_i Z_i \)
Hitting Time Fun!

- $T_n = n + 2(\text{number of total offspring of tree})$
- $T_n = n + 2 \sum_i Z_i$
- Now we take the limit!

$$\lim_{n \to \infty} \frac{T_n}{n} = \lim_{n \to \infty} 1 + 2 \frac{1}{n} \sum_i Z_i$$
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} Z_i = \mathbb{E}_{\pi}[Z_0] = \sum_{k \geq 0} k \pi(k) \]

Where

\[ \pi(k) = \lim_{n \to \infty} P(Z_n = k) \]

\[ = \lim_{n \to \infty} \frac{1}{n} \sum_{i \geq 0} 1_{Z_i = k} \]
Looks tractable!

Theorem (Basdevant and Singh ‘08)

\[ \nu = \frac{1}{1 + 2E_\pi[Z_0]} \]
Truncation of $Z_n$

Denoted $Z_n^{(L)}$

\[ P(Z_{n+1}^{(L)} = j | Z_n^{(L)} = i) = \begin{cases} 
  P(Z_{n+1} = j | Z_n = i) & \text{if } j < L \\
  P(Z_{n+1} \geq L | Z_n = i) & \text{if } j = L 
\end{cases} \]
Transition Matrix

- $i, j \leq 2$

\[
P_{i,j} = \begin{pmatrix}
0 & 1 & 2 \\
0 & pq & pq^2 \\
\frac{1}{2} & \frac{1}{2} pq & \frac{3}{4} (pq^2 + p^2q) \\
\vdots & \vdots & \vdots \\
\end{pmatrix}
\]

- $i > 2$ and/or $j > 2$ and $j < L$

\[
P_{i,j} = \frac{1}{2i+j-2} \left[ \binom{i+j-3}{i-3} p^3 + \binom{i+j-3}{j-3} q^3 + 3 \binom{i+j-3}{i-2} p^2 q + 3 \binom{i+j-3}{j-2} pq^2 \right]
\]
Transition Matrix for Truncation

- \( j = L \) (the last column)

\[
P_{i,j} = 1 - \sum_{j=0}^{L-1} p(i,j)
\]
Upper Bound Using Truncation

**Theorem**

\[ V_M^{(L)} \xrightarrow{L \to \infty} V_{M, \bar{p}} \]
$\mathbb{E}_k[Z_1]$ for $M = 3$ Cookies of Equal Strength

$\mathbb{E}_k[Z_1] = \sum_{n=0}^{\infty} n \cdot P(Z_1 = n | Z_0 = k) = \sum_{n=0}^{\infty} n \cdot P_{k,n}$
$E_k[Z_1]$ for $M = 3$ Cookies of Equal Strength

- $E_k[Z_1] = \sum_{n=0}^{\infty} n \cdot P(Z_1 = n | Z_0 = k) = \sum_{n=0}^{\infty} n \cdot P_{k,n}$

$E_0[Z_1] = 0(p) + 1p(1 - p) + 2p(1 - p)^2 + (1 - p)^3 \sum_{k=3}^{\infty} \frac{k}{2^{k-2}}$

$= 4 - 9p + 7p^2 - 2p^3$
$E_k[Z_1]$ for $M = 3$ Cookies of Equal Strength

$E_k[Z_1] = \sum_{n=0}^{\infty} n \cdot P(Z_1 = n|Z_0 = k) = \sum_{n=0}^{\infty} n \cdot P_{k,n}$

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$= 4 - 9p + 7p^2 - 2p^3$

$E_1[Z_1] = 5 - 6p - p^2 + 2p^3$

$E_k[Z_1] = k + 4 - 6p \quad \forall k \geq M - 1$
Probability Generating Functions

- $G(s) := \sum_{k=0}^{\infty} \pi(k)s^k = \mathbb{E}_\pi[s^{Z_0}]$
- $G'(1) = \mathbb{E}_\pi[Z_0]$
Recursive Formula (Basdevant & Singh)

\[
1 - G \left( \frac{1}{2 - s} \right) = a(s)(1 - G(s)) + b(s)
\]

- \( a(s) = \frac{1}{(2-s)^{M-1}E_{M-1}[s^{Z_1}]} \)
- \( b(s) = 1 - \frac{1}{(2-s)^{M-1}E_{M-1}[s^{Z_1}]} + \sum_{k=0}^{M-2} \pi(k) \left( \frac{E_k[s^{Z_1}]}{(2-s)^{M-1}E_{M-1}[s^{Z_1}]} - \frac{1}{(2-s)^k} \right) \)
Recursive Formula (Basdevant & Singh)

\[ 1 - G \left( \frac{1}{2 - s} \right) = a(s)(1 - G(s)) + b(s) \]

- \( a(s) = \frac{1}{(2-s)^{M-1}E_{M-1}[s^2]} \)
- \( b(s) = 1 - \frac{1}{(2-s)^{M-1}E_{M-1}[s^2]} + \sum_{k=0}^{M-2} \pi(k) \left( \frac{E_k[s^2]}{(2-s)^{M-1}E_{M-1}[s^2]} - \frac{1}{(2-s)^k} \right) \)
- \( E_\pi[Z_0] = \frac{b''(1)}{2(\delta-2)} \)
**b”’(1) for \( M = 3 \) Cookies**

\[
b''(1) = -6 + \mathbb{E}_2[Z_1] + \mathbb{E}_2[Z_1^2]
\]

\[
+ \pi(0) \left\{ -2 \mathbb{E}_2[Z_1](2 + \mathbb{E}_0[Z_1]) + 6 - 2\mathbb{E}_0[Z_1] - \mathbb{E}_0[Z_1^2] + 2\mathbb{E}_0[Z_1]^2 \right\}
\]

\[
+ \pi(1) \left\{ -2 - 2\mathbb{E}_2[Z_1](2 + \mathbb{E}_1[Z_1]) + 6 - 2\mathbb{E}_1[Z_1] - \mathbb{E}_1[Z_1^2] + 2\mathbb{E}_1[Z_1]^2 \right\}
\]

\[
= -2(2\rho - 1)(12\rho - 9) - 2(2\rho - 1)(6\rho^3 - 19\rho^2 + 10\rho)\pi(0) - 2(2\rho - 1)(-6\rho^3 + 4\rho^2)\pi(1)
\]
\( b''(1) \) for \( M = 3 \) Cookies

\[
b''(1) = -6 + E_2[Z_1] + E_2[Z_1^2] \\
\quad + \pi(0) \left\{ -2E_2[Z_1](2 + E_0[Z_1]) + 6 - 2E_0[Z_1] - E_0[Z_1^2] + 2E_0[Z_1]^2 \right\} \\
\quad + \pi(1) \left\{ -2 - 2E_2[Z_1](2 + E_1[Z_1]) + 6 - 2E_1[Z_1] - E_1[Z_1^2] + 2E_1[Z_1]^2 \right\} \\
= -2(2\rho - 1)(12\rho - 9) - 2(2\rho - 1)(6\rho^3 - 19\rho^2 + 10\rho)\pi(0) - 2(2\rho - 1)(-6\rho^3 + 4\rho^2)\pi(1)
\]

\[
V = \frac{6\rho - 5}{6\rho - 5 - 2(2\rho - 1)(12\rho - 9) - 2(2\rho - 1)(6\rho^3 - 19\rho^2 + 10\rho)\pi(0) - 2(2\rho - 1)(-6\rho^3 + 4\rho^2)\pi(1)}
\]
Upper and Lower Bounds

\[ a \pi(0) + b \pi(1) = c \]

\[ \pi(0) = \frac{p(1,0)}{1-p(0,0)} \pi(1) \]

\[ \pi(1) = \frac{p(0,1)}{1-p(1,1)} \pi(0) \]
Upper and Lower Bounds

Speed Bounds for an ERW with 3 Equal Cookies

![Graph showing speed bounds for an ERW with 3 equal cookies.](image)
Future Research

- Find an explicit equation for the speed
- Differentiability of the speed function
Thank you for listening!

We would like to thank Professor Jonathon Peterson, Professor Edray Goins, Dr. Sung Won Ahn, The NSF, Purdue Math Department, and our loving families.