Abstract

A Belyi map \( : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \) is a rational function with at most three critical values, which we may assume these values to be \(0, 1, \infty\). A Dessin d’Enfant is a planar bipartite graph obtained by considering the preimage of a path between two of these critical values, usually taken to be the line segment from 0 to 1. Such graphs can be drawn on the sphere by composing with algebraic projection: \( : \mathbb{P}^1(\mathbb{C}) \to \mathbb{C} \). Replacing \( \) with an elliptic curve \( E \), there is a natural definition of a Belyi map \( : \mathbb{P}^1(\mathbb{C}) \to \mathbb{C} \). Since \( \mathbb{C} \) is homeomorphic to the sphere, we call \( E \) a toroidal Belyi pair. The corresponding Dessin d’Enfant can be drawn on the torus by composing with an elliptic logarithm: \( \log \). In this project, we are interested in the group \( \) for \( a \in \mathbb{P}^1(\mathbb{C}) \). Let \( \) be a toroidal Belyi pair. We lay out a quick algorithm to compute these groups by solving a system of ordinary differential equations and present visualizations of their group actions on the sphere.

Background

Let \( \) be a compact, connected Riemann surface. There are two examples of interest:

The Sphere: the projective line \( \mathbb{P}^1(\mathbb{C}) \) may be embedded into the projective plane using the map \( \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \) which sends \( \{x : x \to \{x : 0 \to 0\} \) so that this curve corresponds to the zeros of the polynomial \( f(x, y) = y \). The set of complex points, namely \( \mathbb{P}^1(\mathbb{C}) \) is a sphere.

Elliptic Curves: an elliptic curve \( E \) is a nonsingular projective variety corresponding to the zeros of the form \( f(x, y) = (x + ay + b)(x - ay + c) = 0 \). Examples of elliptic curves

\[
\begin{align*}
&y^2 = x^3 + 1, \\
&y^2 = x^3 - 1, \\
&y^2 = x^3 + x + 1, \\
&y^2 = x^3 - x + 1.
\end{align*}
\]

The surface defined by an Elliptic curve over the complex numbers is equivalent to a torus.

Belyi Map: a Belyi Map is a rational function \( \) with at most 3 critical values, which we may assume to be \(0, 1, \infty\). Since \( \) may be viewed as the set of zeros of a single polynomial \( f(x, y) \), we can write \( f(x, y) = (x, y)/(x, y) \) as the ratio of two polynomials \( f(x, y) \) and \( g(x, y) \).

Some examples include:

\[
\begin{align*}
&f(x, y) = y - 1 & \text{for} & E : y^2 = x^3 + 1, \\
&f(x, y) = y - 1 & \text{for} & E : y^2 = x^3 + 15xy + 28y = x^3, \\
&f(x, y) = y - 1 & \text{for} & E : y^2 = x^3 + 5x + 10.
\end{align*}
\]

Monodromy Groups

For \( \) in \( \) different from 0, 1, and \( \infty \). For each \( ), \) in the collection of affine points

\[
\begin{align*}
\beta'(y) & = 0 & f(x, y) & = 0 & \Rightarrow & (P_1, P_2, ..., P_N) \in \mathbb{Z}^3(\mathbb{C}) \times \mathbb{Z}^3(\mathbb{C}) \\
\beta'(y) & = 0 & f(x, y) & = 0 & \Rightarrow & (P_1, P_2, ..., P_N) \in \mathbb{Z}^3(\mathbb{C}) \times \mathbb{Z}^3(\mathbb{C}) \\
\end{align*}
\]

there exist unique paths \( \gamma_{\beta'}(y) = 0 \) satisfying

\[
\begin{align*}
\gamma_{\beta'}(y) & = 0 & f(x, y) & = 0 & \Rightarrow & (P_1, P_2, ..., P_N) \in \mathbb{Z}^3(\mathbb{C}) \times \mathbb{Z}^3(\mathbb{C}) \\
\gamma_{\beta'}(y) & = 0 & f(x, y) & = 0 & \Rightarrow & (P_1, P_2, ..., P_N) \in \mathbb{Z}^3(\mathbb{C}) \times \mathbb{Z}^3(\mathbb{C}) \\
\end{align*}
\]

Examples on the Sphere

Say that \( X = \mathbb{P}^1(\mathbb{C}) \) is a Belyi map of degree \( N \). The monodromy group has the generators

\[
\begin{align*}
&\sigma_0 = (1 \ 2 \ \cdots \ N), \\
&\sigma_1 = (1), \\
&\sigma_2 = (N \ 2 \ \cdots \ 1), \\
&\end{align*}
\]

Hence the monodromy group is \( \).

The rational function \( \) is a Belyi map of degree \( N = 5 \). According to our Mathematica code, the monodromy group has the generators

\[
\begin{align*}
&\sigma_0 = (1 \ 2 \ 3 \ 4 \ 5), \\
&\sigma_1 = (1), \\
&\sigma_2 = (1 \ 2 \ 3 \ 4 \ 5), \\
&\end{align*}
\]

Hence the monodromy group is \( \).

The rational function \( \) is a Belyi map of degree \( N = 7 \). According to our Mathematica code, the monodromy group has the generators

\[
\begin{align*}
&\sigma_0 = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
&\sigma_1 = (1), \\
&\sigma_2 = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7), \\
&\end{align*}
\]

Hence the monodromy group is \( \).

Algorithms

The path \( \) \( [0, 1] \to X \) must also satisfy the system of ordinary differential equations

\[
\begin{align*}
\frac{d\gamma_{\beta'}(y)}{dy} & = 2xy - y, \\
\frac{d\lambda_{\beta'}(y)}{dy} & = 0, \\
\frac{d\phi_{\beta'}(y)}{dy} & = 0.
\end{align*}
\]

Algorithm

After solving these equations to suitable numerical precision, we choose \( \sigma_0, \sigma_1, \sigma_2 \in S_N \) as the permutations such that

\[
\begin{align*}
&\sigma_0(i) = i, \\
&\sigma_1(i) = i, \\
&\sigma_2(i) = i, \\
&\sigma_2(i) = i - 1.
\end{align*}
\]

There is preliminary software which partially does this in Mathematica, see figure below for a screenshot.

Future Work

We have preliminary code in Mathematica, which we plan to port to Sage. While Mathematica solves these systems of differential equations very quickly, it cannot determine the structure of groups very well. On the other hand, Sage can determine the structure of groups, but cannot solve systems of differential equations when complex numbers are involved.

Acknowledgements

• Dr. Edray Herber Goins
• Ahluiskab Parab
• Dr. Gregory Bumby / Department of Mathematics
• College of Science
• National Science Foundation (DMS-1565934)

References


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http://www.math.purdue.edu/~egoins/notes/dessin_explorer.mov