Abstract

A Bely˘ı map \( \beta : P^1(\mathbb{C}) \to P^1(\mathbb{C}) \) is a rational function with at most three critical values; we may assume these values are \((0, 1, \infty)\). Replacing \(P^1\) with an elliptic curve \(E\) (this is a smooth curve given by an equation \(y^2 = x^3 + ax + b\)), there is a similar definition of a Bely˘ı map \( \beta : E \to P^1(\mathbb{C}) \). Since \( E \cong \mathbb{C}/\Gamma \) is a torus, we call \((E, \beta)\) a Bely˘ı map.

There are many examples of Bely˘ı maps \( \beta : E \to P^1(\mathbb{C}) \) associated to elliptic curves; several can be found online at LMFDB. Given such a Bely˘ı map \((E, \beta)\), an inverse image \( \Gamma = \beta^{-1}(\{0, 1, \infty\}) \) is a set of \( X \) elements which contains the critical points of the Bely˘ı map. In this project, we investigate when \( \Gamma \) is contained in \( E \).

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Examples of Toroidal Bely˘ı Maps

Example #1: 4T1-4_2.2-a

Consider the Toroidal Bely˘ı pair \((E, \beta)\) in terms of \( E \) and \( \beta(x, y) = y^3 - x^2 - x \). The quasi-critical points are torsion:

- Point: \((0, 0)\) (2, 0) (0, 1) (1, 0) O

These points do not form a group.

Example #2: 4T5-4_3.1-a

Consider the Toroidal Bely˘ı pair \((E, \beta)\) in terms of \( E \) and \( \beta(x, y) = (y^2 + 3x^2 + 2y + 2x + 1)\). The quasi-critical points are torsion:

- Point: \((0, 0)\) (2, 1) (0, 1) (1, 0) O

These points do not form a group.

Example #3: 5T6-5_4.1.4.1-a

Consider the Toroidal Bely˘ı pair \((E, \beta)\) in terms of \( E \) and \( \beta(x, y) = (y^2 + x^2 + 4x + 4)\). The quasi-critical points are not torsion:

- Point: \((-1, 0)\) (1, 0) (1, 1) O

These points do not form a group.

Motivating Questions

- Given the following: \((E, \beta)\), a Toroidal Bely˘ı pair.
- \( \Gamma = \beta^{-1}(\{0, 1, \infty\}) \) as the set of quasi-critical points.

We ask the question:

- When does \( \Gamma \) form a subgroup of \( E \)?

The elements of \( \Gamma \) must be points with finite order whenever \( \Gamma \) is a group. When \( \Gamma \) is in \( \Gamma \), torsion elements in \( E \), regardless of \( \Gamma \) being a group?

Theorem (PRIME 2021)

Given the following:

- \((E, \beta)\), a Toroidal Bely˘ı pair, and \( \Gamma = \beta^{-1}(\{0, 1, \infty\}) \) as the set of quasi-critical points.
- \( \beta = \phi \circ \psi \) where \( \phi : E \to \mathbb{C} \) is any non-constant isogeny, and \( \Gamma \) is \( \beta^{-1}(\{0, 1, \infty\}) \).

We have the main result:

- \((E, \beta)\), a Toroidal Bely˘ı pair.
- \( \Gamma \) is contained in the torsion of \( E \) whenever \( \Gamma \) is contained in the torsion in \( X \).
- \( \Gamma \) is a group whenever \( \Gamma \) is group.

Corollary

There are infinitely many Toroidal Bely˘ı pairs where the set of quasi-critical points forms a group.

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