Elliptic Curves: Minimal Discriminants and Additive Reduction

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Abstract

Elliptic curves over $\mathbb{Q}$ that admit a cyclic isogeny of degree $n$ are parameterizable. In this project, we consider the family of parameterized elliptic curves corresponding to an isogeny class degree of 4. We classify their minimal discriminants and give necessary and sufficient conditions for determining the primes at which additive reduction occurs.

Elliptic Curves

- Let $E$ be the field of rational numbers. We define an elliptic curve $E/\mathbb{Q}$ as a curve given by an affine Weierstrass model $E: y^2 + a_2xy + a_3y = x^3 + a_4x + a_6$ where $a_i \in \mathbb{Q}$ and every point on the curve has a unique tangent. We also include a point at infinity $O$. If such an $E$ exists over $\mathbb{Q}$, then we say that $E$ is given by an integral Weierstrass model.

The signature of an elliptic curve $E$ is $\text{Spec}(E) = (a_1, a_2, \Delta)$ where $a_2, a_3$ and $\Delta$ are the invariants of $E$ defined to be:

$\Delta = -16(4a_4^3 + 27a_6^2) = 6a_2^2 + 36a_4 + 108a_6 = 216(a_3^2 - a_2a_6 - 4a_4)^2$.

If $\Delta = 0$, then we have the following relationship $|a_1| = \text{gcd}(a_2, a_3)$. If $\Delta \neq 0$, then we have that $E$ is isomorphic to $E' = \frac{a_1}{\text{gcd}(a_2, a_3)}x + \frac{a_6}{\text{gcd}(a_2, a_3)}$.

The isogeny class of $E$ is $\text{Spec}(E) = (a_1/a_2, a_2, \Delta/a_2^2)$.

Isogenies

- We say that $\pi : E_1 \rightarrow E_2$ is an isogeny if $\pi$ is a surjective group homomorphism $\pi : E_1 \rightarrow E_2$. The kernel $\pi$ is finite and we define the degree of the isogeny to be $|\ker \pi|$. We say that an isogeny is cyclic if $\ker \pi \approx \mathbb{Z}/n\mathbb{Z}$; and we say that $n$ is an isogeny.