Trace Ideals over Numerical Semigroup Rings

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Pomona Research in Mathematics Experience (PRIME)

Abstract

Numerical semigroups have been studied formally starting in the early 1900s by mathematician Ferdinand Georg Frobenius (1849-1917). Every numerical semigroup can be enriched to have a ring structure called a numerical semigroup ring. Numerical semigroup rings are a more recent object of study and have applications in algebraic geometry as quotients of toric ideals and appear frequently in combinatorial algebra. In this project we study semigroup rings using trace ideals. Trace ideals are a generalization of the trace property and provide a rich theory to detect properties of rings and modules. Our research studies the structure of trace ideals of numerical semigroup rings with few generators. We prove three propositions concerning trace ideals of monomial ideals of numerical semigroup rings.

Trace Ideal

Let $R$ be a commutative ring with unity and $M$ a module over $R$. The trace ideal $\tau_M(R)$ is defined as:

$$\tau_M(R) = \sum_{\alpha \in \text{Hom}_R(M,R)} \alpha(M)$$

The trace ideal reflects important structural features of its module and conveys large amounts of useful information. For example, they can detect important properties of rings such as the Greenstein property and also detect free summands. Furthermore, trace ideals are easy to calculate and behave well under many standard operations.

Computing Trace Ideals

Suppose that

$$K[S]^n \to K[S]^n \to K[S] \to 0$$

is exact so that $[f]$ is the presentation matrix of $I$. Let $[a]$ be a $1 \times n$ matrix representing a map from $R^n$ to $R$.

$$K[S]^n \to K[S] \to [f]$$

• Observe that $[a] = a \in \mathbb{R}$ and if only if $[f] = 0$ (Vasconcelos [3]).

• Then, Hom$_K(I,K[S])$ can be identified with ker$f^T$.

• If $[f]$ is a matrix whose columns generate ker$f^T$, then $\tau_f(K[S])$ is generated by the entries of $[f]$.

This process can be done in Macaulay2 which allows us to automate the computation of trace ideals of semigroup rings.

Example of trace ideals

Consider the numerical semigroup $S = \langle 1, 4 \rangle$ and let $1 \subset I \subset K[S]$ be an ideal of $K[S]$. We list the trace ideal $\tau_I(K[S])$ and the structure of Hom$_K(I,K[S])$ below:

<table>
<thead>
<tr>
<th>$I$</th>
<th>$\text{Hom}_K(I,K[S])$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>{1}</td>
</tr>
<tr>
<td>$I^2$</td>
<td>{1, 2, a, b, c, K[S]}</td>
</tr>
<tr>
<td>$I^3$</td>
<td>{a, b, c \in K[S]}</td>
</tr>
<tr>
<td>$I^4$</td>
<td>{a, b, c \in K[S]}</td>
</tr>
</tbody>
</table>

Motivating Questions

Given the following:

• $S = \langle a_1, \ldots, a_s \rangle$ a numerical semigroup.

• $K$ a field

We ask the questions:

• What are the trace ideals of $K[S]$?

• How are the properties of $S$ related to the trace ideals of $K[S]$?

Conductor Proposition

Let $S = \langle a_1, \ldots, a_s \rangle$ be a numerical semigroup and $I = (\langle d \rangle, F^R) \subset K[S]$. If $d > F(S)$ and $d - a \notin \mathfrak{a}$ then $\tau_I(K[S]) = (\langle d \rangle, F(S))$.

Proof:

For a finitely generated numerical semigroup, the Frobenius number has the following form:

$$F(S) = ab - a - b$$

Since $d > 2\mathfrak{a}$, we have that

$$d - a > ab - a - b = F(S)$$

We have shown that $I = \mathfrak{a}$ is $S$ and we conclude that $I$ is a principal ideal of $K[S]$ since $F(G) = (\langle d \rangle, F(S))$.

Maximal Ideal Proposition

Let $S = \langle a_1, \ldots, a_s \rangle$ with $n \geq 2$. The ideal $I = (\langle d \rangle, F^R, \ldots, F^R) \subset K[S]$ is a trace ideal of $K[S]$.

Proof Sketch A:

• Note: $I \subset K[S]$.

• $I$ is a maximal ideal of $K[S]$ hence $\tau_I(K[S]) = I$ or $\tau_I(K[S]) = K[S]$.

• $K[S]$ is local therefore $\tau_I(K[S]) = K[S]$ implies $I$ is a principal ideal.

• However, $I$ is not principal by assumption.

• We conclude that $\tau_I(K[S]) = I$.

Corollary to Maximal Ideal Proposition

Let $S = \langle a_1, \ldots, a_s \rangle$ with $n \geq 2$, and $I \subset K[S]$ be a non-principal ideal of $K[S]$. If $\langle d \rangle, F^R, \ldots, F^R \subset K[S]$ then $\tau_I(K[S]) = (\langle d \rangle, F(S))$.

Power to Ring Proposition

Let $S = \langle a_1, \ldots, a_s \rangle$ be a numerical semigroup. Denote $I_k = (\langle d \rangle, F(S), \ldots, F(S), F(S))$. There exists some $M \in \mathbb{N}$ such that $m > M$ implies $\tau_{I_k}(K[S]) = K[S]$.

Proof Sketch A:

• There exists a large enough $M \in \mathbb{N}$ such that $m(M - a_k) > F(S)$ for all $1 \leq k \leq m$ with $a_k > 0$.

• $m > M$ implies that $m_i = (a_i - a_k) \in S$ for all $1 \leq i, j \leq n$.

• Fix some $a_k$ then $m_{i,k} = m_i - a_k$ for every $i$.

• Consequently, $I_k$ is a principal ideal of $K[S]$ and $\tau_I(K[S]) = K[S]$.

Future Work

All of our conjectures focus on ideals generated by monomials. We chose to focus on these ideals because the homomorphic image of these ideals to the semigroup ring are often generated by simple fractions. Furthermore, it is possible to relate the possible homomorphisms of numerical ideals to the gaps of the underlying numerical semigroup. The next step in our work is to develop techniques to classify the trace ideals of non-monomial ideals. There is new development in the theory of trace ideals of semigroup rings by Kobayashi et al. [7] and we hope to apply this theory to understand the structure of trace ideals of non-monomial ideals.

References


Acknowledgements

• Dr.brero Hendricks (Harvey Mudd College)
• Olivia del Guercio (Rice University)
• Ronan Ramirez (University of California, Irvine)
• Dr. Erika R. Becker (Pomona College)
• Dr. Alex Barrientos (University of St. Thomas)
• Dr. Evan Galicia (University of Colorado Boulder)
• Dr. Lara Kass (Harvey Mudd College)
• Department of Mathematics, Pomona College
• National Science Foundation (DMS-2125172)

Methods

We utilized Macaulay2 to compute trace ideals over semigroup rings. Macaulay2 is a mathematics software system for computational algebraic geometry and commutative algebra. It has core algorithms for computing trace ideals including calculating free resolutions of modules and other important linear algebra subroutines. We used two important functions for this project to define semigroup rings and compute their trace ideals. We created the semigroup rings code and modified the trace ideals code in Macaulay2’s paper. The code is below.

```plaintext
semigroupRing := S => (f = unfold f; t = length f; 0 = (x, y) list t x, y; f = apply f, p2; return (transpose p2))
```

```plaintext
traceIdeal := I => (f = f; s = 3; if m > s then return (1)); (m, n, v) = (s, s, s) return (1))
```

Figure 1: The numerical semigroup $S = \langle 1, 4 \rangle$.

Figure 2: The numerical semigroup $S = \langle 1, 4 \rangle$ mapping into the polynomial ring $K[X]$. 

Figure 3: The trace ideal of $M$ over $R$.

Figure 4: The trace ideal $(I, M)$ of $K[S]$.