Homework 1: Review of Metric Spaces

"topology [topo- (from Greek topos place + -logy] 1. Topographical study of a particular place; specif., the history of a region as indicated by its topography. 2. Anat. The anatomy of a particular region of the body. 3. Math. The doctrine of those properties of a figure unaffected by any deformation without tearing or joining." - Webster's New Collegiate Dictionary

1. We can define a metric space more succinctly than we did in 131 as follows:

Definition. *Let M be a set and $d : M \times M \to \mathbb{R}$ be a function such that*

(1) $d(a, b) = 0$ *iff* $a = b$ ***(nondegeneracy)***

(2) $\forall a, b, c \in M$, $d(b, c) \leq d(a, b) + d(a, c)$ ***(triangle inequality)****.*

Then we say d is a ***metric*** *or distance and (M, d) denotes a* ***metric space****.*

Using this above definition show that if d is a metric for A, then $d(a, b) \geqslant 0$ and $d(a, b) = d(b, a)$ for all a and b in A. These are properties that were part of the definition of a metric in Math 131.

2. Show that $d(x, y) = (x - y)^2$ does not define a metric on $\mathbb{R}$.

3. Let M denote the metric space $(\mathbb{R}, d)$ where d is the usual metric. Let M_0 denote the metric space $(\mathbb{R}, d_0)$ where d_0 is the discrete metric. Show that all functions $f : M_0 \to M$ are continuous. Show that there does not exist any injective continous function from M to M_0.

4. Give an example of a continuous function between metric spaces which does not take open sets to open sets, and give an example of a discontinuous function between metric spaces which does take open sets to open sets.

5. Let M denote the metric space of continuous real valued functions on the interval $[0, 1]$, where $d(f, g) = \max\{|f(x) - g(x)|\}$. Let $p \in [0, 1]$, and let $F : M \to \mathbb{R}$ be given by $F(f) = f(p)$. Prove that F is continuous.