**Homework 2: Topological Spaces**

"Point set topology is a disease from which later generations will regard themselves as having recovered." - Henri Poincaré

1. Let $X$ be a topological space and let $A$ and $B$ be subsets of $X$. Prove or disprove the following statements.

a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

b) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

2. Let $X$ be a topological space and let $U$ be an open set in $X$. Is $U = \mathrm{Int}(\overline{U})$? Give a proof or a counterexample.

3a) Consider $(\mathbb{R}, F)$ where $U \in F$ iff $U$ is a union of intervals $(a, b)$ where $a$, $b \in \mathbb{Q}$. Is $F$ the usual topology on $\mathbb{R}$? Prove your answer.

b) Consider $(\mathbb{R}, F)$ where $U \in F$ iff $U$ is the union of intervals $[a, b)$ where $a$, $b \in \mathbb{Q}$. Is $F$ the half-open interval topology? Prove your answer.

4. Let $X$ be a topological space and let $A \subseteq X$ such that $\overline{A} = X$. Let $O$ be open in $X$. Prove that $O \subseteq \overline{A \cap O}$.

5. Let $(X, d)$ denote a metric space, and let $S$ and $T$ be disjoint closed subsets of $X$. Prove that there exist disjoint open sets $U$ and $V$ such that $S \subseteq U$ and $T \subseteq V$.