Homework 3: Continuous Functions and Subspaces

"Today the angel Topology and the demon Abstract Algebra struggle for the soul of each of the mathematical domains." - Herman Weyl

1. Let X and Y be topological spaces and let $X = A \cup B$ where A and B are both closed subspaces of X. Suppose that $f : A \to Y$ and $g : B \to Y$ are continuous functions such that for every $x \in A \cap B$ then $f(x) = g(x)$. Let $h : X \to Y$ be defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

Prove that h is a continuous function.

2. Let $f : X \to Y$ be a continuous function between topological spaces X and Y. If f is (a) injective (b) surjective (c) bijective, which of the following four cases can arise: (1) f is neither closed nor open (2) f is open but not closed (3) f is closed but not open (4) f is both open and closed. Give an example or prove that no such example can exist.

3. A topological space (X, F) is said to be *metrizable* if there is a metric d for X such that the open sets of (X, d) are precisely the sets in F. Let X and Y be homeomorphic topological spaces. Suppose that X is metrizable, prove that Y is also metrizable.

4. Let X be a topological space and let Y denote $\mathbb{R}^2$ with the dictionary order topology. Let $f : X \to Y$ and $g : X \to Y$ be continuous. Let $C = \{x \in X | f(x) \leqslant g(x)\}$. Prove that C is closed in X.

5. Let X be a topological space and let Y denote $\mathbb{R}^2$ with the dictionary order topology. Let $f : X \to Y$ and $g : X \to Y$ be continuous. Define $h : X \to Y$ by $h(x) = \min\{f(x), g(x)\}$. Prove that h is continuous.